A fully well-balanced scheme for shallow water equations with Coriolis force

Vivien DESVEAUX [1]   Alice MASSET [1]

[1]LAMFA, UMR 7352 CNRS
Université de Picardie Jules Verne

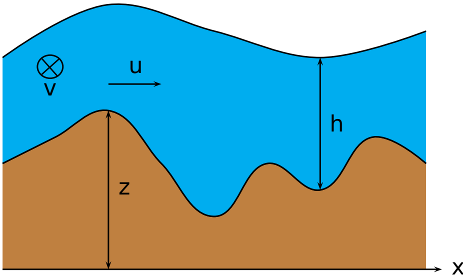
8ème école EGRIN, 25-28 mai 2021

Introduction

Shallow water equations with Coriolis force :

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{gh^2}{2}\right) = -gh\partial_x z + fhv, \\ \partial_t(hv) + \partial_x(huv) = -fhu. \end{cases}$$

- $h(x,t)$: fluid height
- $u(x,t)$: tangential velocity
- $v(x,t)$: transverse velocity
- $z(x)$: topography
- f : Coriolis force
- g : gravity constant



Notations : $w = (h, hu, hv)^T, \widetilde{w} = (w, z)$.

Existing works

With Coriolis force : steady states at rest preserved
[Bouchut et al., 2004],
[Lukáčová-Medvid'ová et al., 2007],
[Audusse et al., 2011],
[Chertock et al., 2018].

Goals :

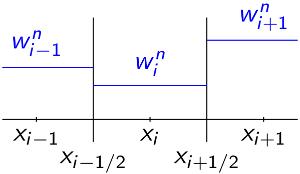
- preserve positivity of h,
- get a fully well-balanced scheme (FWB), [Noelle et al., 2007], [Berthon and Chalons, 2016], [Berthon et al., 2021],
- increase precision, [Michel-Dansac et al., 2016], [Ghitti et al., 2020].

Outline

Outline

Godunov-type scheme principle

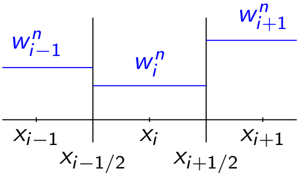
Finite volume discretisation at time t^n is a juxtaposition of Riemann problems



Exact Riemann solver : $\mathcal{W}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$
Approximate Riemann solver : $\widehat{\mathcal{W}}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$

Godunov-type scheme principle

Finite volume discretisation at time t^n is a juxtaposition of Riemann problems



Exact Riemann solver : $\mathcal{W}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$
Approximate Riemann solver : $\widehat{\mathcal{W}}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$

Approximation at time t^{n+1} obtained in two steps :

- $w_{\Delta x}(x, t^n + t) = \widehat{\mathcal{W}}_R\left(\dfrac{x - x_{i+1/2}}{t}, \widetilde{w}_i^n, \widetilde{w}_{i+1}^n\right)$ on $[x_i, x_{i+1}]$.
- $w_i^{n+1} = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} w_{\Delta x}(x, t^n + \Delta t)dx.$

Steady states discretisation

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(\dfrac{u^2}{2} + g(h+z)\right) = fv, \\ u\partial_x v = -fu. \end{cases} \qquad \longrightarrow \qquad \begin{cases} h_R u_R = h_L u_L = q, \\ \left[\dfrac{u^2}{2} + g(h+z)\right] = \Delta x f \overline{v}, \\ q[v] = -\Delta x f q, \end{cases}$$

discretisation

$$[X] = X_R - X_L, \qquad \overline{X} = \frac{X_L + X_R}{2}.$$

Steady states discretisation

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x \left(\frac{u^2}{2} + g(h+z) \right) = fv, \\ u\partial_x v = -fu. \end{cases} \qquad \begin{cases} h_R u_R = h_L u_L = q, \\ \left[\frac{u^2}{2} + g(h+z) \right] = \Delta x f \overline{v}, \\ q[v] = -\Delta x f q, \end{cases}$$

$\longrightarrow$

discretisation

Steady state indicator :

$$\mathcal{E}(\widetilde{w}_L, \widetilde{w}_R, \Delta x) := \sqrt{\left| [hu] \right|^2 + \left| \left[\frac{u^2}{2} + g(h+z) \right] - \Delta x f \overline{v} \right|^2 + \left| \overline{hu}([v] + f\Delta x) \right|^2}.$$

Steady states discretisation

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(\dfrac{u^2}{2} + g(h+z)\right) = fv, \\ u\partial_x v = -fu. \end{cases} \qquad \longrightarrow \qquad \begin{cases} h_R u_R = h_L u_L = q, \\ \left[\dfrac{u^2}{2} + g(h+z)\right] = \Delta x f \overline{v}, \\ q[v] = -\Delta x f q, \end{cases}$$

discretisation

Steady state indicator :

$$\mathcal{E}(\widetilde{w}_L, \widetilde{w}_R, \Delta x) := \sqrt{\left|[hu]\right|^2 + \left|\left[\frac{u^2}{2} + g(h+z)\right] - \Delta x f \overline{v}\right|^2 + \left|\overline{hu}([v] + f\Delta x)\right|^2}.$$

Definition

A couple $(\widetilde{w}_L, \widetilde{w}_R)$ defines a local steady state for the system if $\mathcal{E}(\widetilde{w}_L, \widetilde{w}_R, \Delta x) = 0$.

Steady states discretisation

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(\frac{u^2}{2} + g(h+z)\right) = fv, \\ u\partial_x v = -fu. \end{cases} \qquad \begin{cases} h_R u_R = h_L u_L = q, \\ \left[\frac{u^2}{2} + g(h+z)\right] = \Delta x f \overline{v}, \\ q[v] = -\Delta x f q, \end{cases}$$

$$\longrightarrow$$

discretisation

Steady state indicator :

$$\mathcal{E}_{LR} = \sqrt{\left|[hu]\right|^2 + \left|\left[\frac{u^2}{2} + g(h+z)\right] - \Delta x f \overline{v}\right|^2 + \left|\overline{hu}([v] + f\Delta x)\right|^2}.$$

Definition

A couple $(\widetilde{w}_L, \widetilde{w}_R)$ defines a local steady state for the system if $\mathcal{E}(\widetilde{w}_L, \widetilde{w}_R, \Delta x) = 0$.

Consistency

Consistency condition :

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx$$

Consistency

Consistency condition :

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx$$

$$= \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}(f(w_R) - f(w_L))$$

$$+ \frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} s\left(\mathcal{W}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right), z(x)\right) dxdt.$$

Consistency

Consistency condition :

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx$$
$$= \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}(f(w_R) - f(w_L))$$
$$+ \frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} s\left(\mathcal{W}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right), z(x)\right) dxdt.$$

Weak consistency condition :

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}(f(w_R) - f(w_L)) + \frac{\Delta t}{\Delta x} S(\widetilde{w}_L, \widetilde{w}_R),$$

where $S(\widetilde{w}_L, \widetilde{w}_R)$ is such that

$$S((w, z_L), (w, z_R)) = s_{cor}(w)\Delta x + s_{topo}(w)[z],$$

for the continuous source term $s(\widetilde{w}) = s_{cor}(w) + s_{topo}(w)\partial_x z$.

Source term discretisation

Let $(\widetilde{w}_L, \widetilde{w}_R)$ be a local steady state.
Weak consistency and FWB condition + steady state discretisation $\Longrightarrow$

$$S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g\mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1-\mathrm{Fr})^2},$$

$$S^{hv}(\widetilde{w}_L, \widetilde{w}_R) = -\Delta x f q.$$

where $\mathrm{Fr} = \frac{\overline{h}|u_L u_R|}{g h_L h_R}$ is a discrete Froude number.

Source term discretisation

Let $(\widetilde{w}_L, \widetilde{w}_R)$ be a local steady state.
Weak consistency and FWB condition + steady state discretisation $\Longrightarrow$

$$S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g\mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1-\mathrm{Fr})^2},$$
$$S^{hv}(\widetilde{w}_L, \widetilde{w}_R) = -\Delta x f q.$$

where $\mathrm{Fr} = \frac{\overline{h}|u_L u_R|}{g h_L h_R}$ is a discrete Froude number.

Remark : $\lim\limits_{\mathrm{Fr}\to 1} S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g}{4\overline{h}}[h]^3.$

Source term discretisation

Let $(\widetilde{w}_L, \widetilde{w}_R)$ be a local steady state.
Weak consistency and FWB condition + steady state discretisation $\Longrightarrow$

$$S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h} \overline{v} - g \overline{h}[z] + \frac{g \mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1-\mathrm{Fr})^2},$$
$$S^{hv}(\widetilde{w}_L, \widetilde{w}_R) = -\Delta x f q.$$

where $\mathrm{Fr} = \frac{\overline{h}|u_L u_R|}{g h_L h_R}$ is a discrete Froude number.
Remark : $\lim\limits_{\mathrm{Fr}\to 1} S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h} \overline{v} - g \overline{h}[z] + \frac{g}{4\overline{h}}[h]^3.$

For any $(\widetilde{w}_L, \widetilde{w}_R)$ we define

$$S^{hu}(\widetilde{w}_L, \widetilde{w}_R) = \Delta x f \overline{h} \overline{v} - g \overline{h}[z] + \frac{g \mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1-\mathrm{Fr})^2 + \mathcal{E}_{LR}},$$
$$S^{hv}(\widetilde{w}_L, \widetilde{w}_R) = -\Delta x f \widetilde{q}.$$

Approximate Riemann solver

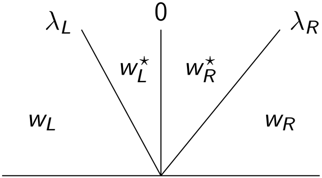


Figure: Approximate Riemann solver $\widehat{\mathcal{W}}_R$

To determine $w_L^\star$ and $w_R^\star$ (6 unknowns) we use

- the weak consistency condition (3 relations).
- the fully well-balanced condition (3 relations):
  if $(\widetilde{w}_L, \widetilde{w}_R)$ is a local steady state then $w_L^\star = w_L$ and $w_R^\star = w_R$.

Approximate Riemann solver

We set $\alpha_{LR} = g\bar{h} - |u_L u_R|$.
If $(\widetilde{w}_L, \widetilde{w}_R)$ is a local steady state, the FWB property implies

$$
\begin{aligned}
&q = h_L u_L = h_R u_R, \\
&\alpha_{LR}(h_R - h_L) = S^{hu}(\widetilde{w}_L, \widetilde{w}_R), \\
&q(v_R - v_L) = S^{hv}(\widetilde{w}_L, \widetilde{w}_R).
\end{aligned}
$$

Approximate Riemann solver

We set $\alpha_{LR} = g\bar{h} - |u_L u_R|$.
If $(\widetilde{w}_L, \widetilde{w}_R)$ is a local steady state, the FWB property implies

$$
\begin{aligned}
& q = h_L u_L = h_R u_R, \\
& \alpha_{LR}(h_R - h_L) = S^{hu}(\widetilde{w}_L, \widetilde{w}_R), \\
& q(v_R - v_L) = S^{hv}(\widetilde{w}_L, \widetilde{w}_R).
\end{aligned}
$$

$$\downarrow$$

$$
\begin{aligned}
& q^\star = h_L^\star u_L^\star = h_R^\star u_R^\star, \\
& \alpha_{LR}(h_R^\star - h_L^\star) = S^{hu}(\widetilde{w}_L, \widetilde{w}_R), \\
& \widetilde{q}(v_R^\star - v_L^\star) = S^{hv}(\widetilde{w}_L, \widetilde{w}_R).
\end{aligned}
$$

Approximate Riemann solver

We set $\alpha_{LR} = g\bar{h} - |u_L u_R|$.
If $(\widetilde{w}_L, \widetilde{w}_R)$ is a local steady state, the FWB property implies

$$\begin{aligned} q &= h_L u_L = h_R u_R, \\ \alpha_{LR}(h_R - h_L) &= S^{hu}(\widetilde{w}_L, \widetilde{w}_R), \\ q(v_R - v_L) &= S^{hv}(\widetilde{w}_L, \widetilde{w}_R). \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} q^\star &= h_L^\star u_L^\star = h_R^\star u_R^\star, \\ \alpha_{LR}(h_R^\star - h_L^\star) &= S^{hu}(\widetilde{w}_L, \widetilde{w}_R), \\ \widetilde{q}(v_R^\star - v_L^\star) &= S^{hv}(\widetilde{w}_L, \widetilde{w}_R). \end{aligned}$$

$$h_R^\star - h_L^\star = \begin{cases} \dfrac{\alpha_{LR} S^{hu}(\widetilde{w}_L, \widetilde{w}_R)}{\alpha_{LR}^2 + \mathcal{E}_{LR}} & \text{if } \mathcal{E}_{LR} \neq 0, \\ h_R - h_L & \text{if } \mathcal{E}_{LR} = 0, \end{cases}$$

$$v_R^\star - v_L^\star = \begin{cases} \dfrac{\widetilde{q}\, S^{hv}(\widetilde{w}_L, \widetilde{w}_R)}{\widetilde{q}^2 + \mathcal{E}_{LR}} & \text{if } \mathcal{E}_{LR} \neq 0, \\ v_R - v_L & \text{if } \mathcal{E}_{LR} = 0, \end{cases}$$

Scheme properties

Let ε be a positive small parameter. We set

$$\delta := \min(\varepsilon, h_L, h_R, h^{HLL}),$$

with

$$h^{HLL} = \frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{h_R u_R - h_L u_L}{\lambda_R - \lambda_L}.$$

If $h_L^\star < \delta$, we set $h_L^\star = \delta$ and we define $h_R^\star$ with respect to the consistency relation.

$$h_R^\star = \left(1 - \frac{\lambda_L}{\lambda_R}\right) h^{HLL} + \frac{\lambda_L}{\lambda_R} h_L^\star \geq \delta.$$

We proceed similarly if $h_R^\star < \delta$.

Remark : This procedure preserves the consistency and the fully well-balanced property.

Scheme properties

Let ε be a positive small parameter. We set

$$\delta := \min(\varepsilon, h_L, h_R, h^{HLL}),$$

with

$$h^{HLL} = \frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{h_R u_R - h_L u_L}{\lambda_R - \lambda_L}.$$

If $h_L^\star < \delta$, we set $h_L^\star = \delta$ and we define $h_R^\star$ with respect to the consistency relation.

$$h_R^\star = \left(1 - \frac{\lambda_L}{\lambda_R}\right) h^{HLL} + \frac{\lambda_L}{\lambda_R} h_L^\star \geq \delta.$$

We proceed similarly if $h_R^\star < \delta$.

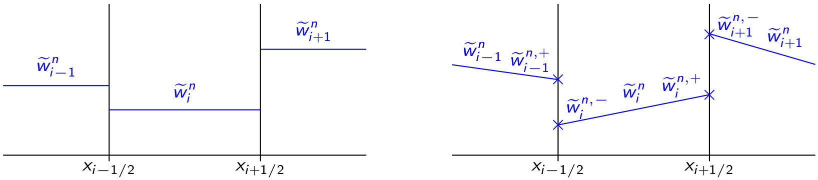
Remark : This procedure preserves the consistency and the fully well-balanced property.

Theorem

Under a **first-order CFL** *restriction, the approximate Riemann solver* $\widehat{\mathcal{W}}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$ *leads to a* **fully well-balanced** *Godunov-type scheme which also* **preserves the positivity of** h.

Outline

Piecewise reconstruction



$$\sigma_i^n(\widetilde{w}) = \mathrm{minmod}\left(\frac{\widetilde{w}_i^n - \widetilde{w}_{i-1}^n}{\Delta x}, \frac{\widetilde{w}_{i+1}^n - \widetilde{w}_i^n}{\Delta x}\right)$$

$$\mathrm{minmod}(\sigma_L, \sigma_R) = \begin{cases} \min(\sigma_L, \sigma_R) & \text{if } \sigma_L > 0 \text{ and } \sigma_R > 0, \\ \max(\sigma_L, \sigma_R) & \text{if } \sigma_L < 0 \text{ and } \sigma_R < 0, \\ 0 & \text{otherwise.} \end{cases}$$

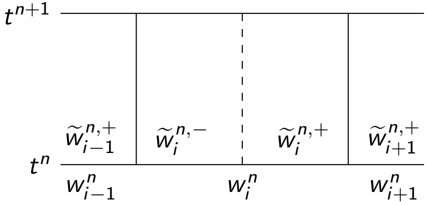
$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \frac{\Delta x}{2}\sigma_i^n(\widetilde{w}),$$

MUSCL method

First-order scheme :

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right) - F\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right)\right) + \frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right) + S\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right)\right).$$

Second-order scheme :

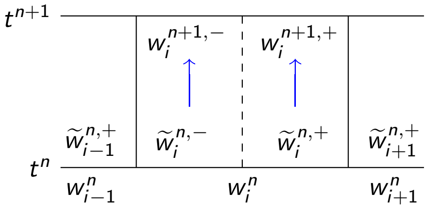


MUSCL method

First-order scheme : ↑

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right) - F\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right)\right) + \frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right) + S\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right)\right).$$

Second-order scheme :

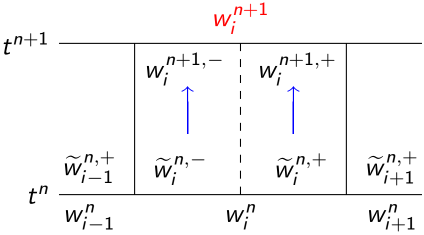


MUSCL method

First-order scheme : $\uparrow$

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right) - F\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right)\right) + \frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right) + S\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right)\right).$$

Second-order scheme :



$$w_i^{n+1} = \frac{1}{2}\left(w_i^{n+1,-} + w_i^{n+1,+}\right)$$

MUSCL method

Second-order scheme :

$$w_i^{n+1} = \frac{w_i^{n,+} + w_i^{n,-}}{2} - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right)\right)$$
$$+\frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \frac{\Delta x}{2}\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right)\right).$$

MUSCL method

Second-order scheme :

$$w_i^{n+1} = \frac{w_i^{n,+} + w_i^{n,-}}{2} - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right)\right)$$
$$+\frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \frac{\Delta x}{2}\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right)\right).$$

Required reconstruction properties :

- conservative : $\frac{w_i^{n,+}+w_i^{n,-}}{2} = w_i^n$,

MUSCL method

Second-order scheme :

$$w_i^{n+1} = \frac{w_i^{n,+} + w_i^{n,-}}{2} - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right)\right)$$

$$+\frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \frac{\Delta x}{2}\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right)\right).$$

Required reconstruction properties :

- conservative : $\frac{w_i^{n,+} + w_i^{n,-}}{2} = w_i^n$,
- fully well-balanced : If $(\widetilde{w}_i)_{i\in\mathbb{Z}}$ is a discrete steady state, then

$$\mathcal{E}(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x/2) = \mathcal{E}(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x/2) = 0, \text{ for all } i \in \mathbb{Z}.$$

MUSCL method

Second-order scheme :

$$w_i^{n+1} = \frac{w_i^{n,+} + w_i^{n,-}}{2} - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right)\right)$$

$$+\frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \frac{\Delta x}{2}\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right)\right).$$

Required reconstruction properties :

- conservative : $\frac{w_i^{n,+}+w_i^{n,-}}{2} = w_i^n$,
- fully well-balanced : If $(\widetilde{w}_i)_{i\in\mathbb{Z}}$ is a discrete steady state, then

$$\mathcal{E}(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x/2) = \mathcal{E}(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x/2) = 0, \text{ for all } i \in \mathbb{Z}.$$

Incompatible conditions

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

New reconstruction :

$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\widetilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

New reconstruction :

$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\widetilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

$$\theta(x) = \frac{x^2}{x^2 + \Delta x^2}.$$

$$\theta_i^n = \theta(\mathcal{E}(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x) + \mathcal{E}(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x))$$

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

New reconstruction :

$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\widetilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

Second-order scheme :

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right)\right)$$
$$+\frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \frac{\Delta x}{2}\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \frac{\Delta x}{2}\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2}\right)\right).$$

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

New reconstruction :

$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\widetilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

Second-order scheme :

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)$$
$$+ \frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right) + 2S\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_2\right) + S\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right)\right).$$

$$\text{with } \Delta x_1 = \Delta x\left(1 - \frac{\theta_i^n}{2}\right) \quad \text{and} \quad \Delta x_2 = \theta_i^n \frac{\Delta x}{2}.$$

Scheme properties

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)$$

Scheme properties

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)$$

$$= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)\right)$$

$$+ \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right)\right)\right)$$

Scheme properties

$$
\begin{aligned}
h_i^{n+1} &= h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right) \\
&= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)\right) \\
&\quad + \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right)\right)\right) \\
&\geq 0,
\end{aligned}
$$

since the cut-off procedure does not depend on Δx_1.

Scheme properties

$$
\begin{aligned}
h_i^{n+1} &= h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right) \\
&= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)\right) \\
&\quad + \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right)\right)\right) \\
&\geq 0,
\end{aligned}
$$

since the cut-off procedure does not depend on Δx_1.

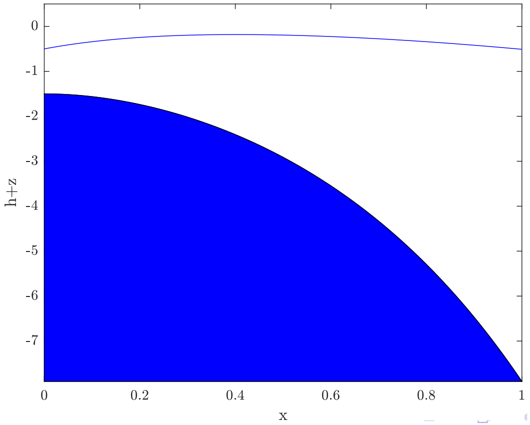
Theorem

Under a **second-order CFL** *restriction, the second-order scheme is* **fully well-balanced** *and* **preserves the positivity of** *h.*

Outline

Numerical results : moving steady state

$$h_0(x) = \exp^{2x}, u_0(x) = \exp^{-2x}, v_0(x) = -fx,$$
$$z(x) = -\frac{1}{2}f^2x^2 - \exp^{2x} - \frac{1}{2}\exp^{-4x},$$
$$f = g = 1.$$



Numerical results : moving steady state

$$h_0(x) = \exp^{2x}, u_0(x) = \exp^{-2x}, v_0(x) = -fx,$$
$$z(x) = -\frac{1}{2}f^2x^2 - \exp^{2x} - \frac{1}{2}\exp^{-4x},$$
$$f = g = 1.$$

Steady state distance at time t^n :

$$\mathcal{E}^n_{\infty,j} = \max_{1\leq i\leq N} \mathcal{E}(\widetilde{w}^n_i, \widetilde{w}^n_{i+1}, \Delta x).$$

Numerical results : moving steady state

$$h_0(x) = \exp^{2x}, u_0(x) = \exp^{-2x}, v_0(x) = -fx,$$
$$z(x) = -\frac{1}{2}f^2x^2 - \exp^{2x} - \frac{1}{2}\exp^{-4x},$$
$$f = g = 1.$$

Steady state distance at time t^n :

$$\mathcal{E}^n_{\infty,j} = \max_{1\leq i\leq N} \mathcal{E}(\widetilde{w}^n_i, \widetilde{w}^n_{i+1}, \Delta x).$$

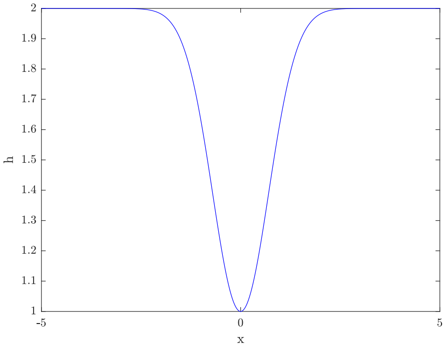
$T_{\max} = 1, N = 200.$

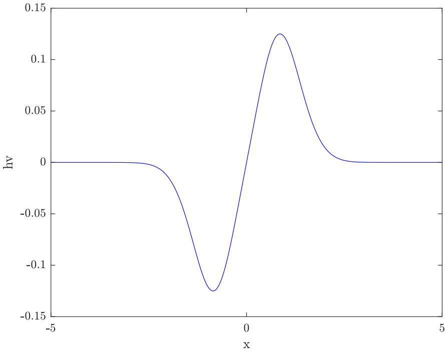
- $\mathcal{E}^0_{\infty,1} = \mathcal{E}^0_{\infty,2} = 8.87 \times 10^{-16}$
- $\mathcal{E}^{T_{\max}}_{\infty,1} = 5.19 \times 10^{-14}$
- $\mathcal{E}^{T_{\max}}_{\infty,2} = 8.86 \times 10^{-15}$

Numerical results : geostrophic steady state

$$h_0(x) = \frac{2}{g} - e^{-x^2}, u_0(x) = 0, v_0(x) = \frac{2g}{f} x e^{-x^2},$$

$$f = 10, g = 1.$$





Numerical results : geostrophic steady state

$T_{\max} = 200, N = 200.$

- $\mathcal{E}^0_{\infty,1} = \mathcal{E}^0_{\infty,2} = 4.06 \times 10^{-5}$
- $\mathcal{E}^{T_{\max}}_{\infty,1} = 1.12 \times 10^{-7}$
- $\mathcal{E}^{T_{\max}}_{\infty,2} = 2.53 \times 10^{-12}$

Numerical results : geostrophic steady state

$T_{\max} = 200, N = 200.$

- $\mathcal{E}^0_{\infty,1} = \mathcal{E}^0_{\infty,2} = 4.06 \times 10^{-5}$
- $\mathcal{E}^{T_{\max}}_{\infty,1} = 1.12 \times 10^{-7}$
- $\mathcal{E}^{T_{\max}}_{\infty,2} = 2.53 \times 10^{-12}$

Table: L^1-error, first-order scheme

N	h		hv	
200	5.25×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.30×10^{-5}	1.99
800	3.30×10^{-6}	1.99	1.38×10^{-5}	1.94
1600	8.58×10^{-7}	1.94	3.73×10^{-6}	1.88
3200	2.30×10^{-7}	1.91	1.02×10^{-6}	1.87
6400	6.01×10^{-8}	1.93	2.73×10^{-7}	1.90

Table: L^1-error, second-order scheme

N	h		hv	
200	5.26×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.27×10^{-5}	2.00
800	3.29×10^{-6}	2.00	1.32×10^{-5}	2.00
1600	8.22×10^{-7}	2.00	3.30×10^{-6}	2.00
3200	2.05×10^{-7}	2.00	8.25×10^{-7}	2.00
6400	5.14×10^{-8}	2.00	2.06×10^{-7}	2.00

Perspectives

In sight :

- Considering f variable.
- Build a scheme for the 2D system with Coriolis force.
- Take temperature into consideration.

Perspectives

In sight :

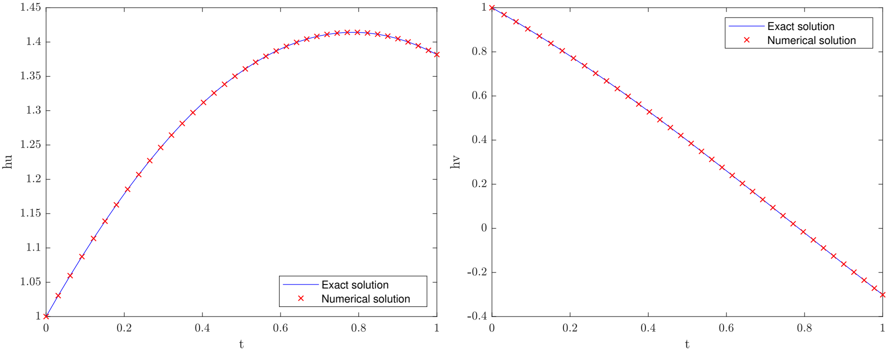
- Considering f variable.
- Build a scheme for the 2D system with Coriolis force.
- Take temperature into consideration.

Thanks for your attention

Stationary state in space

For a constant initial condition (h_0, u_0, v_0) fixed, the exact solution of RSW equations writes

$$\begin{aligned} h(x,t) &= h_0, \\ u(t) &= u_0 \cos(ft) + v_0 \sin(ft), \\ v(t) &= v_0 \cos(ft) - u_0 \sin(ft). \end{aligned}$$



Stationary state in space

$T_{\max} = 1, N = 200.$

Table: L^1-error, first-order scheme

N	hu		hv	
200	3.82×10^{-4}	0.99	8.06×10^{-5}	0.99
400	1.91×10^{-4}	0.99	4.03×10^{-5}	0.99
800	9.56×10^{-5}	0.99	2.01×10^{-5}	0.99
1600	4.78×10^{-5}	0.99	1.01×10^{-5}	0.99
3200	2.39×10^{-5}	0.99	5.04×10^{-6}	0.99
6400	1.20×10^{-5}	0.99	2.52×10^{-6}	0.99

Table: L^1-error, second-order scheme

N	hu		hv	
200	7.71×10^{-9}	1.99	3.58×10^{-8}	1.99
400	1.92×10^{-9}	1.99	8.95×10^{-9}	1.99
800	4.82×10^{-10}	2.00	2.24×10^{-9}	1.99
1600	1.20×10^{-10}	2.00	5.60×10^{-10}	1.99
3200	3.01×10^{-11}	2.00	1.40×10^{-10}	1.99
6400	7.52×10^{-12}	2.00	3.50×10^{-11}	1.99