Simulation numérique de l’équation de Richards : une stratégie de Galerkine discontinue adaptative pour des applications exigeantes

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Exposé – Huitième école EGRIN, GdR « Ecoulements Gravitaires et RIsques Naturels »

25 mai 2021

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Context

Groundwater dynamics in sedimentary beaches controls various processes:

- Exchanges of fresh/salt water between ocean and coastal aquifers
- Diffusion of dissolved materials (nutrients, pollutants)
- Biogeochemical cycles
- Sediment transport

Relevant questions in the context of global warming (sea level rise) and coastal urbanization:

- Sandy beaches represent around 33% of coastline
- Erosion affects around 25% of sandy beaches
- Important benefits (tourism and ecosystem services)

Luijendijk *et al.* 2018, Vousdoukas *et al.* 2020

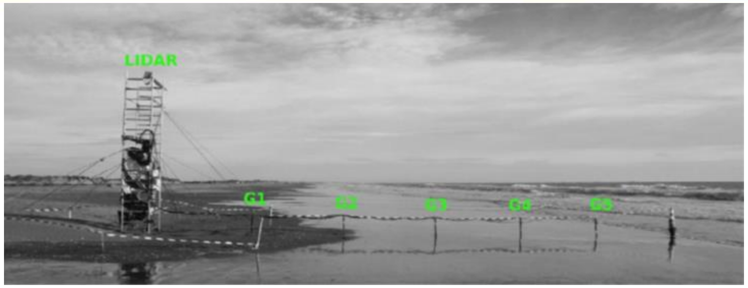
Waikiki beach, Hawaii

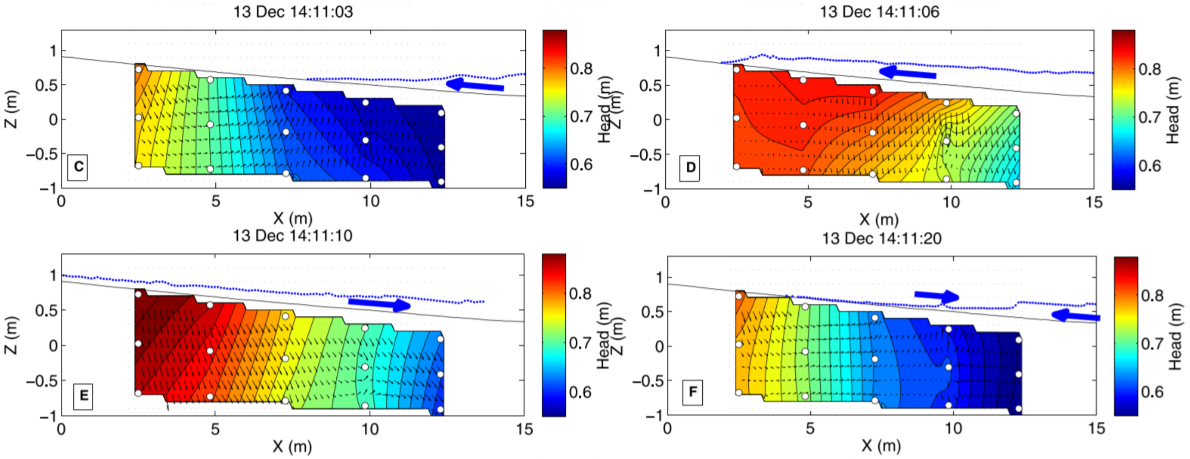
Daytona beach, Florida

Motivations

Modelling starting point:

- Recent but good experimental understanding
  Steenhauer *et al.* 2016, Turner *et al.* 2016, Heiss et al. 2015
- Experimental limitations and few models Li *et al.* 2002, Malott *et al.* 2016





Experiment in Rousty beach, France, Sous *et al*. 2016, 2017
Velocity magnitudes are between 10^{-4} and 10^{-6} m/s

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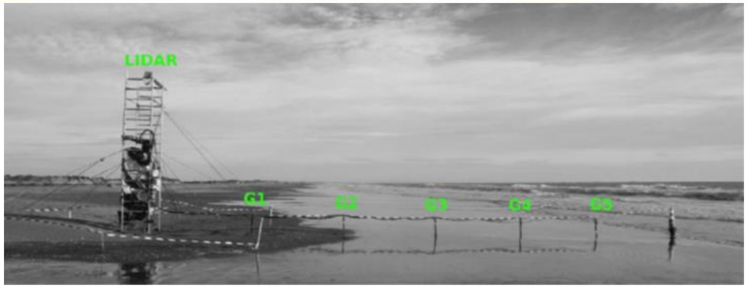
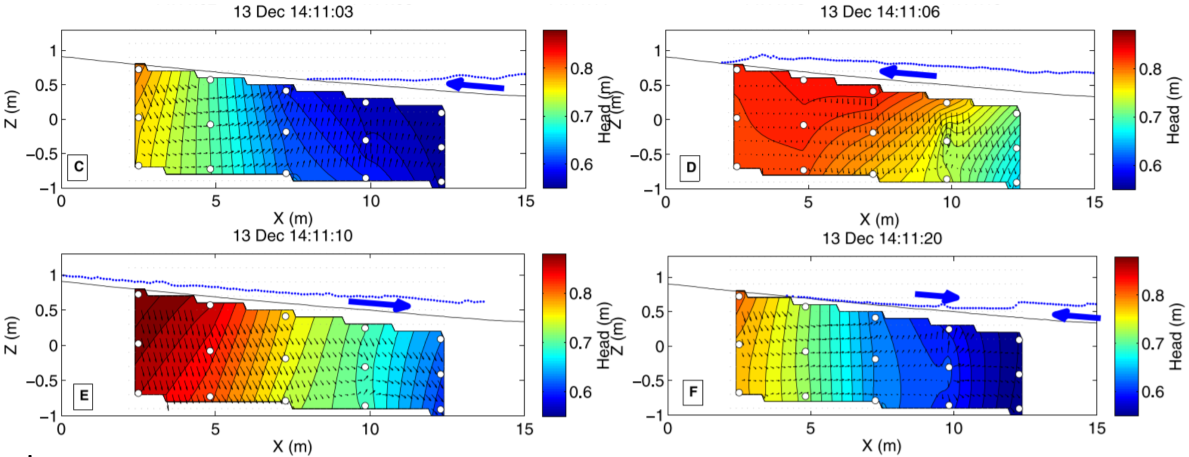


Numerical starting point:

- 3D hyperbolic solver for wave propagation and wave breaking
- Saint-Venant + bifluid Euler + Serre-Green-Naghdi + FSI
- Coupling, interface sharpening, etc
- Unstructured finite volume (MUSL, Riemann solver, Godunov scheme)
- 2nd order RK, Adams-Bashforth, local time stepping
- Block-based adaptive mesh refinement
- Domain decomposition, parallel computing

Ersoy *et al.* 2013, Golay *et al.* 2015



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Li and Raichlen experiment

Yasuda test-case

Outline

I. Modelling
- Richards' equation
- Derivation
- Hydraulic properties
- Seepage
- Challenges and solving strategy

II. Numerical methods
- Discontinuous Galerkin (DG) methods
- Backward differentiation formula (BDF) methods
- Linearization

III. Adaptation
- Adaptive mesh refinement (AMR)
- *A posteriori* error estimates
- Weighted discontinuous Galerkin (WDG) methods

IV. Numerical results
- Polmann's test-case
- Tracy's benchmark
- Simulation of La Verne dam wetting
- Simulations for BARDEX II

Conclusion
- Summary
- Perspectives

Modelling

- Richards’ equation
- Derivation
- Hydraulic properties
- Seepage
- Challenges and solving strategy

Richards' equation

Richards' equation is a classic nonlinear parabolic equation to describe flow in variably-saturated porous media Richardson 1922, Richards 1931

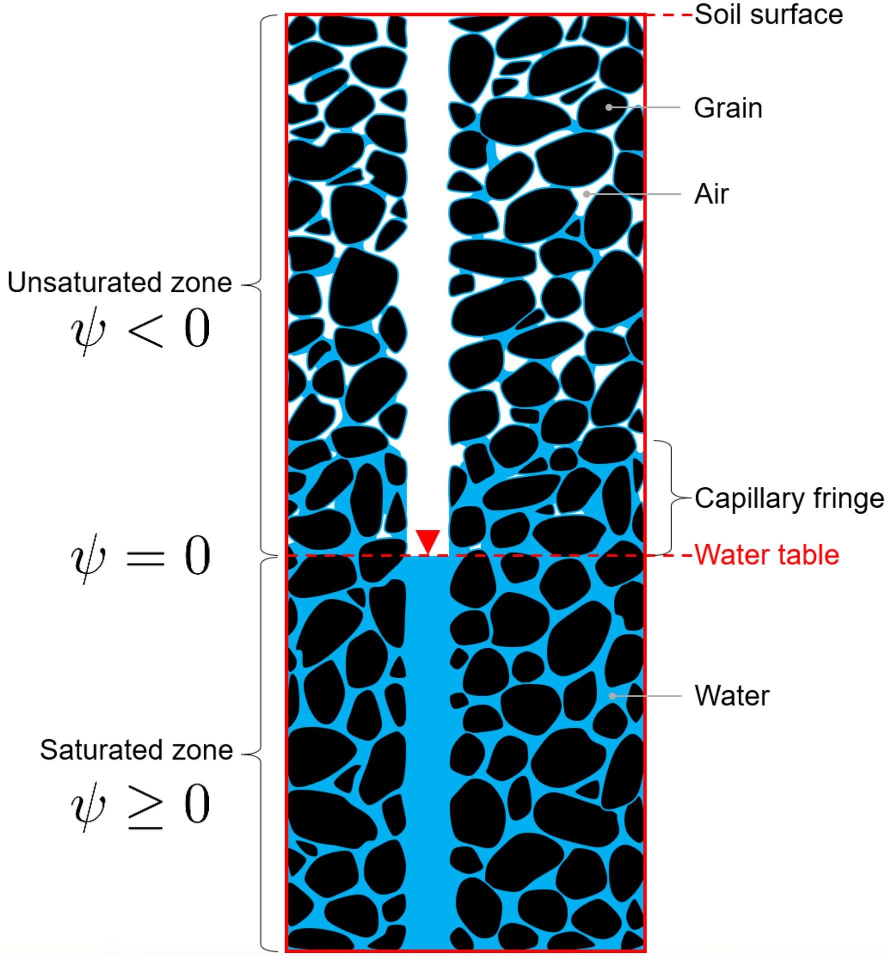
$$\partial_t(\theta(\psi)) - \nabla \cdot (\mathbb{K}(\psi)\nabla(\psi + z)) = 0$$

Mixed form of Richards' equation:

- Pressure head ψ (m)
- Water content θ

Elevation z (m)
Hydraulic conductivity tensor $\mathbb{K}$ (m/s)
Hydraulic head $h = \psi + z$ (m)
Darcy flux $\boldsymbol{q} = -\mathbb{K}(\psi)\nabla h$ (m/s)

Farthing and Ogden 2017, Zha *et al*. 2019



Modelling assumptions

Mass conservation principle applied on a two-phase flow in porous medium:

- Flux modelled by empiric observations Darcy 1851, Buckingham 1907
- Theoretical derivations of flux are possible but problem-dependant (homogenization, volume averaging)

$$\alpha \in \{\text{air}; \text{water}\}, \qquad \begin{cases} \partial_t(\rho_\alpha \Phi S_\alpha) + \nabla \cdot (\rho_\alpha \boldsymbol{q_\alpha}) = 0 \\ \boldsymbol{q_\alpha} = -\dfrac{\mathbb{k}(S_\alpha)}{\mu_\alpha} \nabla(p_\alpha + \rho_\alpha g z) \end{cases}$$

ρ_α density $[\mathrm{M} \cdot \mathrm{L}^{-3}]$

S_α saturation

μ_α dynamic viscosity $[\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-1}]$

$\boldsymbol{q_\alpha}$ Darcy velocity $[\mathrm{M} \cdot \mathrm{T}^{-1}]$

t time $[\mathrm{T}]$

Φ porosity

p_α pressure $[\mathrm{M} \cdot \mathrm{L}^{-1} \cdot \mathrm{T}^{-2}]$

$\mathbb{k}_\alpha$ tensor of permeability $[\mathrm{L}^2]$

g gravitational acceleration $[\mathrm{L} \cdot \mathrm{T}^{-2}]$

z the elevation $[\mathrm{L}]$

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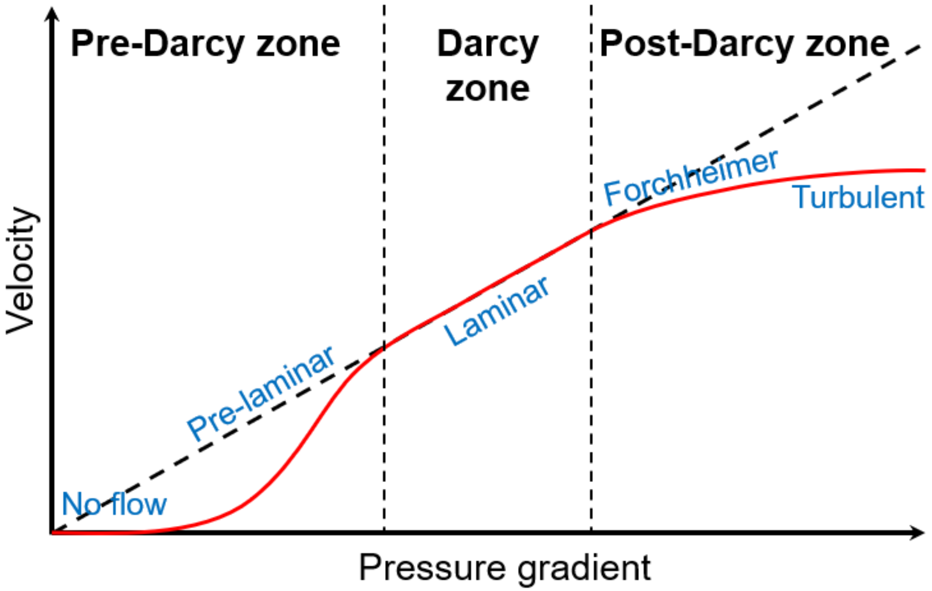

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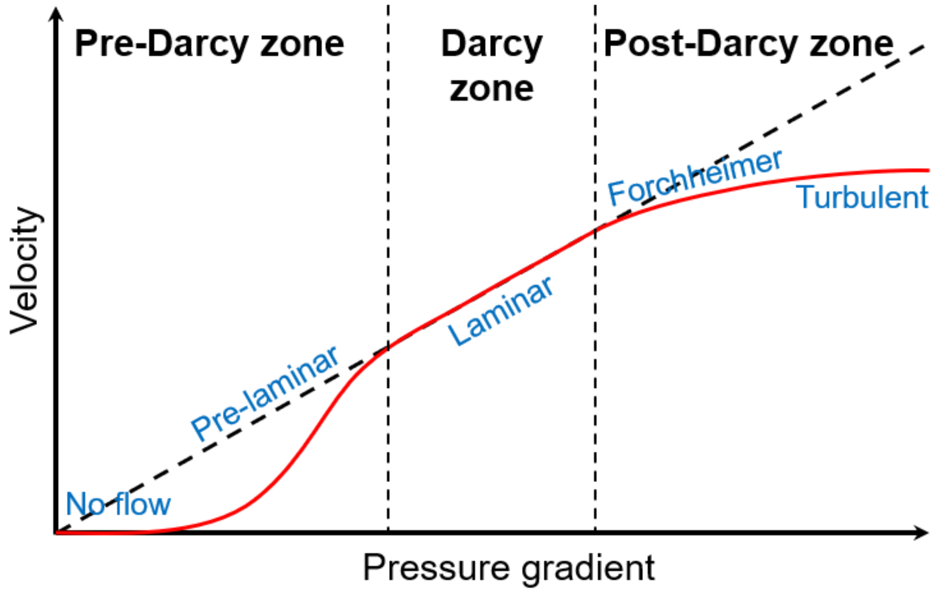

→ 2 equations for 4 unknows

➢ Two closure conditions: $\begin{cases} S_{\text{air}} + S_{\text{water}} = 1, & \text{by definition} \\ p_{\text{air}} - p_{\text{water}} = P_{\text{c}}(S_{\text{water}}) \end{cases}$

Capillary pressure: empiric invertible function

Derivation of Richards' equation

Main hypothesis

- Air viscosity smaller than water viscosity so air pressure balances faster = hydrostatic

$$\nabla(p_{\text{air}} + \rho_{\text{air}} g z) = 0$$

$$\iff p_{\text{air}} = p_0 - \rho_{\text{air}} g z$$

Szymkiewicz 2013, Baron PhD 2015

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(H1) Homogeneous water $\nabla \rho = 0$

(H2) Incompressible water $\partial_t \rho = 0$

(H3) Not deformable solid skeleton $\partial_t \Phi = 0$

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$$\partial_t(\Phi S) - \nabla \cdot \left(\frac{\rho g}{\mu} \mathbb{k}(S) \nabla \left(\frac{p}{\rho g} + z \right) \right) = 0$$

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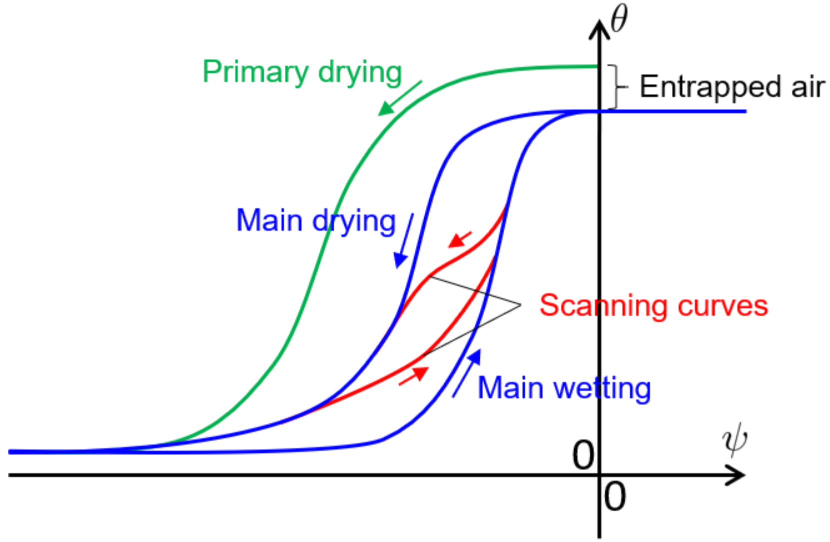
Mixed-form

Szymkiewicz 2013, Baron PhD 2015

Hydraulic properties

Water content and hydraulic conductivity are dynamic and hysteretic functions of pressure head:



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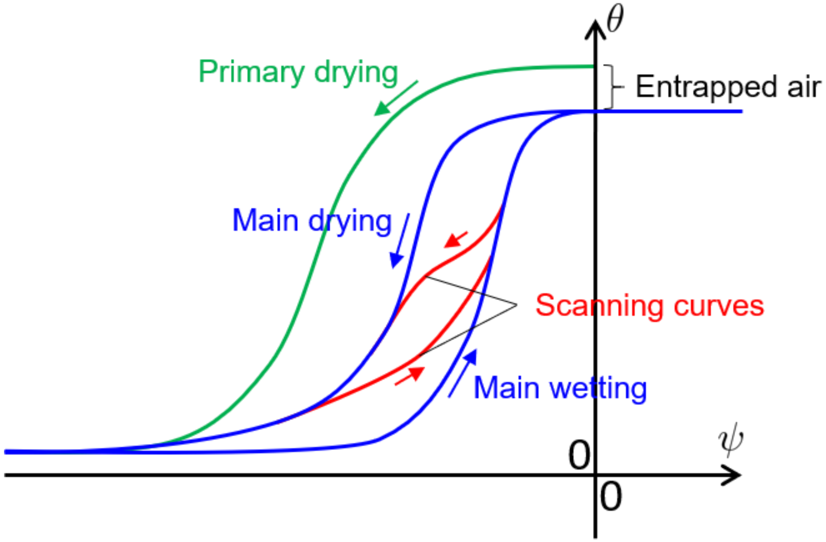

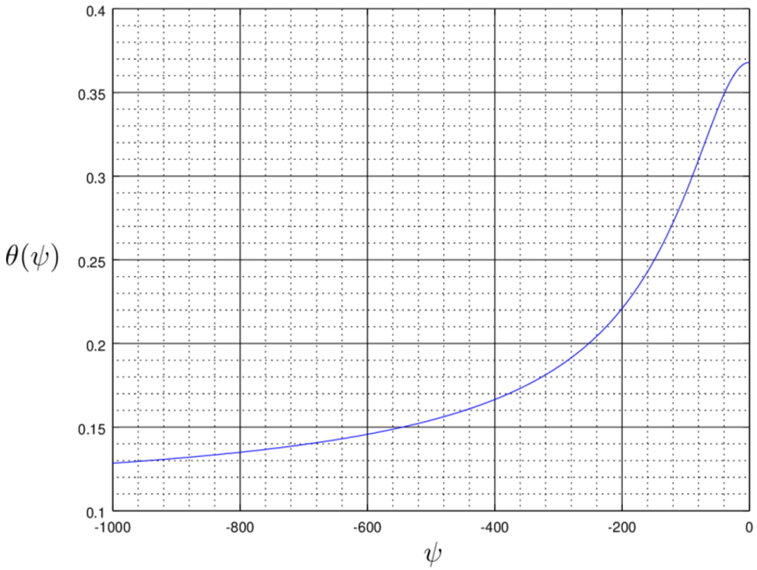

But models are *"simpler"*: $\mathbb{K}(\psi) = \mathbb{K}_{\mathrm{s}} K_{\mathrm{r}}(\psi)$

$$S_{\mathrm{e}}(\psi) = \frac{\theta(\psi) - \theta_{\mathrm{r}}}{\theta_{\mathrm{s}} - \theta_{\mathrm{r}}}$$

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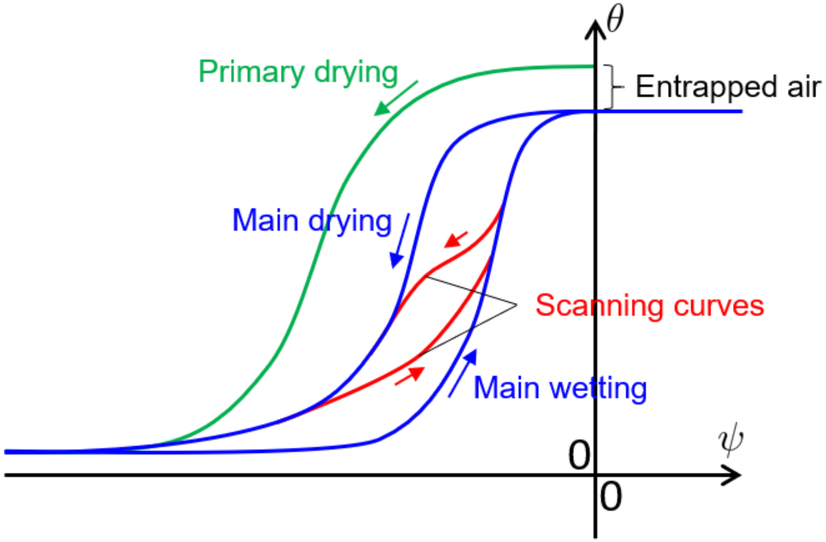

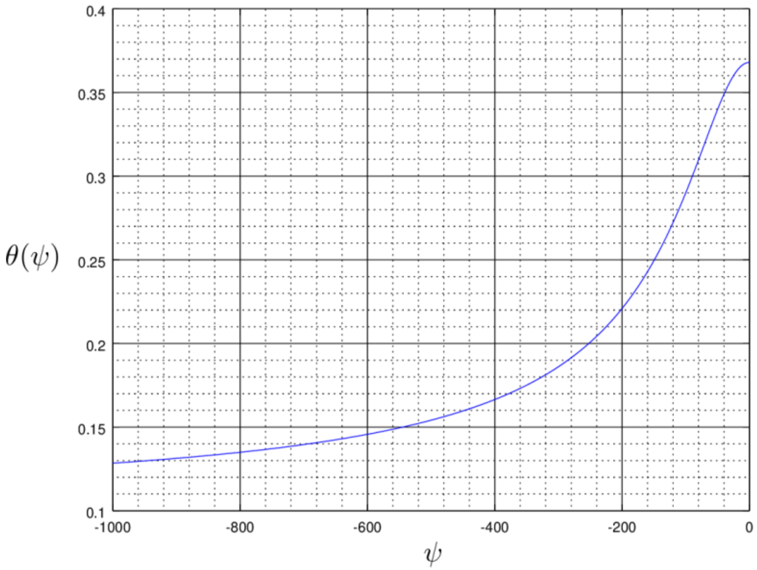

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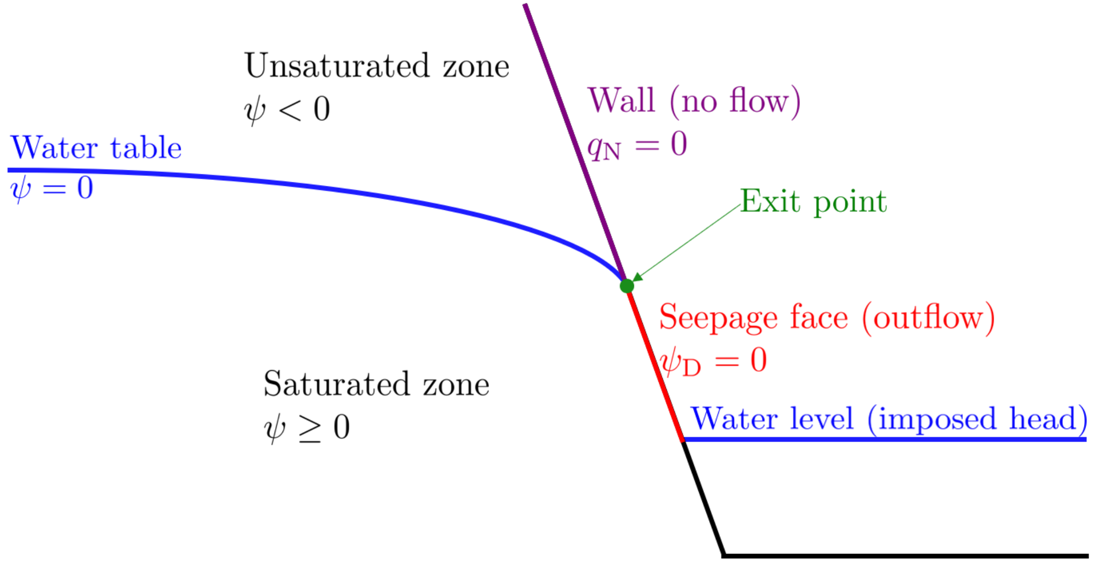
Name	Expression	Parameters
Van Genuchten-Mualem relations (1980)	$S_\mathrm{e} = (1 + (\alpha\lvert\psi\rvert)^n)^{-m}$ $K_\mathrm{r} = S_\mathrm{e}^l\left(1 - \left(1 - S_\mathrm{e}^{\frac{1}{m}}\right)^m\right)^2$	$l = \begin{cases} 0.5 & \text{for Mualem} \\ 1 & \text{for Burdine} \end{cases}$: pore connectivity [-] α: parameter linked to air entry pressure inverse $[\mathrm{L}^{-1}]$ $n > 1$: pore-size distribution [-] $m = 1 - \frac{1}{n}$: pore-size distribution [-]

Seepage boundary condition

- Seepage boundary condition   Scudeler *et al*. 2017

  = mix of Neumann and Dirichlet boundary conditions

  = outflow boundary condition $\begin{cases} h = z & \text{if } h \geq z \text{ and } -\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} > 0 \\ -\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} = 0 & \text{otherwise} \end{cases}$



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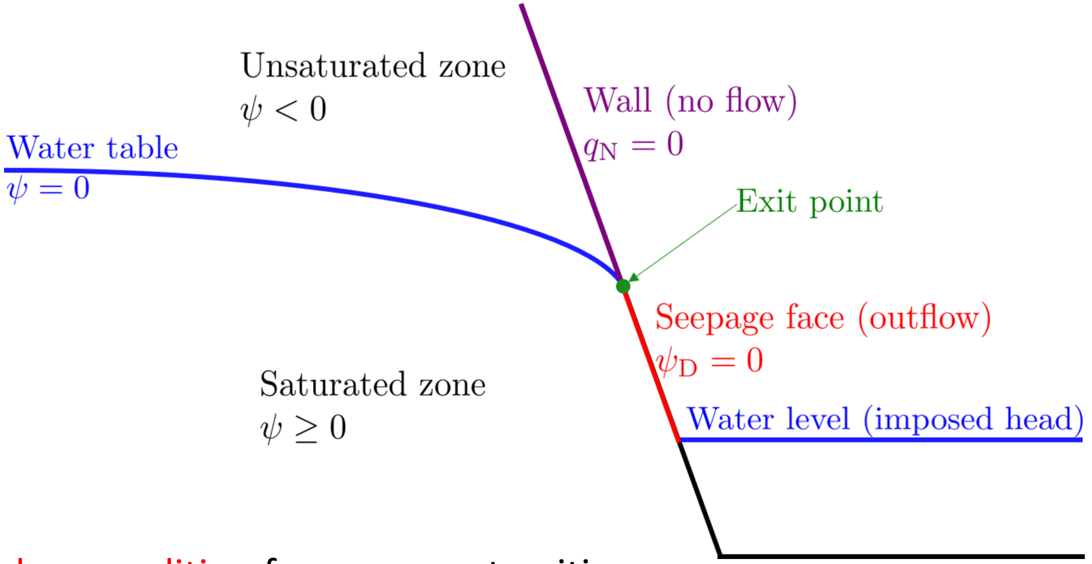

Nonlinear Robin boundary condition for a compact writing:

$$\mathbb{1}_{\mathrm{S}}(h)(h-z) - (1 - \mathbb{1}_{\mathrm{S}}(h))\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} = 0$$

where $\mathbb{1}_{\mathrm{S}} : \Gamma_{\mathrm{S}} \to \{0, 1\}$

$$h \mapsto \begin{cases} 1 & \text{if } h \geq z \text{ and } -\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Behaviours and challenges

$$\partial_t(\theta(\psi)) - \nabla \cdot (\mathbb{K}(\psi)\nabla(\psi + z)) = 0$$

Degeneracy possibilities:

- Complete saturation → θ and $\mathbb{K}$ are constant → elliptic equation → fast diffusion
- Almost complete unsaturation → $\mathbb{K}$ and θ go to 0 rapidly → stopped diffusion
- For $\psi \to 0^-$ → $\mathbb{K}$ and θ can exhibit very steep gradients

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Other strong nonlinearities could happen due to:

- Steep or dynamic boundary conditions
- Steep initial condition
- Heterogeneous porous medium
- Seepage boundary condition (≈ nonlinear Robin boundary condition)

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Groundwater flow features:

- Wetting front moving dynamically and possibly very sharp ↔ nonlinear varying diffusivity
- Internal layer is static and linked to a discontinuity ↔ heterogeneous and anisotropic diffusivity
- Spurious oscillations = non-physical effect (undershoot/overshoot)

Issues and solving strategy

$$\partial_t(\theta(\psi)) - \nabla \cdot (\mathbb{K}(\psi)\nabla(\psi + z)) = 0$$

Richards' equation:

- Extremely sharp fronts (spatial smoothness)
- Stiff partial differential equation (time discretization)
- Nonlinear solver can fail to converge (linearization)

How to get an accurate (*physical*), efficient (*cost-effective*) and robust (*convergent*) simulation?

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Chosen strategy :

- High-order adaptive mesh refinement
- Implicit time scheme
- Iterative nonlinear process with adaptive time stepping + stopping criteria
- Flexible discrete approximation

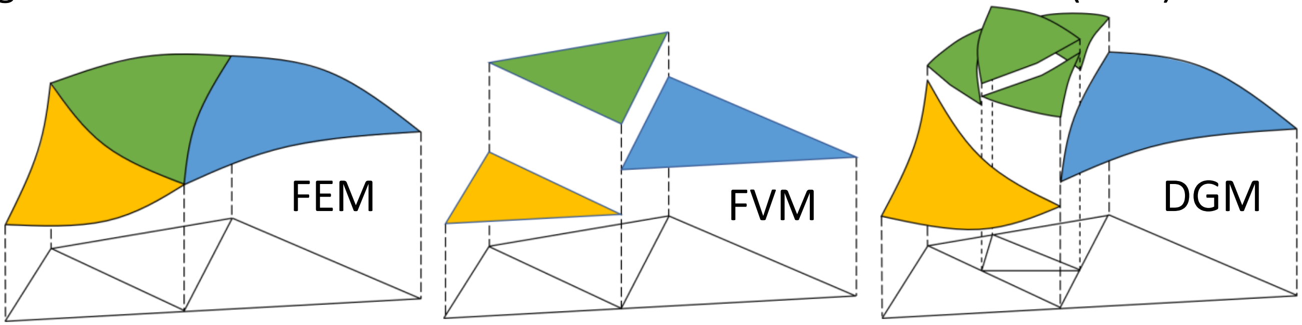
Numerical methods

- Discontinuous Galerkin (DG) methods
- Backward differentiation formula (BDF) methods
- Linearization

Discontinuous Galerkin methods

DG methods: Reed & Hill 1971, Di Pietro, Cockburn/Arnold, Dolejší/Feistauer, Rivière

- Based on variational formulation as in Finite Element Methods (FEM)
- Designed in an element-wise fashion as in Finite Volume Methods (FVM)



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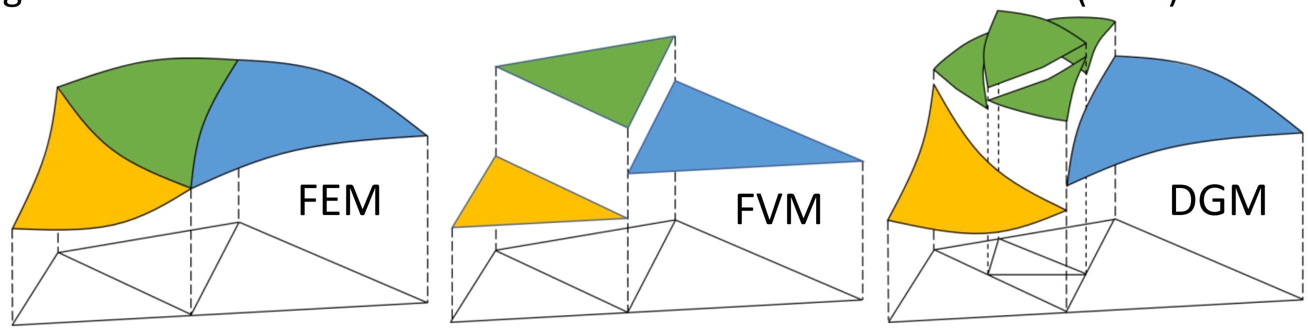

Advantages:

- Compact stencil → high-order approximation
- Element-wise formulation
  - Possible discontinuity
  - Unstructured, non-conforming and hybrid mesh → mesh adaptation
- Weak formulation → boundary condition, analysis
- Local mass balance
- Algebraic system of decoupled equations
  - Sparse structure by blocks (solving)
  - Parallelization

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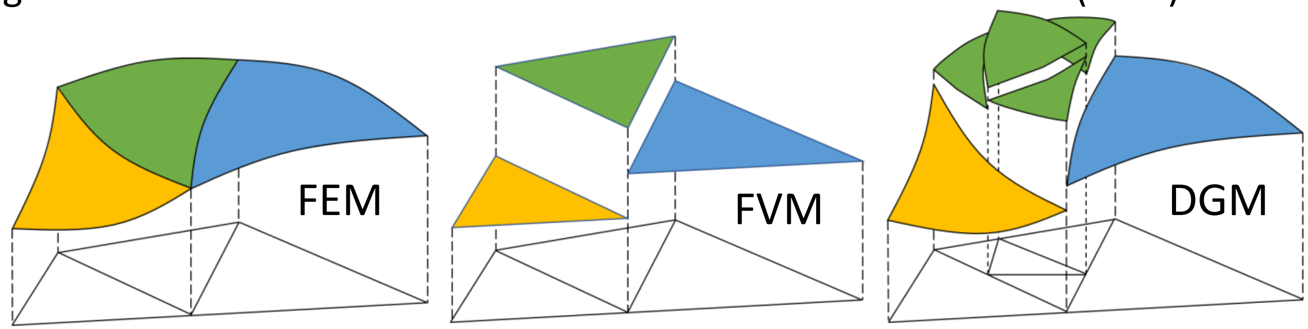

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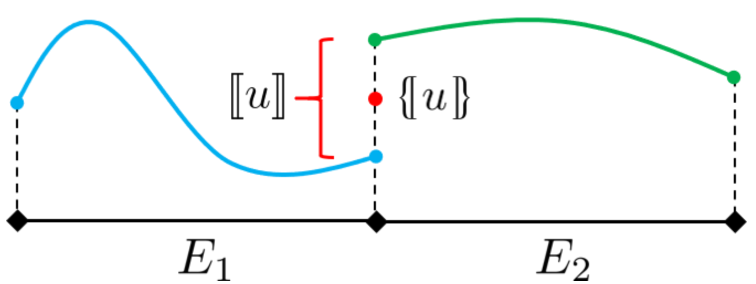
Drawbacks:

- Oscillations (lack of robustness)
  - Stability by penalty parameters
  - Dispersion at discontinuity
- Computational cost → more DoF
- Implementation may be difficult
  - Quadrature formula
  - Visualization
  - Less available codes
  - DoF with no physical meaning
- Still gaps in theoretical analysis

Space semidiscretization: DG methods

Find $h(\boldsymbol{x}, t) : \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \partial_t \theta(h-z) - \nabla \cdot (\mathbb{K}(h-z)\nabla h) = 0, & \text{in } \Omega \times (0,T) \\ h = h_0, & \text{in } \Omega \times \{0\} \\ h = h_{\mathrm{D}}, & \text{on } \Gamma_{\mathrm{D}} \times (0,T) \\ -\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} = q_{\mathrm{N}}, & \text{on } \Gamma_{\mathrm{N}} \times (0,T) \\ \mathbb{1}_{\mathrm{S}}(h)(h-z) - (1-\mathbb{1}_{\mathrm{S}}(h))\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n} = 0, & \text{on } \Gamma_{\mathrm{S}} \times (0,T) \end{cases}$$



Weak formulation:

- Multiply the problem by a test function
- Integrate on each element
- Use Green's theorem
- Sum over all elements

Rivière 2008
Dolejší and Feistauer 2015

Penalization

Find $h \in S_p(\mathcal{E}_\mathfrak{h}^n)$ such that $\forall v \in S_p(\mathcal{E}_\mathfrak{h}^n)$,

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$$

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) = \sum_{E \in \mathcal{E}_\mathfrak{h}} \int_E \partial_t \theta(h-z) v \, \mathrm{d}E$$

$$a_{\mathfrak{h},n}(h,v) = \sum_{E \in \mathcal{E}_\mathfrak{h}} \int_E \mathbb{K}(h-z) \nabla h \cdot \nabla v \, \mathrm{d}E$$

Volume

Interior face

Boundary faces

Dirichlet face

Neumann face

Seepage face

Penalization

Find $h \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$ such that $\forall v \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$,
$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$

Volume

Interior face

Boundary faces
Dirichlet face
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$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) = \sum_{E \in \mathcal{E}_{\mathfrak{h}}} \int_E \partial_t \theta(h-z) v \,\mathrm{d}E$$

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$$- \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n_F} v \,\mathrm{d}F - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \mathbb{1}_{\mathrm{S}}(h)\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n_F} v \,\mathrm{d}F$$

$$l_{\mathfrak{h},n}(v) = - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{N}}} \int_F q_{\mathrm{N}} v \,\mathrm{d}F$$

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$$- \sum_{F\in\mathcal{F}_\mathfrak{h}^{\mathrm{D}}} \int_F \mathbb{K}(h-z)\nabla h\cdot \boldsymbol{n_F} v\,\mathrm{d}F - \sum_{F\in\mathcal{F}_\mathfrak{h}^{S}} \int_F \mathbb{1}_{\mathrm{S}}(h)\mathbb{K}(h-z)\nabla h\cdot \boldsymbol{n_F} v\,\mathrm{d}F$$

Penalty for continuity constraint

$$l_{\mathfrak{h},n}(v) = -\sum_{F\in\mathcal{F}_\mathfrak{h}^{\mathrm{N}}} \int_F q_{\mathrm{N}} v\,\mathrm{d}F$$

$$\varrho_F^{\mathrm{I}} := \frac{\sigma_F^{\mathrm{I}}\gamma_F}{\mu_F}$$

$$\mu_F := \frac{{\mathfrak{h}_F}^{\beta}}{{p_F}^2}$$

Penalization

Find $h \in S_p(\mathcal{E}_\mathfrak{h}^n)$ such that $\forall v \in S_p(\mathcal{E}_\mathfrak{h}^n)$,

$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$

Volume

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Boundary faces

Dirichlet face

Neumann face

Seepage face

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) = \sum_{E \in \mathcal{E}_\mathfrak{h}} \int_E \partial_t \theta(h-z) v \, \mathrm{d}E$$

$$a_{\mathfrak{h},n}(h,v) = \sum_{E \in \mathcal{E}_\mathfrak{h}} \int_E \mathbb{K}(h-z) \nabla h \cdot \nabla v \, \mathrm{d}E - \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{I}} \int_F \{\!\!\{ \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} \}\!\!\} [\![v]\!] \, \mathrm{d}F + \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{I}} \int_F \varrho_F^\mathrm{I} [\![h]\!] [\![v]\!] \, \mathrm{d}F$$

$$- \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{D}} \int_F \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} v \, \mathrm{d}F - \sum_{F \in \mathcal{F}_\mathfrak{h}^S} \int_F \mathbb{1}_\mathrm{S}(h) \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} v \, \mathrm{d}F$$

$$+ \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{D}} \int_F \varrho_F^\mathrm{D} h v \, \mathrm{d}F + \sum_{F \in \mathcal{F}_\mathfrak{h}^S} \int_F \varrho_F^\mathrm{D} \mathbb{1}_\mathrm{S}(h) h v \, \mathrm{d}F$$

Penalty for continuity constraint

$$l_{\mathfrak{h},n}(v) = - \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{N}} \int_F q_\mathrm{N} v \, \mathrm{d}F$$

$$+ \sum_{F \in \mathcal{F}_\mathfrak{h}^\mathrm{D}} \int_F \varrho_F^\mathrm{D} h_\mathrm{D} v \, \mathrm{d}F + \sum_{F \in \mathcal{F}_\mathfrak{h}^S} \int_F \varrho_F^\mathrm{D} \mathbb{1}_\mathrm{S}(h) z v \, \mathrm{d}F$$

Penalty for Dirichlet and seepage boundary conditions

$$\varrho_F^\mathrm{I} := \frac{\sigma_F^\mathrm{I} \gamma_F}{\mu_F}$$

$$\mu_F := \frac{{\mathfrak{h}_F}^\beta}{{p_F}^2}$$

Symmetrization

Find $h \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$ such that $\forall v \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$,

$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h, v) = l_{\mathfrak{h},n}(v)$

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) = \sum_{E \in \mathcal{E}_{\mathfrak{h}}} \int_E \partial_t \theta(h-z) v \, \mathrm{d}E$$

$$a_{\mathfrak{h},n}(h, v) = \sum_{E \in \mathcal{E}_{\mathfrak{h}}} \int_E \mathbb{K}(h-z) \nabla h \cdot \nabla v \, \mathrm{d}E - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \int_F \{\!\!\{ \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} \}\!\!\} [\![v]\!] \, \mathrm{d}F$$

$$+ \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \int_F \varrho_F^{\mathrm{I}} [\![h]\!] [\![v]\!] \, \mathrm{d}F$$

$$- \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} v \, \mathrm{d}F + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \varrho_F^{\mathrm{D}} h v \, \mathrm{d}F$$

$$- \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \mathbb{1}_{\mathrm{S}}(h) \mathbb{K}(h-z) \nabla h \cdot \boldsymbol{n_F} v \, \mathrm{d}F + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \varrho_F^{\mathrm{D}} \mathbb{1}_{\mathrm{S}}(h) h v \, \mathrm{d}F$$

$$l_{h,n}(v) = \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \varrho_F^{\mathrm{D}} h_{\mathrm{D}} v \, \mathrm{d}F - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{N}}} \int_F q_{\mathrm{N}} v \, \mathrm{d}F$$

$$+ \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \varrho_F^{\mathrm{D}} \mathbb{1}_{\mathrm{S}}(h) z v \, \mathrm{d}F$$

Symmetrization

Find $h \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$ such that $\forall v \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$,

$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) = \sum_{E \in \mathcal{E}_{\mathfrak{h}}} \int_E \partial_t \theta(h-z) v \,\mathrm{d}E$$

$$\begin{aligned}
a_{\mathfrak{h},n}(h,v) = & \sum_{E \in \mathcal{E}_{\mathfrak{h}}} \int_E \mathbb{K}(h-z)\nabla h \cdot \nabla v \,\mathrm{d}E - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \int_F \{\!\!\{\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n_F}\}\!\!\} [\![v]\!] \,\mathrm{d}F \\
& + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \int_F \varrho_F^{\mathrm{I}} [\![h]\!] [\![v]\!] \,\mathrm{d}F \\
& - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n_F} v \,\mathrm{d}F + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \varrho_F^{\mathrm{D}} h v \,\mathrm{d}F \\
& - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \mathbb{1}_{\mathrm{S}}(h)\mathbb{K}(h-z)\nabla h \cdot \boldsymbol{n_F} v \,\mathrm{d}F + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \varrho_F^{\mathrm{D}} \mathbb{1}_{\mathrm{S}}(h) h v \,\mathrm{d}F
\end{aligned}$$

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l_{h,n}(v) = & \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \int_F \varrho_F^{\mathrm{D}} h_{\mathrm{D}} v \,\mathrm{d}F - \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{\mathrm{N}}} \int_F q_{\mathrm{N}} v \,\mathrm{d}F \\
& + \sum_{F \in \mathcal{F}_{\mathfrak{h}}^{S}} \int_F \varrho_F^{\mathrm{D}} \mathbb{1}_{\mathrm{S}}(h) z v \,\mathrm{d}F
\end{aligned}$$

		Symmetrization		
		$\Theta = -1$	$\Theta = 0$	$\Theta = 1$
Penalization	$\forall F \in \mathcal{F}_{\mathfrak{h}}, \sigma_F^{\mathrm{I}} = \sigma_F^{\mathrm{D}} = 0$	OBB method	-	global element method
	$\forall F \in \mathcal{F}_{\mathfrak{h}}, \sigma_F^{\mathrm{I}} \neq 0, \sigma_F^{\mathrm{D}} \neq 0$	NIPG	IIPG	SIPG

NIPG: Non-symmetric Interior Penalty Galerkin
SIPG: Symmetric Interior Penalty Galerkin
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OBB method: Oden-Baumann-Babuška method

Epshteyn and Rivière 2006
Dolejší 2008

Symmetrization

Find $h \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$ such that $\forall v \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$,

$m_{\mathfrak{h},n}(\partial_t\theta(h-z),v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$

Chosen method: (primal) IIPG

$$m_{\mathfrak{h},n}(\partial_t\theta(h-z),v) = \sum_{E\in\mathcal{E}_{\mathfrak{h}}} \int_E \partial_t\theta(h-z)v\,\mathrm{d}E$$

$$\begin{aligned}
a_{\mathfrak{h},n}(h,v) = & \sum_{E\in\mathcal{E}_{\mathfrak{h}}} \int_E \mathbb{K}(h-z)\nabla h\cdot\nabla v\,\mathrm{d}E - \sum_{F\in\mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \int_F \{\!\!\{\mathbb{K}(h-z)\nabla h\cdot \boldsymbol{n_F}\}\!\!\}[\![v]\!]\,\mathrm{d}F \\
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\end{aligned}$$

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		$\Theta=-1$	$\Theta=0$	$\Theta=1$
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Epshteyn and Rivière 2006
Dolejší 2008

Symmetrization

Find $h \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$ such that $\forall v \in S_p(\mathcal{E}_{\mathfrak{h}}^n)$,

$$m_{\mathfrak{h},n}(\partial_t \theta(h-z), v) + a_{\mathfrak{h},n}(h,v) = l_{\mathfrak{h},n}(v)$$

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DG for Richards' equation:

- 1D LDG Li *et al.* 2007
- 2D mixed SIPG Sochala 2009
- 2D STDG Dolejší *et al.* 2019

		Symmetrization		
		$\Theta = -1$	$\Theta = 0$	$\Theta = 1$
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Epshteyn and Rivière 2006
Dolejší 2008

DG solution representation

$$\forall t \in (0,T), \forall \boldsymbol{x} \in \Omega, \quad h(\boldsymbol{x},t) = \sum_{k=1}^{N_E} \sum_{l=1}^{N_{\text{dof}}^E} \chi_k^l(t) \Phi_k^l(\boldsymbol{x}) \text{ where } \Phi_k^l(\boldsymbol{x}) = \begin{cases} \phi_k^l(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in E \\ 0 & \text{otherwise} \end{cases}$$

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Choice of the polynomial basis:

- Orthogonal (numerical property)
- Hierarchical (adaptation)

Nodal basis:

- Difficulty to adapt for high-order
- DoF have physical meaning
- Examples : Lagrange, Hermite

Modal basis:

- Easy implementation
- DoF are just coefficients in the DG expansion
- Examples : monomial, Taylor, Legendre, Dubiner

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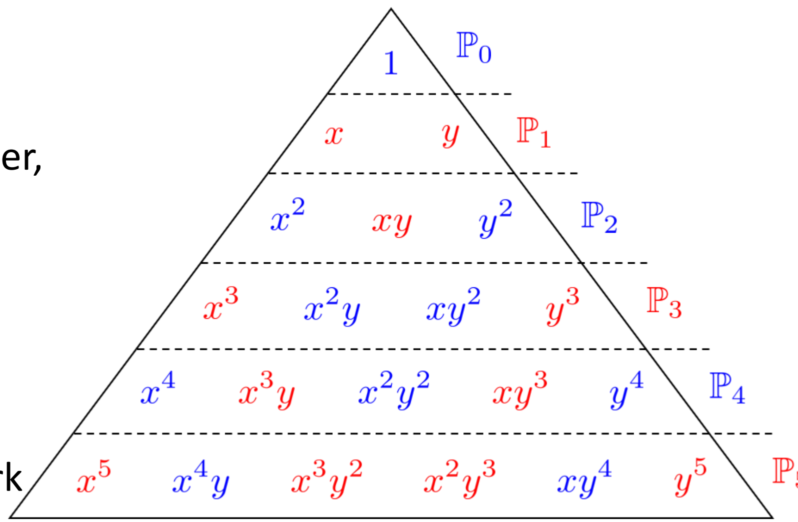
DG numerical approximation:

What is the relation with the number of DoF, basis order, element shape?

What is computational cost?

Monomial basis = each stage

Tensorized monomial basis = each coloured *V*-shaped mark



Time discretization: Backward Differentiation Formula

BDF methods: Dolejší *et al*. 2008, Hay *et al*. 2015

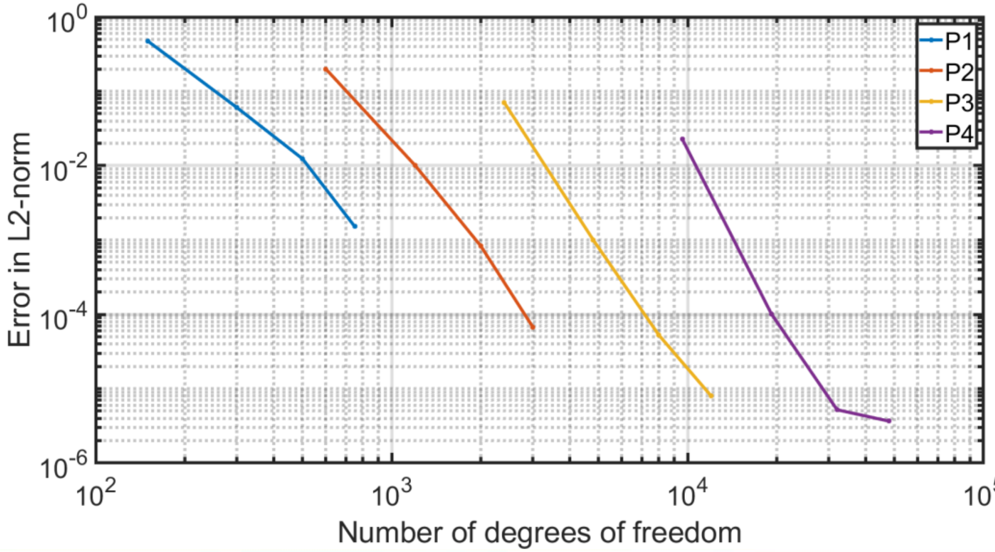
- Multistep implicit schemes
- High-order (up to 6)
- Good stability properties for stiff equations
- Divided differences for adaptive time stepping
- Implicit Euler scheme = 1-order BDF method

Find a sequence of $(h^n)_{n\in\mathbb{N}_+} \in S_p(\mathcal{E}_\mathfrak{h}^n)$ such that

$$\begin{cases} h^0 = h_0 \\ \forall v \in S_p(\mathcal{E}_\mathfrak{h}^n), \quad m_{\mathfrak{h},n}\left(\sum_{k=0}^{q} \frac{\alpha_{q,k}}{\tau^n}\theta(h^{n+1-k} - z), v\right) + a_{\mathfrak{h},n}(h^{n+1}, v) = l_{\mathfrak{h},n}(v, t^{n+1}) \end{cases}$$

Convergence for Tracy's benchmark

$$\|e\|_{L^2} \approx c_\mathfrak{h}\mathfrak{h}^p + c_\tau \tau^q$$



Linearization: Newton-Raphson method and adaptive time step

Newton-Raphson method → Fixed-point iteration

$$r_{\mathfrak{h},n}(h^{n+1},v) := m_{\mathfrak{h},n}\left(\sum_{k=0}^{q}\frac{\alpha_{s,k}}{\tau^n}\theta(h^{n+1-k}-z),v\right)+a_{\mathfrak{h},n}(h^{n+1},v)-l_{\mathfrak{h},n}(v;t^{n+1})$$

$$\begin{cases}\dfrac{\mathrm{d}r_{\mathfrak{h},n}(h^{n+1,m},v)}{\mathrm{d}h^{n+1,m}}\delta_h^{n+1,m} = -r_{\mathfrak{h},n}(h^{n+1,m},v)\\ h^{n+1,m+1} = h^{n+1,m}+\delta_h^{n+1,m}\end{cases}$$

Paniconi & Putti 1994
Lehmann & Ackerer 1998
List & Radu 2016

Relaxation and stopping criteria:

- Damped Newton-Raphson method possibly in combination with fixed-point iteration
- Two convergence criteria: residual and incremental

$$\frac{\left\|r_{\mathfrak{h},n}(h^{n+1,m},v)\right\|_{L^2(\Omega)}}{\left\|a_{\mathfrak{h},n}(h^{n+1,m},v)\right\|_{L^2(\Omega)}} < \varepsilon_1 \quad \text{and} \quad \frac{\left\|\delta_h^{n+1,m}\right\|_{L^2(\Omega)}}{\left\|h^{n+1,m}\right\|_{L^2(\Omega)}} < \varepsilon_2$$

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Adaptive time stepping → nonlinear iterations (heuristic methods)

$$\begin{cases}\tau^{n+1} = \begin{cases}\lambda_{\text{amp}}\tau^n & \text{if } N_{\text{it}} \le m_{\text{it}}\\ \tau^n & \text{if } m_{\text{it}} < N_{\text{it}} \le M_{\text{it}}\\ \lambda_{\text{red}}\tau^n & \text{if } M_{\text{it}} < N_{\text{it}} \le W_{\text{it}}\end{cases}\\ \tau^n = \lambda_{\text{red}}\tau^n \text{ if } W_{\text{it}} < N_{\text{it}} \text{ or if the solver has failed (time step is started again)}\end{cases}$$

Convergence is strengthened but not guaranteed. What about performance?

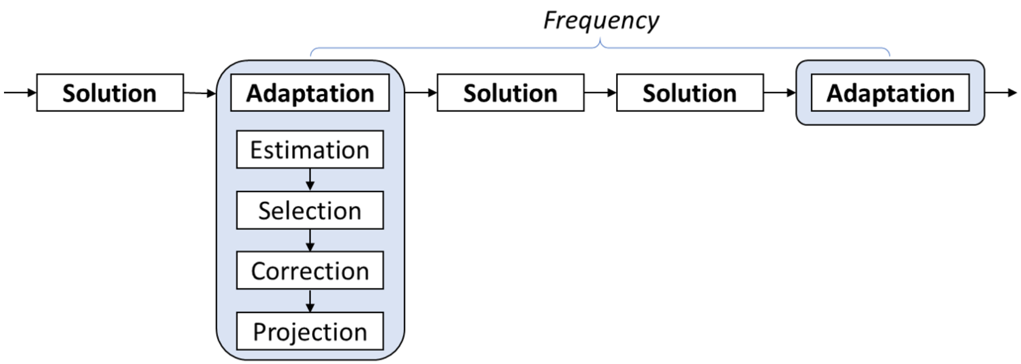
Adaptation

- Adaptive mesh refinement (AMR)
- *A posteriori* error estimates
- Weighted discontinuous Galerkin (WDG) methods

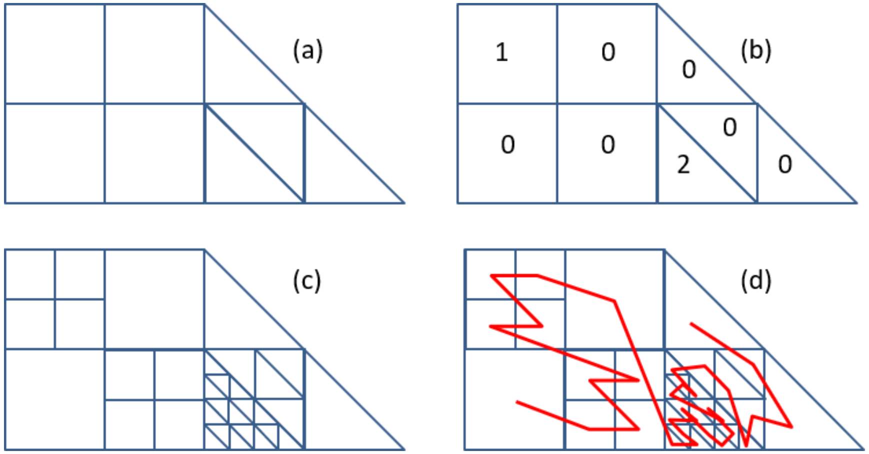
Adaptive mesh refinement (AMR)

Adaptivity:

- Block-based AMR
- Sequential process
- Threshold values for selection
- Projection by local DG problem



(a) Block mesh
(b) Refinement level
(c) Mesh generation using quadtree
(d) Morton numbering



BB AMR Altazin *et al*. 2016, Pons & Ersoy 2019
hp Mitchell 2014
Richards' Li et al. 2006/2007, Miller et al. 2006, Šolín & Kuraz 2011, Dolejší 2019

A posteriori error estimate

Nonlinear parabolic FEM Verfürth 2013
hp elliptic FEM Melenk & Wohlmuth 2001
hp (steady) convection-diffusion DG
Houston et al. 2007
Schötzhau & Zhu 2009/2010

A posteriori error estimate

Error indicator based on *a posteriori* estimation

Energy norm $$\|\!|u|\!\|^2_{\mathcal{E}(E)} := \|u\|^2_{\mathrm{R}(E)} + \sum_{F \in \partial E} \|u\|^2_{\mathrm{J}(F)}$$

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"Residual derivation"

$$\left(\eta_E^n\right)^2 = \left(\eta_{E,\mathrm{R}}^n\right)^2 + \left(\eta_{E,\mathrm{F}}^n\right)^2 + \left(\eta_{E,\mathrm{J}}^n\right)^2$$

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Volume residual $$\left(\eta_{E,\mathrm{R}}^n\right)^2 = \frac{\mathfrak{h}_E^2}{p_E^2 \lambda_\mathrm{m}(\mathbb{K})} \left\| \frac{\theta\left(u_\mathfrak{h}^{n+1}\right) - \theta\left(u_\mathfrak{h}^n\right)}{\tau^n} - \nabla \cdot \left(\mathbb{K}\left(u_\mathfrak{h}^{n+1}\right) \nabla u_\mathfrak{h}^{n+1}\right) \right\|^2_{L^2(E)}$$

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Face residual (flux jump) $$\left(\eta^n_{E,\mathrm{F}}\right)^2 = \sum_{F\in\partial E\cap\mathcal{F}^{\mathrm{I}}_{\mathfrak{h}}} \frac{\mathfrak{h}_F}{2 p_F \kappa_{\mathrm{m}}} \left\| \left[\!\left[\mathbb{K}\left(u_{\mathfrak{h}}^{n+1}\right)\nabla u_{\mathfrak{h}}^{n+1}\cdot \boldsymbol{n}\right]\!\right]\right\|^2_{L^2(F)} + \sum_{F\in\partial E\cap\mathcal{F}^{\mathrm{N}}_{\mathfrak{h}}} \frac{\mathfrak{h}_F}{p_F \kappa_{\mathrm{l}}} \left\| q_{\mathrm{N}} - \mathbb{K}\left(u_{\mathfrak{h}}^{n+1}\right)\nabla u_{\mathfrak{h}}^{n+1}\cdot \boldsymbol{n}\right\|^2_{L^2(F)}$$

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$$\|u\|^2_{\mathrm{R}(E)} := \left\|(\mathbb{K}(u))^{\frac{1}{2}} \nabla u\right\|^2_{L^2(E)}$$

$$\|u\|^2_{\mathrm{J}(F)} := \varrho_F \|[\![u]\!]\|^2_{L^2(F)}$$

$$\kappa_{\mathrm{l}} := \min(\kappa_{\mathrm{l}}, \kappa_{\mathrm{r}})$$

"Residual derivation"

$$\left(\eta_E^n\right)^2 = \left(\eta_{E,\mathrm{R}}^n\right)^2 + \left(\eta_{E,\mathrm{F}}^n\right)^2 + \left(\eta_{E,\mathrm{J}}^n\right)^2$$

Volume residual
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Solution jump
$$\left(\eta_{E,\mathrm{J}}^n\right)^2 = \sum_{F \in \partial E \cap \mathcal{F}_{\mathfrak{h}}^{\mathrm{I}}} \frac{1}{2} \left(\varrho_F^{\mathrm{I}} + \frac{\mathfrak{h}_F}{p_F \kappa_{\mathrm{m}}}\right) \left\| \left[\!\left[u_{\mathfrak{h}}^{n+1} \right]\!\right] \right\|^2_{L^2(F)} + \sum_{F \in \partial E \cap \mathcal{F}_{\mathfrak{h}}^{\mathrm{D}}} \left(\varrho_F^{\mathrm{D}} + \frac{\mathfrak{h}_F}{p_F \kappa_{\mathrm{l}}}\right) \left\| u_{\mathrm{D}} - u_{\mathfrak{h}}^{n+1} \right\|^2_{L^2(F)}$$

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Weighted DG framework

How to avoid spurious oscillations without use of demanding techniques (slope limiters)? **WDG!**

- developed for convection-diffusion equation
- heterogeneous diffusivity matching the mesh

It is not expected to work for nonlinear diffusivity…

Ern & Proft 2006
Proft & Rivière 2006/2009
Ern/Di Pietro 2008

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Two key ingredients:

- Weighted averages to decide the amount of diffusive flux $\{\!\!\{u\}\!\!\}_\omega := \omega_\mathrm{l} u_\mathrm{l} + \omega_\mathrm{r} u_\mathrm{r}$

$$\omega_l + \omega_r := 1 \qquad \begin{cases} \omega_\mathrm{l} = \dfrac{\kappa_\mathrm{r}}{\kappa_\mathrm{l} + \kappa_\mathrm{r}}, \quad \omega_\mathrm{r} = \dfrac{\kappa_\mathrm{l}}{\kappa_\mathrm{l} + \kappa_\mathrm{r}} & \text{if } \kappa_\mathrm{l} + \kappa_\mathrm{r} \neq 0, \\ \omega_\mathrm{l} = \omega_\mathrm{r} = \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

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- Relaxation of penalty to enforce continuity: a discontinuity should approximate better sharp fronts.

$$\gamma_F = \{\!\{\kappa\}\!\}_\omega = \frac{2\kappa_\mathrm{l}\kappa_\mathrm{r}}{\kappa_\mathrm{l} + \kappa_\mathrm{r}}$$

$$\varrho_F^\mathrm{I} := \frac{\sigma_F^\mathrm{I}\gamma_F}{\mu_F} \qquad \mu_F := \frac{\mathfrak{h}_F{}^\beta}{p_F{}^2}$$

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AMR and WDG are (ideally) working in synergy through the estimation-based error indicator:

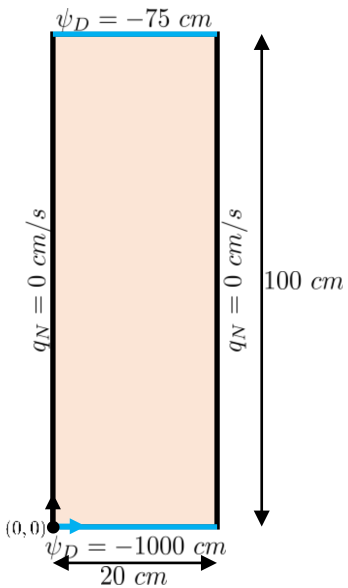
- ➢ AMR is used as a capturing technique
- ➢ WDG adjusts the local numerical smoothness (oscillations are reduced)

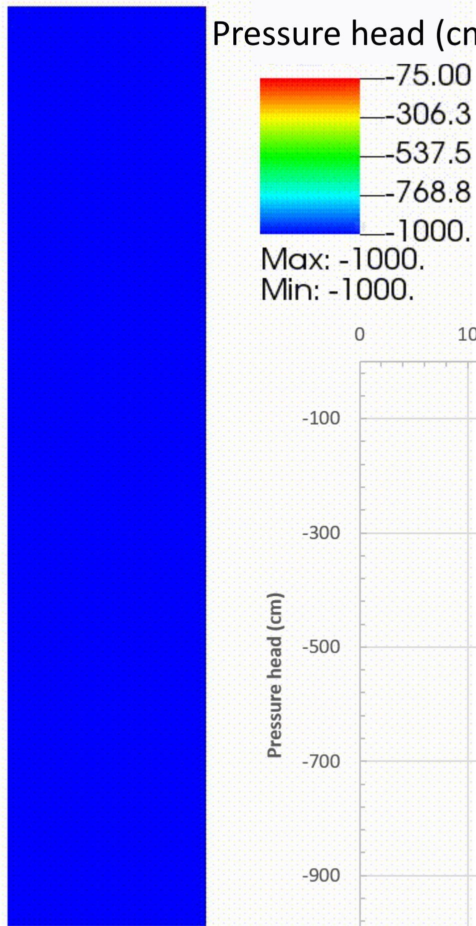
Numerical results

- Polmann’s test-case
- Tracy’s benchmark
- Simulation of La Verne dam wetting
- Simulation of an idealized beach
- Simulation of one case from BARDEX II

Polmann's test-case

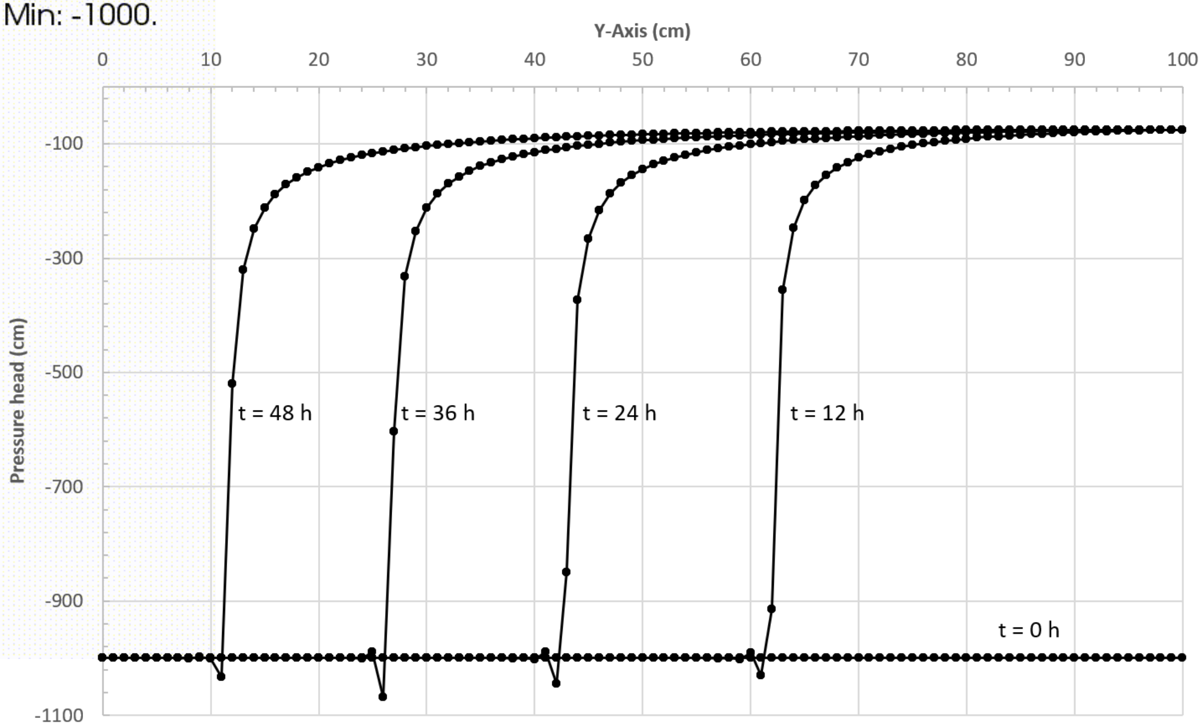
1D vertical infiltration into a soil column (Van Genuchten-Mualem relations)





Computations:

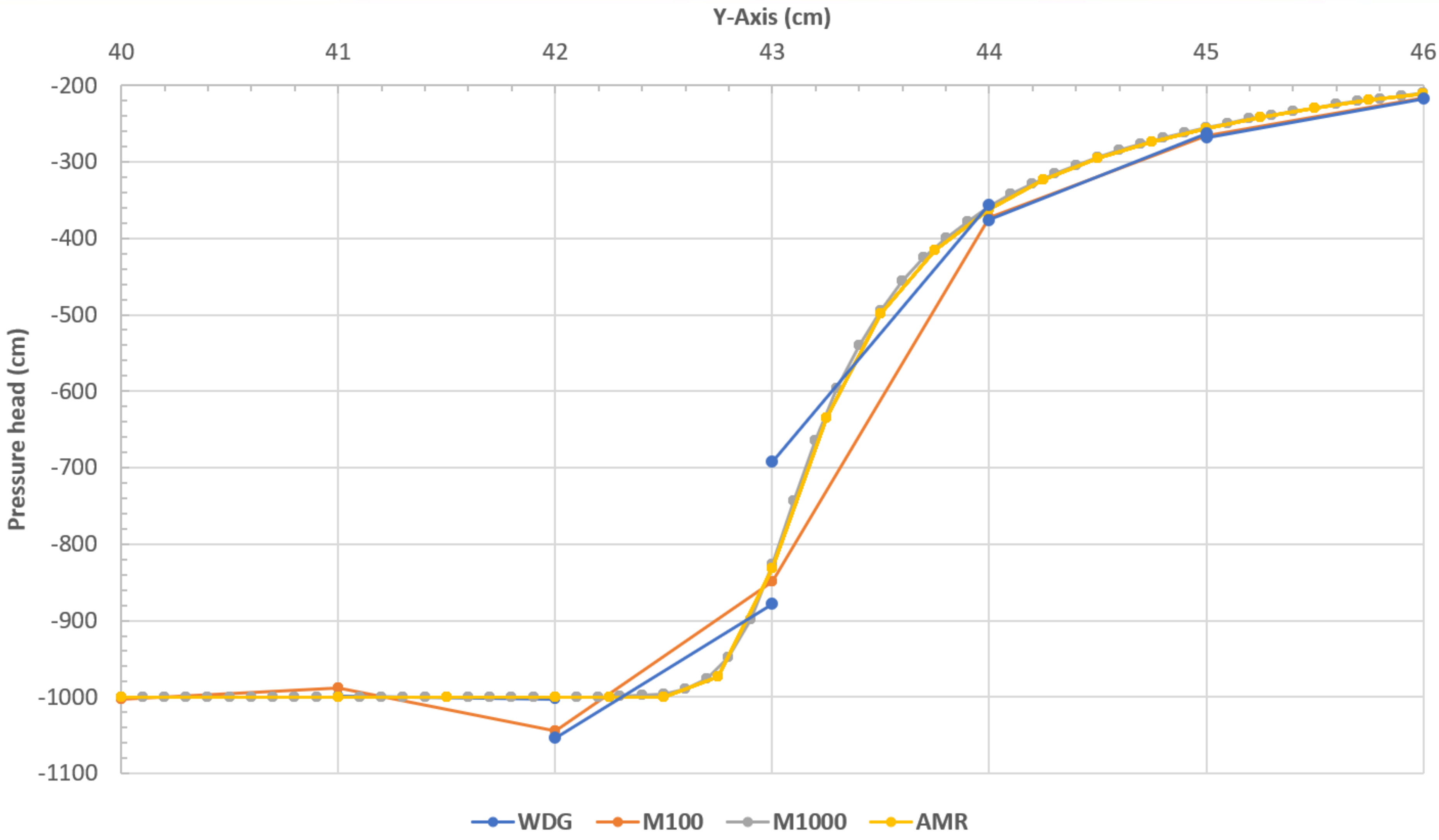
- M100 = mesh with 100 elements
- M1000 = mesh with 1000 elements
- AMR (211 elements)
- WDG (100 elements)



Polmann *et al*. 1988
Celia *et al*. 1990
Manzini & Ferraris 2004
Sochala 2008

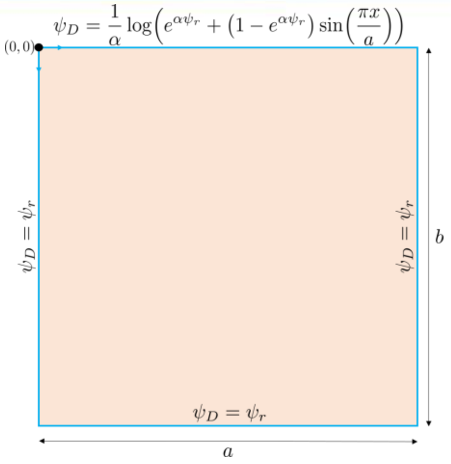
M100 computation: spurious oscillation at the wetting front

Polmann's test-case comparison



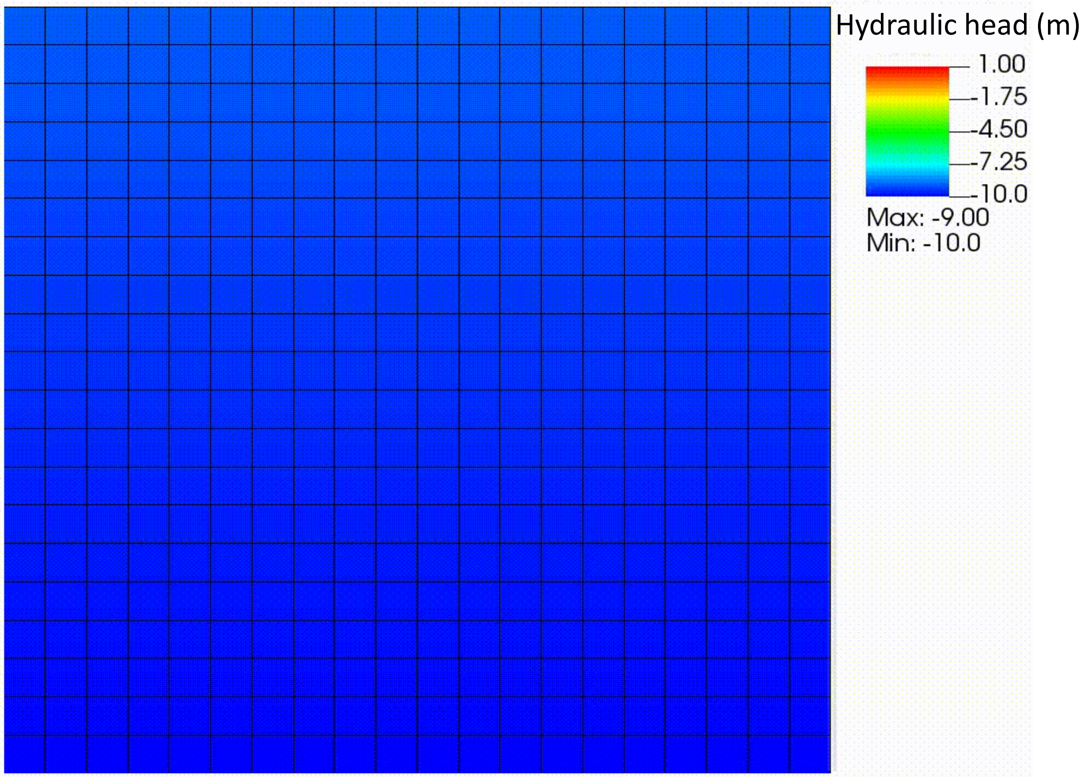
Wetting front at t = 24h

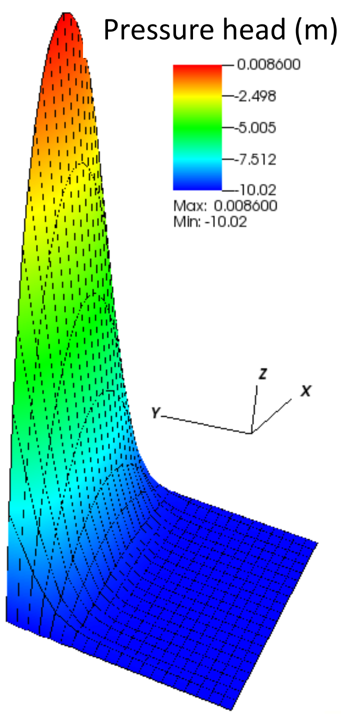
Tracy's benchmark



Analytical solution is known Tracy 2006:

- Steep top boundary condition
- Inconsistency between boundary condition and initial condition

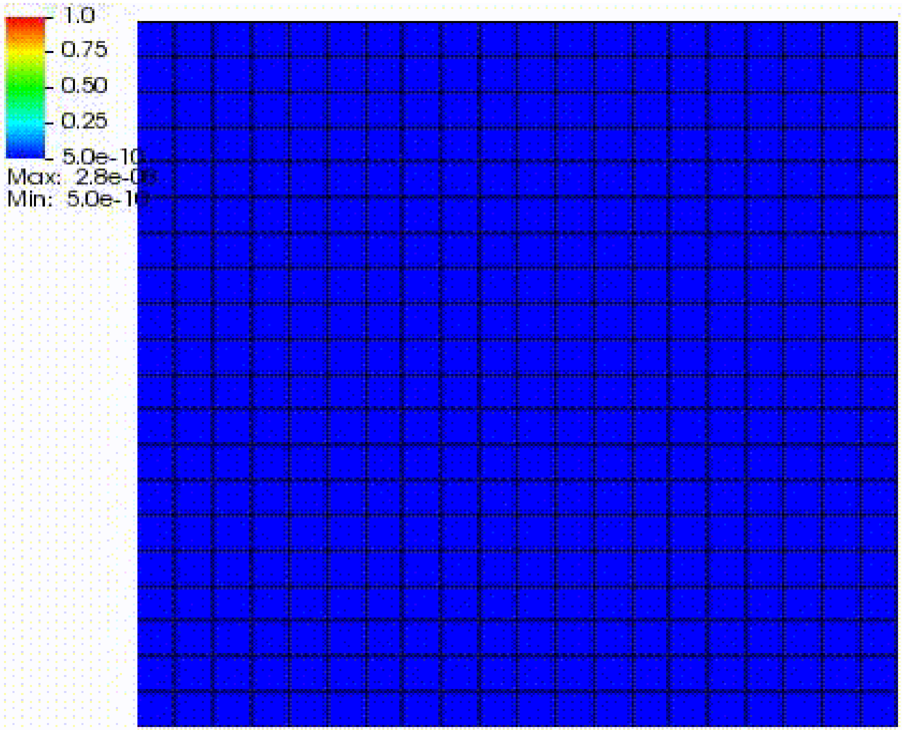




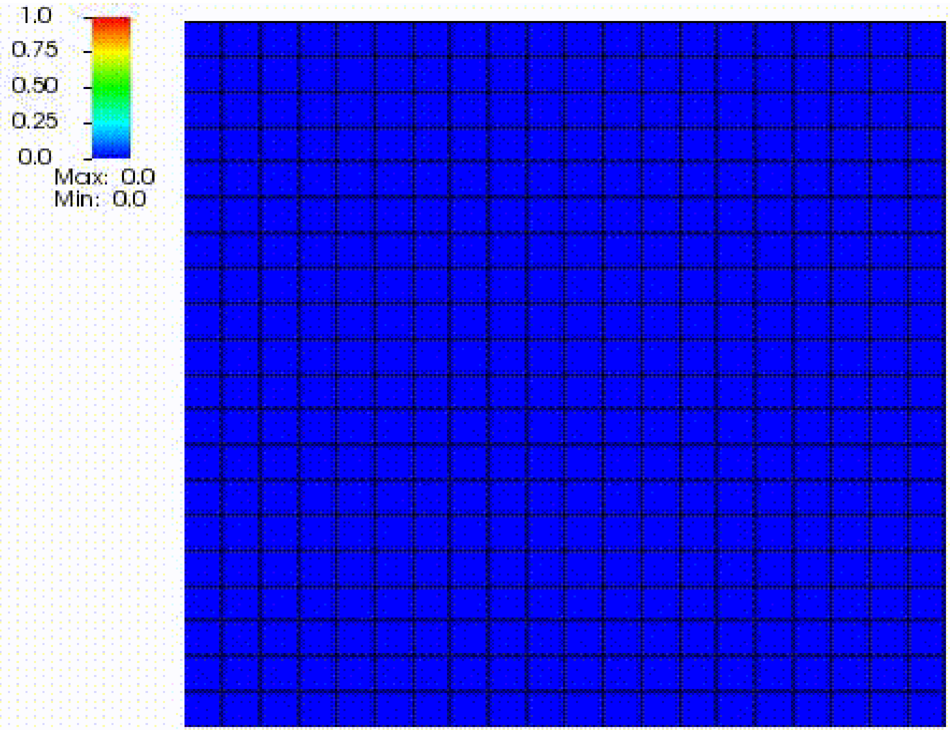
Šolín & Kuraz 2011
Dolejší *et al*. 2019

Error indicator effectivity

True error in energy norm $\|\|e\|\|_{\mathcal{E}(E)}$



Estimation-based error indicator η_E^n



Wetting of La Verne dam

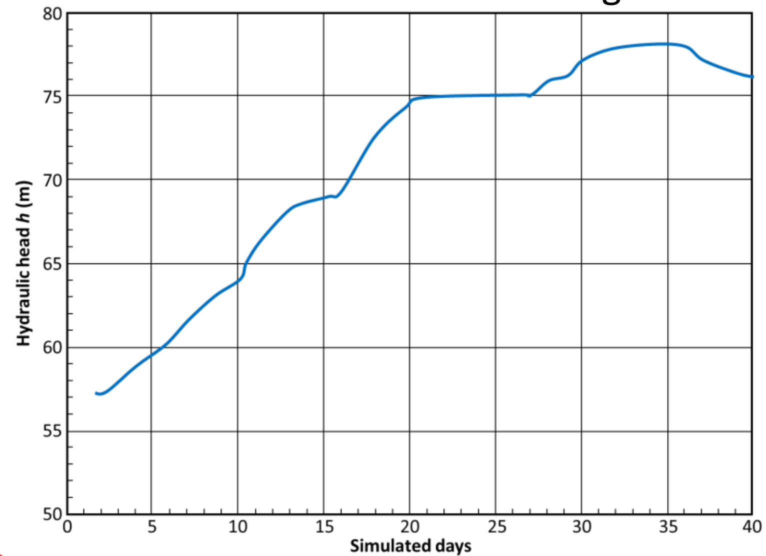
La Verne dam is an earth-filled dam:

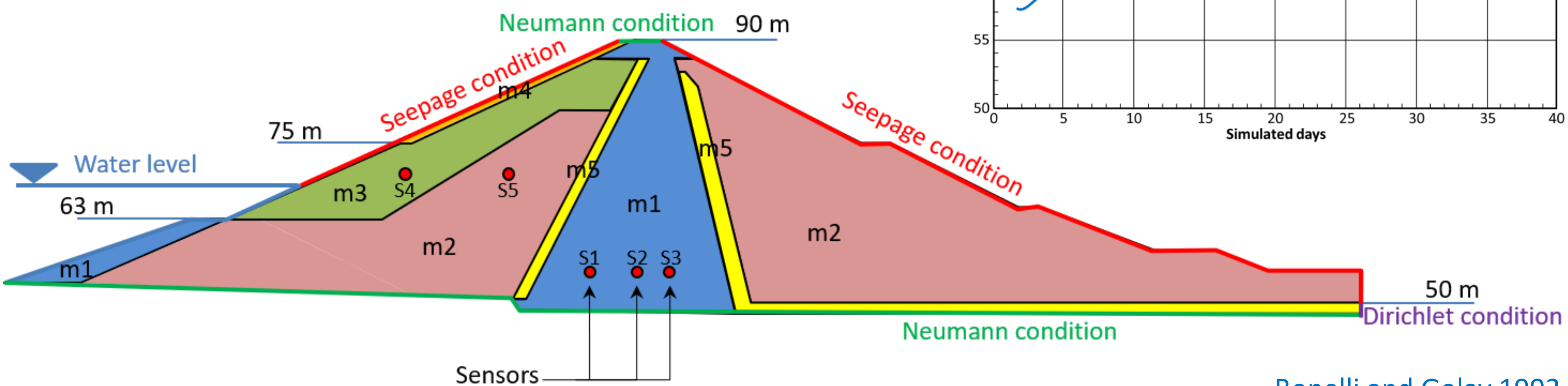
- 40 days of reservoir impoundment (dynamic forcing)
- Experimental data available
- Multi-materials configuration (Vachaud's and Van Genuchten-Mualem relations)
- Seepage

Challenging simulation:

- 1-order BDF and quadratic approximation
- AMR and WDG

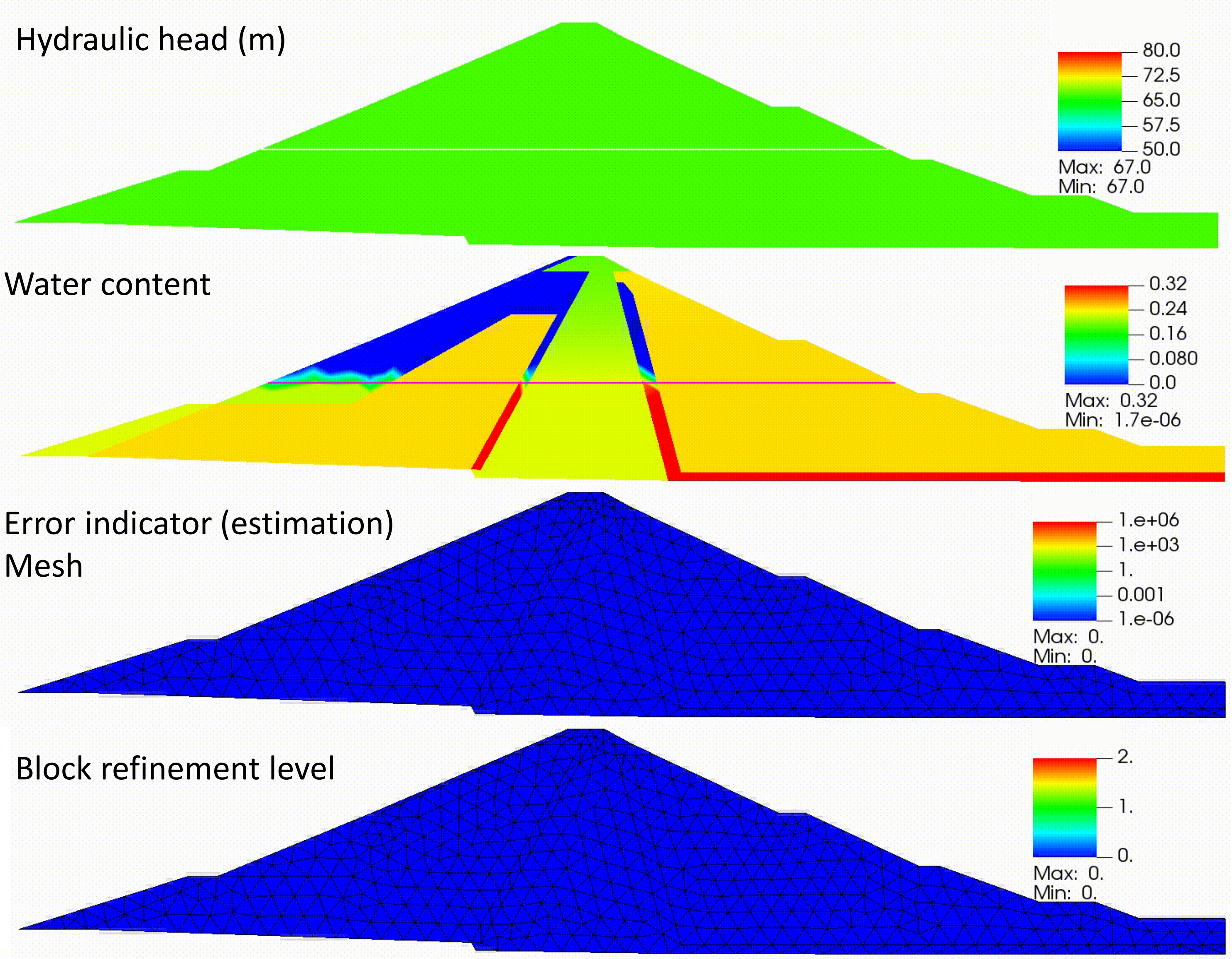
Reservoir height





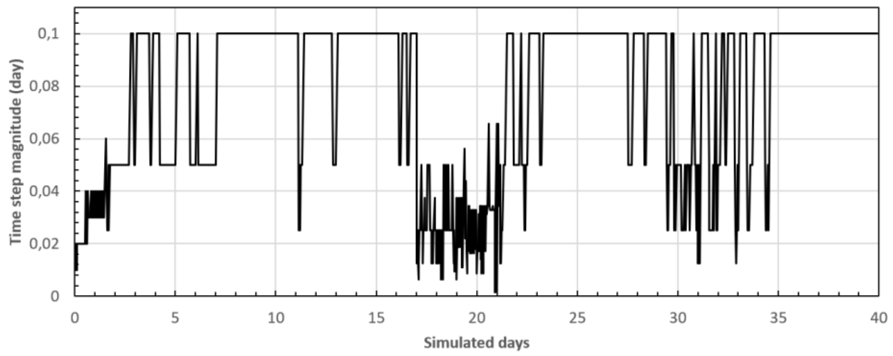
Bonelli and Golay 1993
Fleureau and Fry 1991

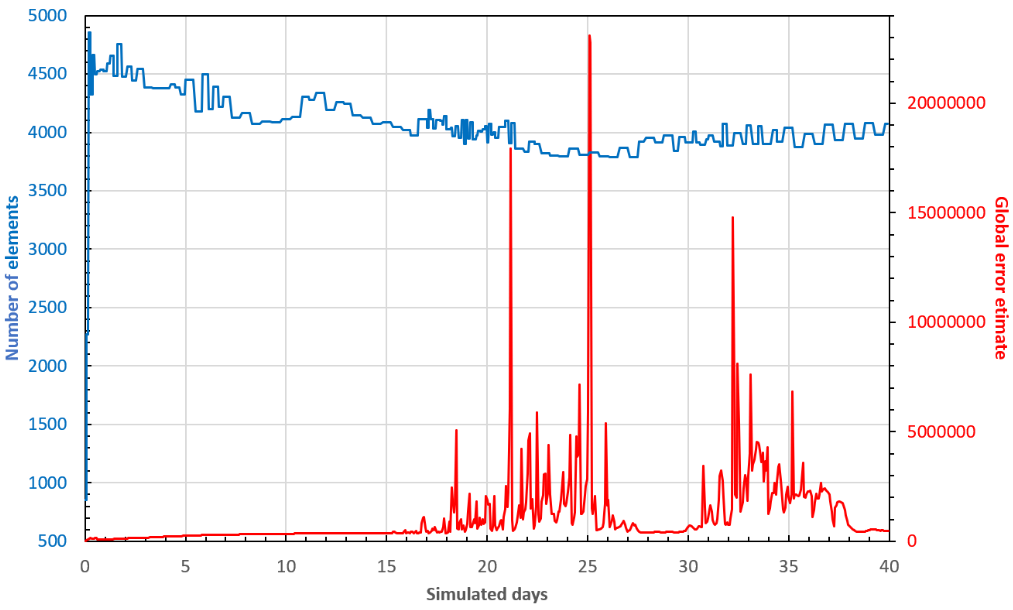
Results of La Verne dam simulation



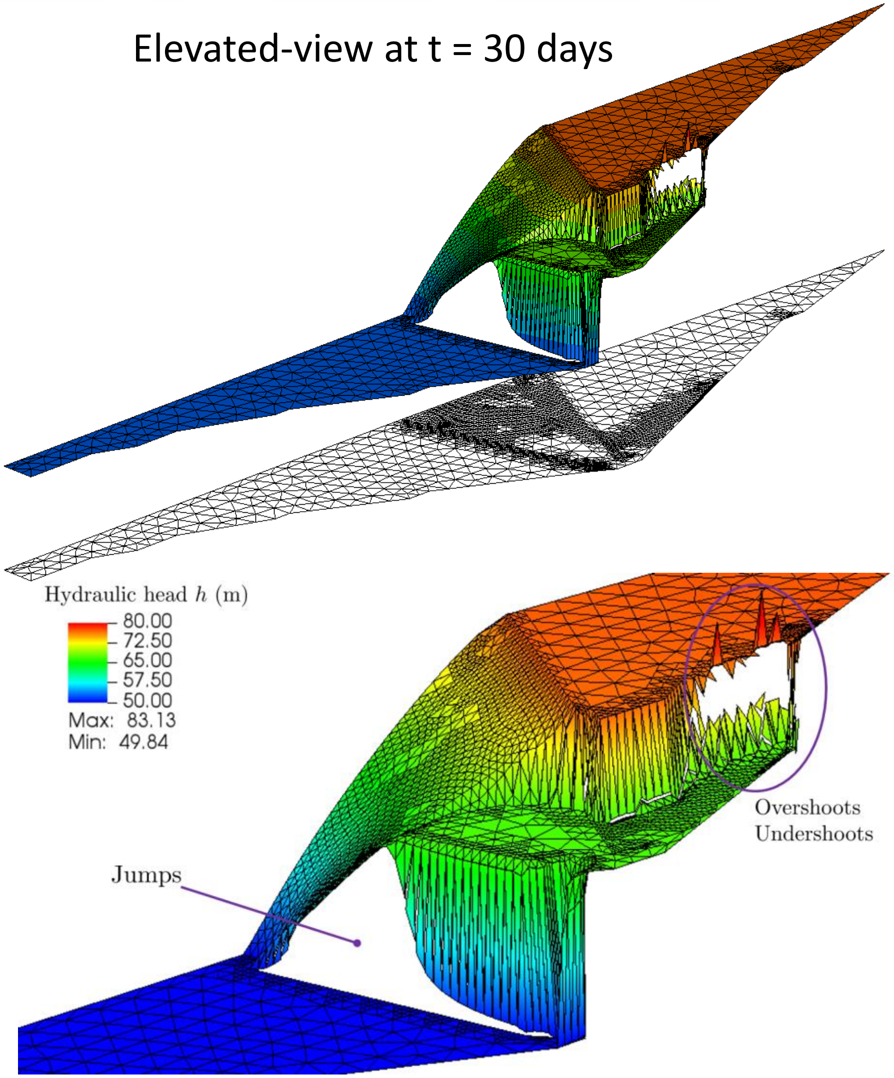
T = 40 days

Discussion





Elevated-view at t = 30 days

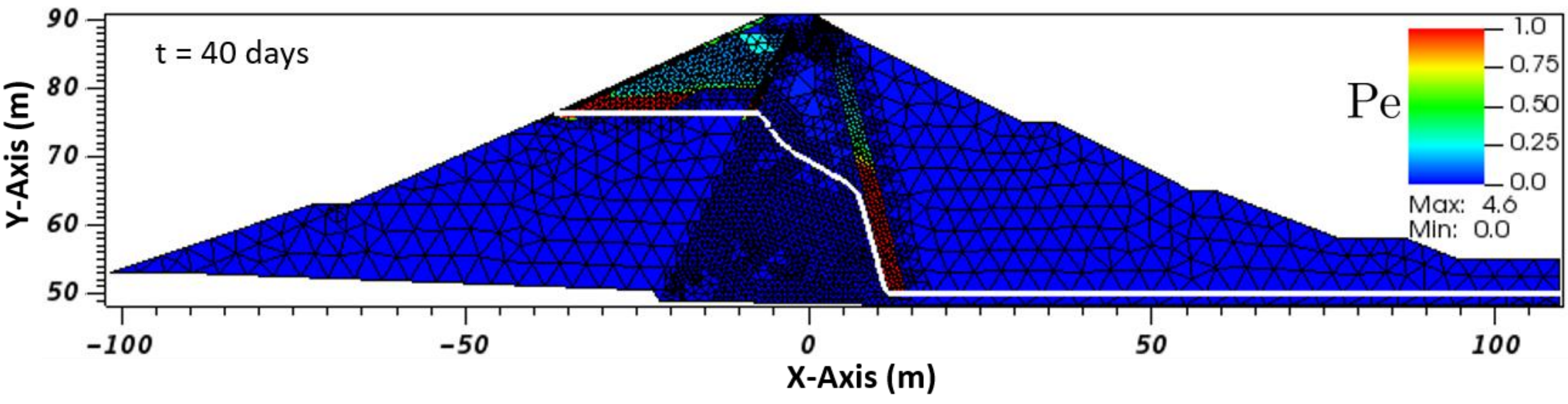


Some background analysis

$$\partial_t \theta(\psi) - \nabla \cdot (\mathbb{K}(\psi) \nabla(\psi + z)) = 0$$

- Developed saturation-based Richards equation: $$\partial_t \theta \underbrace{- \nabla \cdot (\mathbb{K}(\theta) \nabla z)}_{\text{gravitational effects (advection)}} \underbrace{- \nabla \cdot \left(\frac{\mathbb{K}(\theta)}{C(\theta)} \nabla \theta \right)}_{\text{capillary effects (diffusion)}} = 0$$

$$\mathrm{Pe} = \frac{\|\mathbb{K}(\theta)\| L}{\|\mathbb{D}(\theta)\| \theta} = \frac{C(\theta) L}{\theta}$$

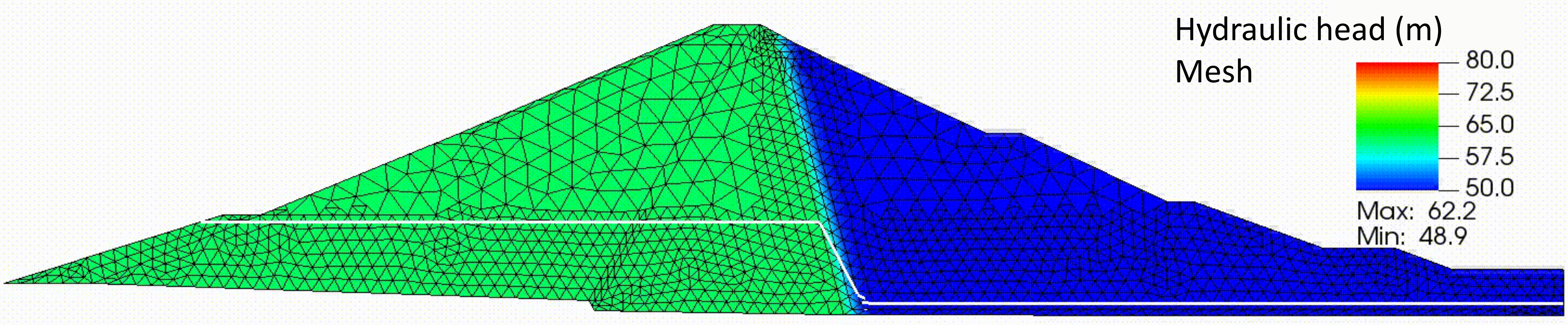


Investigation

An *augmented* simulation:

- No WDG
- Finer discretization in permeable materials
- Refinement for both gradient- and estimation-based error indicators
- Refinement around water table
- 4-order BDF

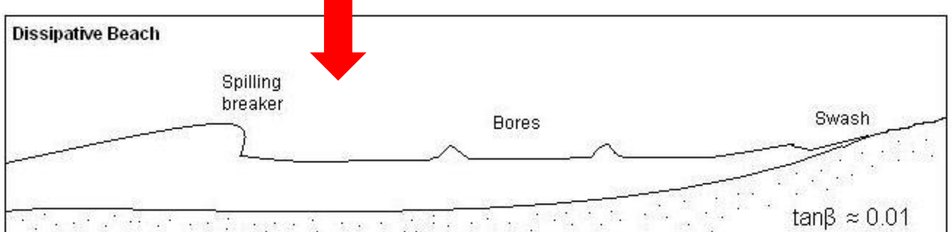
No spurious oscillations! But 13.5 times longer than the previous simulation… (Intel Xeon CPU 2.4 GHz)
→ 42h37min VS 3h11min



Wave-forced beach groundwater dynamics

Scope of the study:

- Sedimentary beaches (sand, loamy sand) with gentle slope
- Long infragravity waves + large fluctuations
- Low groundwater velocity, mainly pressure waves, wide range of saturation



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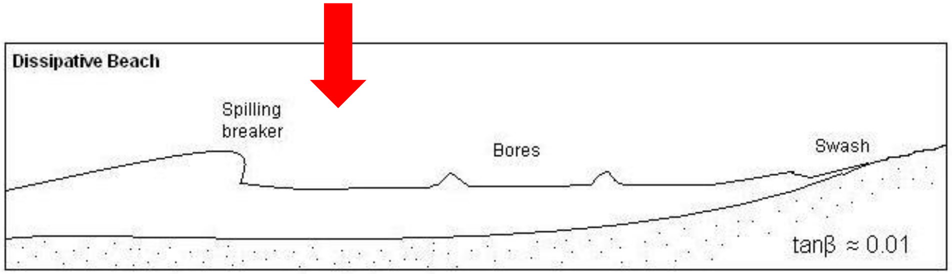
Issues on beach response to swash event:

- Global/local space scale + Time-averaged/resolved scale
- Infiltration/exfiltration (coupling), seepage
- Sediment transport (accretion/erosion)
- Morphodynamics (hydroporomechanics) … and more!



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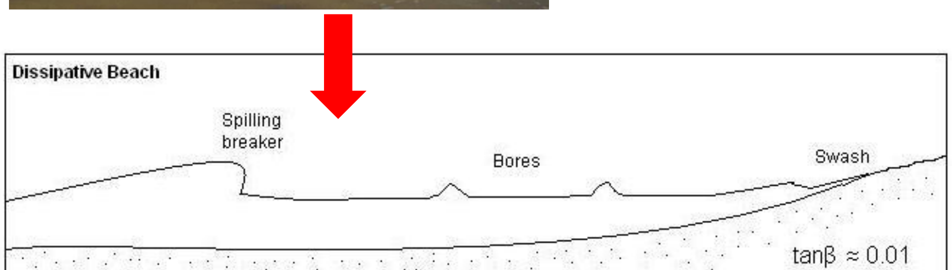
Seven Mile beach, Australia



Slea Head beach, Ireland

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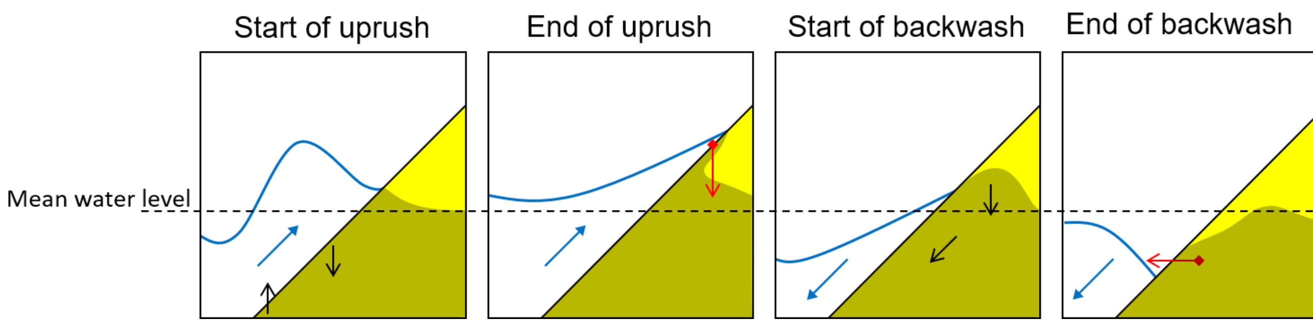

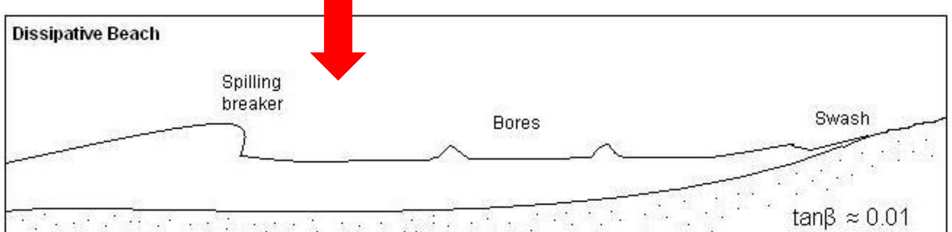

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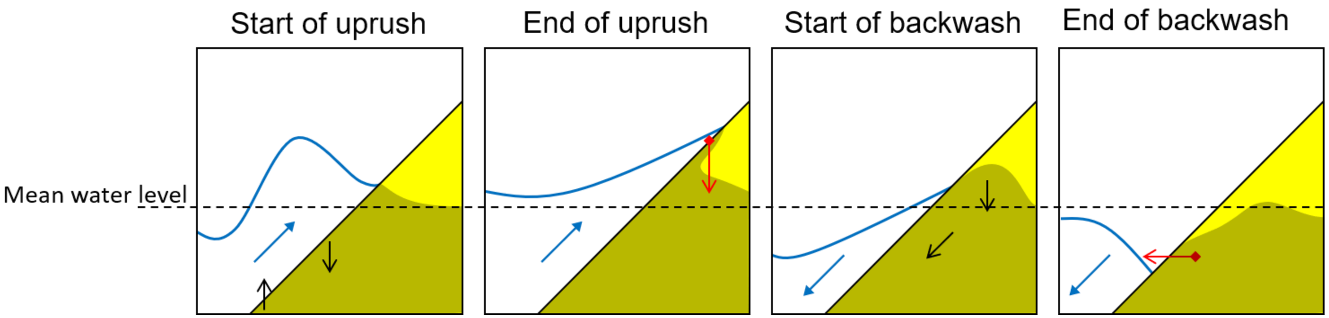

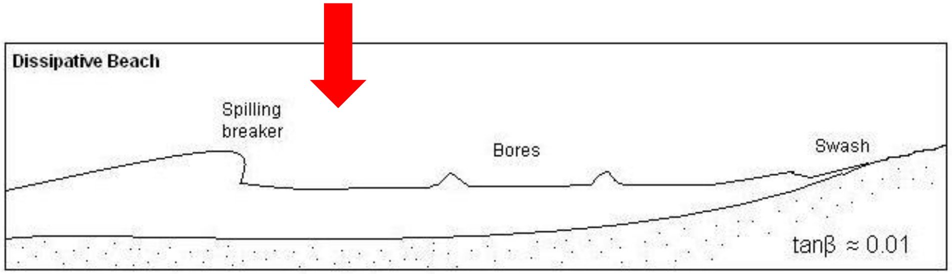

Slea Head beach, Ireland

Shallow water equations! (SWASH code, Zijlema *et al*. 2011)
Darcy's law/Dupuit-Forchheimer? → Richards' equation
Coupling? → Forcing

Large-scale experiment: BARDEX II

Turner *et al*. 2016

BARDEX = BARrier Dynamics EXperiment II at Delta Flume, Netherlands, 2012:

- Sand barrier of 95 m (flume ≈ 140 m) whose crest reaches 4.5 m
- Van Genuchten-Mualem relations taken for medium-sized sand
- T = 750 s

Video from Hachem Kassem (2015, YouTube ©)

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- T = 750 s

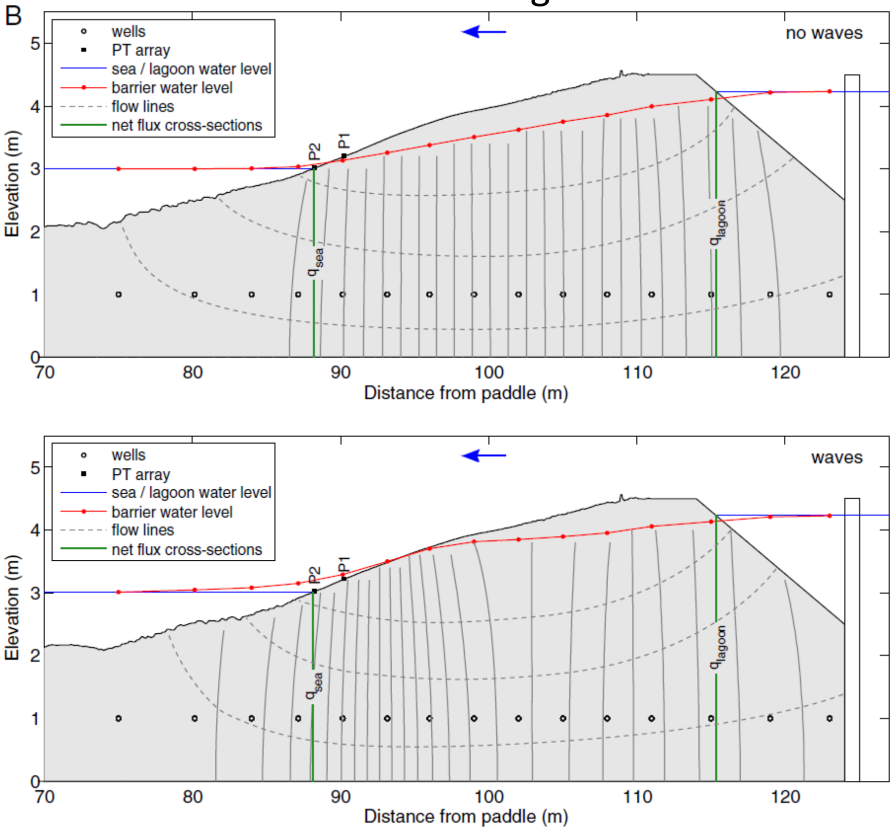
For each case, two conditions:

- No wave
- Irregular waves

Study of hydro- and morphodynamic effects for a sand barrier:

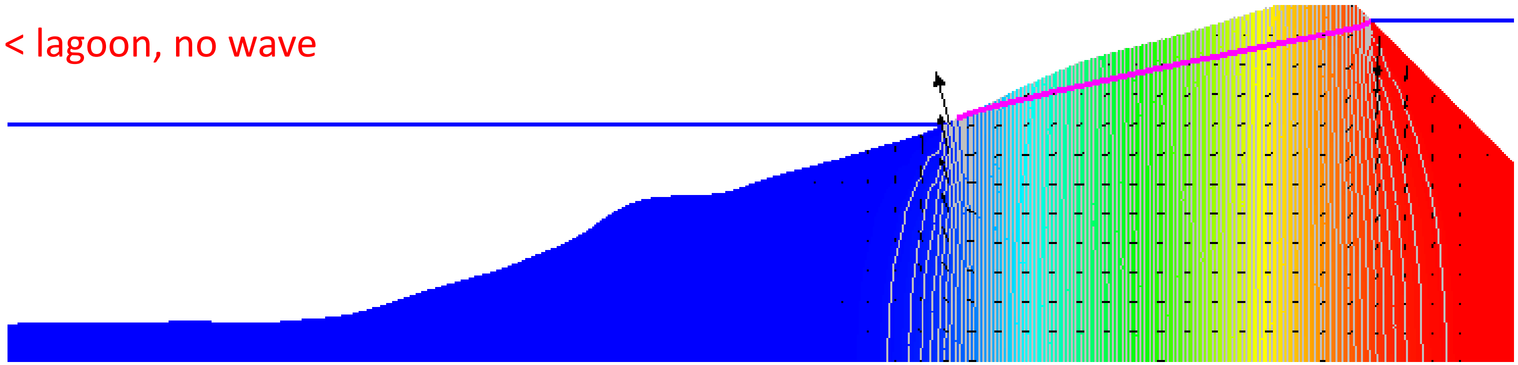
- Case A2: Sea < lagoon
- Case A4: Sea > lagoon
- Case A6: Sea = lagoon

Video from Hachem Kassem (2015, YouTube ©)



Simulation of case A2 from BARDEX II

Hydraulic head (m)
Flux (m/s)



Hydraulic head (m)
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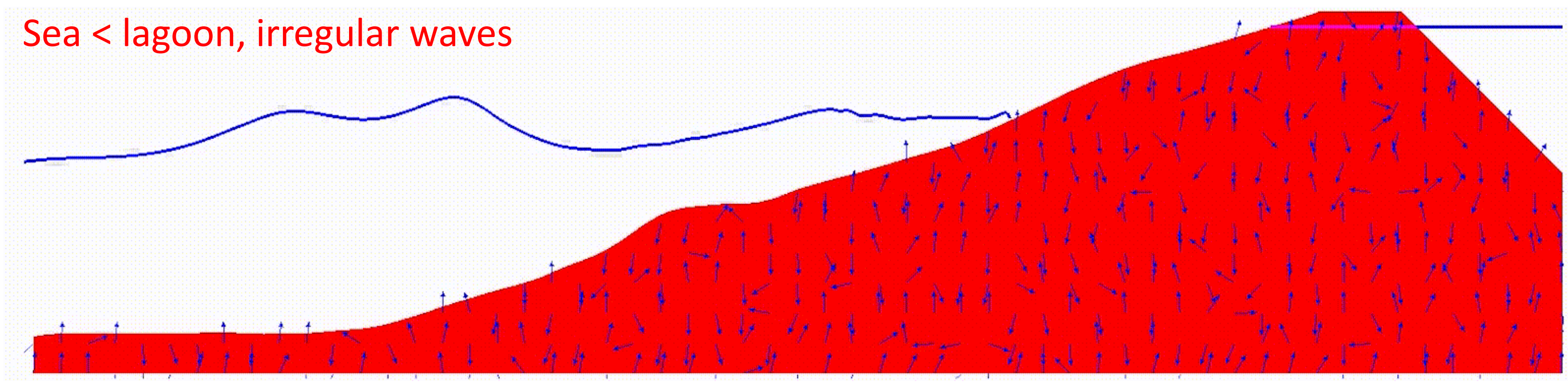
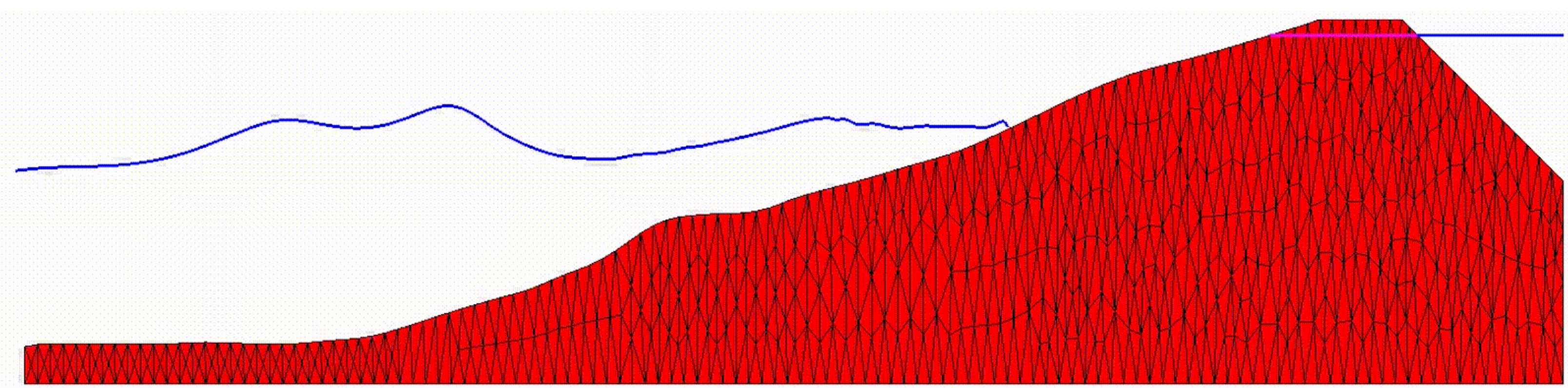


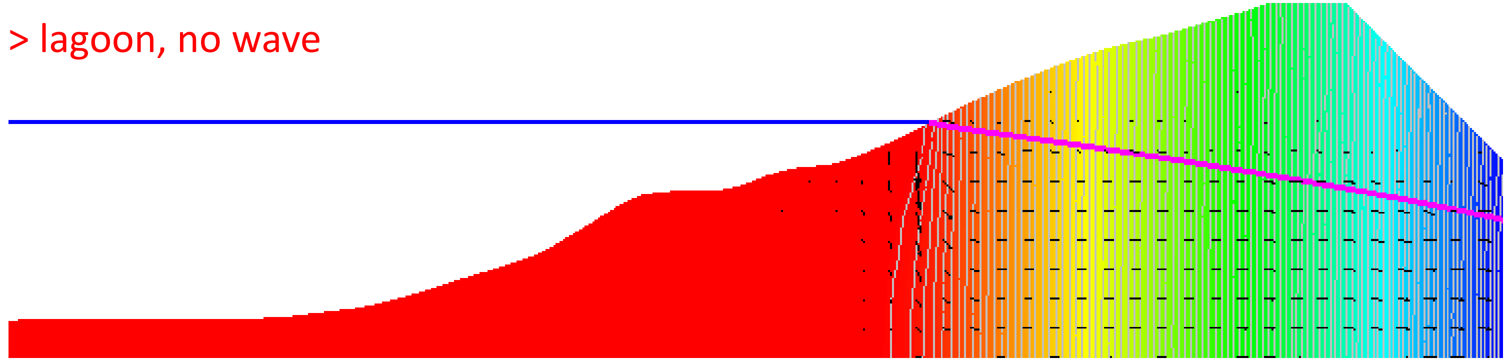
Mesh



Simulation of case A4 from BARDEX II

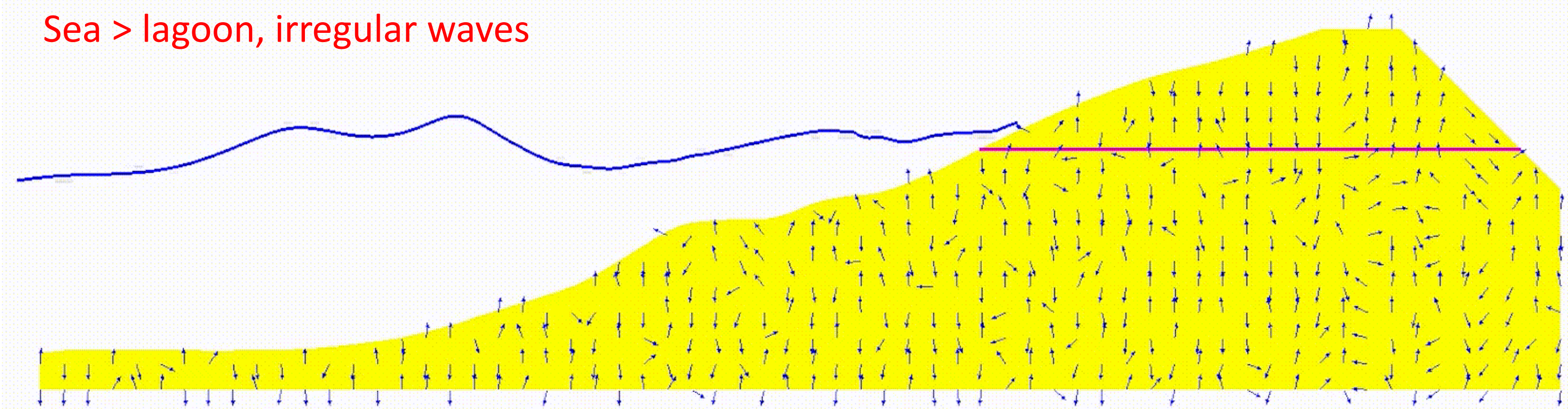
Sea > lagoon, no wave

Hydraulic head (m)
Flux (m/s)



Sea > lagoon, irregular waves

Hydraulic head (m)
Flux (m/s)



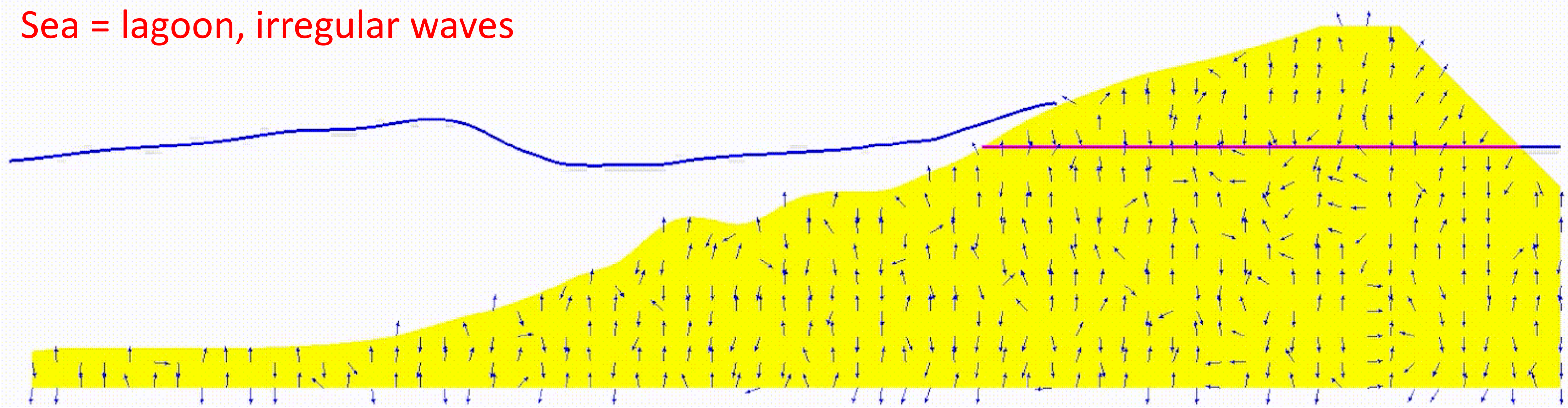
Simulation of case A6 from BARDEX II

Sea = lagoon, no wave

Hydraulic head (m)
Flux (m/s)

Sea = lagoon, irregular waves

Hydraulic head (m)
Flux (m/s)



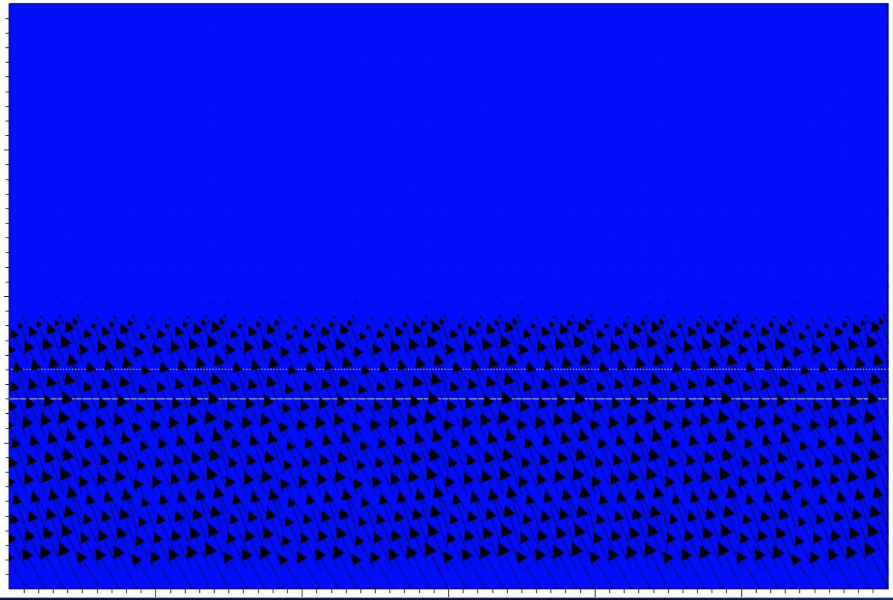
Conclusion

- Summary
- Perspectives

Summary

Richards' equation:

- Model for groundwater flows in **variably-saturated** porous media
- **Capillary** and **gravity** effects but no air-phase
- **Seepage**
- Numerical solution can be troublesome because of **nonlinearities, degeneracies, multiple space-time scales** leading to **convergence problems** and **spurious oscillations**



Water table recharge (Vauclin *et al*. 1979)

Summary

Richards' equation:

- Model for groundwater flows in **variably-saturated** porous media
- **Capillary** and **gravity** effects but no air-phase
- **Seepage**
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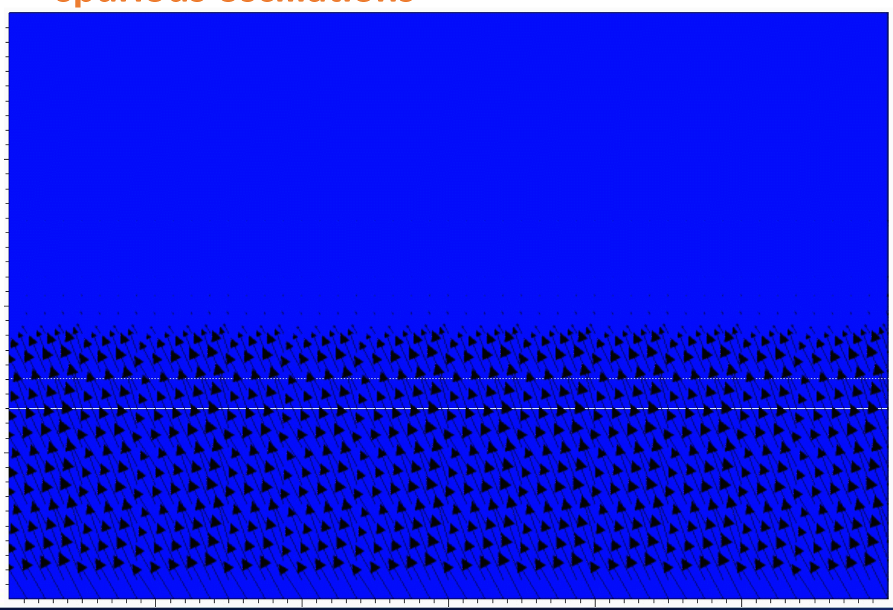

High-order adaptive DG strategy: my thesis (HAL) + Clément et al., Advances in Water Resources, 2021

- **High-order** method (*hp*-adaptation + BDF)
- Nonlinear robustness with adaptive time stepping
- Unstructured **non-conforming** hybrid mesh
- Local mass balance
- **Flexibility by weak penalization** to enforce continuity, stability, boundary conditions, projection
- **Block-based AMR** (capturing techniques) and **WDG** (smoothness adjustment) work in synergy through **error indicator to resolve sharp fronts/layers**
- **Heuristic parameters of adaptation** should be investigated. **Improvement** is supported by the *augmented* simulation.

Water table recharge (Vauclin *et al*. 1979)

Perspectives

Simulations:

- Confrontation with beach groundwater experiments (BARDEX II, Rousty beach)

Mathematical modelling and numerical methods:

- Coupling of surface/groundwater flow (iterative coupling and/or enhanced interface boundary condition)
- Improvements
  - Nonlinear convergence: robustness and speed by relaxations/accelerations (scheme) or regularizations (Richards' equation)
  - Adaptivity algorithm (threshold values, refinement level, frequency, blocks, solution process)
  - And more with DG? Penalty values, flux schemes, interpolation basis
- *hp*-Adaptation, error estimation

Programming for *Rivage* code:

- Optimization, parallelization, domain decomposition
- 3D
- DG hyperbolic solver for Saint-Venant's equation + two-fluid equation

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