UNIVERSITÄT BIELEFELD

Faculty of Mathematics

Boundary conditions for the time-discrete Green–Naghdi equations

joint work with

Sebastian Noelle (RWTH Aachen University)
Martin Parisot (Cardamom team – Inria Bordeaux)

Tabea Tscherpel

Bielefeld Seoul
International Research Training Group 2235


Michel Griffon (commonswiki, 2011)


Antony Colas (2014)


serc.carleton.edu (2017)

UNIVERSITÄT BIELEFELD

Faculty of Mathematics

Serre/Green–Naghdi equations

⇝ derived by vertical averaging from free surface incompressible Euler equations

$$
\begin{aligned}
\partial_t h + \nabla \cdot (h\overline{u}) &= 0 \\
\partial_t (h\overline{u}) + \nabla \cdot \left(h\overline{u} \otimes \overline{u} + \frac{g}{2} h^2 \mathrm{I} \right) &= -\nabla (h\overline{q}) - (gh + q_B)\nabla B \\
\partial_t (h\overline{w}) + \nabla \cdot (h\overline{w}\ \overline{u}) &= q_B \\
\partial_t (h\sigma) + \nabla \cdot (h\sigma\, \overline{u}) &= \sqrt{3}(2\overline{q} - q_B)
\end{aligned}
$$

$$
\overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \qquad \text{and} \qquad \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u}
$$

h	water depth
$\overline{u}$	horizontal...
$\overline{w}$	vertical...
σ	standard deviation ...velocity
$\overline{q}$	hydrodynamic pressure
q_B	... at the bottom
B	bottom (given)

- cf. Aïssiouene '16, Gavrilyuk '17, Parisot '19, Popinet '20
- alternative formulation for $(h, \overline{u})$, cf. Peregrine'67, ...

⇝ due to its **non-linear** and **dissipative** nature handling non-standard boundary conditions is a challenge

cf. Audiard '12, Kazolea '14, Lannes '20, Kazakova '20, Aïssiouene '20

Semi-discrete equations and splitting

$U := (\overline{u}, \overline{w}, \sigma)^\top$ velocity • $\delta_t^n > 0$ time step • $h^{n+1} = h^{n*}$

(I) Shallow water / advection step

$$\begin{pmatrix} h^{n*} \\ h^{n*} U^{n*} \end{pmatrix} = \begin{pmatrix} h^{n} \\ h^{n} U^{n} \end{pmatrix} - \delta_t^n \nabla \cdot F^n + \delta_t^n S^n$$

(II) Correction step $h^{n*} U^{n+1} = h^{n*} U^{n*} - \delta_t^n h^{n*} \pi^{n+1}(\overline{q}^{n+1}, q_B^{n+1})$

$$\overline{w}^{n+1} = \overline{u}^{n+1} \cdot \nabla B - \frac{h^{n*}}{2} \nabla \cdot \overline{u}^{n+1} \quad \text{and} \quad \sigma^{n+1} = -\frac{h^{n*}}{2\sqrt{3}} \nabla \cdot \overline{u}^{n+1}$$

splitting schemes e.g. in Bonneton '11, Aïssiouene '16, Favrie '17, ...

- $L^2(\Omega; h)$ weighted Lebesgue space with inner product $\langle f, g \rangle_h := \int_\Omega f\, g\, h \,\mathrm{d}x$, if $0 < c \leq h \leq C < \infty$
- space of admissible functions

$$\mathbb{A}_h := \left\{ U \in L^2(\Omega; h) \colon \ \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \ \text{ and } \ \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \right\}$$

Projection structure for $\Omega = \mathbb{R}^d$

Assume that $0 < c \leq h \leq C < \infty$ is fixed and $B \in W^{1,\infty}(\Omega)$.

Orthogonality

$$\langle U, \pi(\overline{q}, q_B) \rangle_h = -\int_\Omega \nabla \cdot (h\overline{q}\,\overline{u})\,\mathrm{d}x = 0$$

Let $\Pi_h \colon L^2(\Omega; h) \to \mathbb{A}_h$ be the $\langle \cdot, \cdot \rangle_h$-orthogonal projection onto

$$\mathbb{A}_h := \left\{ U \in L^2(\Omega; h) \colon \overline{w} = \overline{u} \cdot \nabla B - \frac{h}{2} \nabla \cdot \overline{u} \quad \text{and} \quad \sigma = -\frac{h}{2\sqrt{3}} \nabla \cdot \overline{u} \right\}.$$

Recall: We want to decompose $U^* = U + (\delta_t)\pi$ with $U \in \mathbb{A}_h$ and $\pi \in \mathbb{A}_h^\perp$.

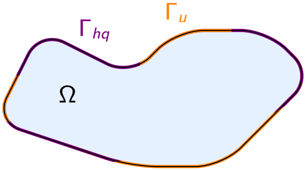
Wellposedness of ‘Projection solution’

$U = \Pi_h(U^*) \in \mathbb{A}_h$ and $\delta_t \pi(\overline{q}, q_B) = U^* - \Pi_h(U^*) \in \mathbb{A}_h^\perp$ form a (unique) solution of the correction step and then $\overline{u} \in H(\mathrm{div}; \Omega)$ and $h\overline{q} \in H^1(\Omega)$.

$\rightsquigarrow$ On Ω **bounded** $\overline{u}$ has normal trace in $H^{-1/2}(\partial\Omega)$ and $h\overline{q}$ has trace in $H^{1/2}(\partial\Omega)$.

Projection structure for bounded Ω

Assume $0 < c \leq h \leq C < \infty$, $B \in W^{1,\infty}(\Omega)$ and $\partial\Omega =: \Gamma_{hq} \cup \Gamma_u$.



Orthogonality

$$\langle U, \pi(h\overline{q}, q_B)\rangle_h = -\int_\Omega \nabla \cdot (h\overline{q}\,\overline{u})\,\mathrm{d}x = -\int_{\partial\Omega} \overbrace{h\overline{q}}^{=0 \text{ on } \Gamma_{hq}} \overbrace{\overline{u}\cdot\nu}^{=0 \text{ on } \Gamma_u} \,\mathrm{d}s(x) = 0$$

Let $\Pi_{h,0}\colon L^2(\Omega; h) \to \mathbb{A}_{h,0}$ be the $\langle\cdot,\cdot\rangle_h$-orthogonal projection onto

$$\mathbb{A}_{h,0} := \{U \in \mathbb{A}_h \colon \overline{u} \in H_{\Gamma_u}(\mathrm{div};\Omega)\}$$

with $H^1_{\Gamma_{hq}} = \{f \in H^1(\Omega)\colon f|_{\Gamma_{hq}} = 0\}$,

$$H_{\Gamma_u}(\mathrm{div};\Omega), = \{v \in H(\mathrm{div};\Omega)\colon \langle v\cdot\nu, f\rangle_{\partial\Omega} = 0 \quad \forall f \in H^1_{\Gamma_{hq}}(\Omega)\}.$$

Wellposedness of 'Projection solution'

$U = \Pi_{h,0}(U^*) \in \mathbb{A}_{h,0}$ and $\delta_t\pi(\overline{q}, q_B) = U^* - \Pi_h(U^*) \in \mathbb{A}_{h,0}^\perp$ form a (unique) solution of the correction step and $h\overline{q} \in H^1_{\Gamma_{hq}}(\Omega)$.

Boundary conditions

Inhomogeneous boundary conditions (formally)

$$h\overline{q} = \widetilde{hq} \quad \text{on } \Gamma_{hq} \qquad \text{and} \qquad \overline{u} \cdot \nu = \widetilde{u} \quad \text{on } \Gamma_u$$

More rigorously: given $\widetilde{hq} \in H^{1/2}(\Gamma_{hq})$ and $\widetilde{u} \in H^{-1/2}(\partial\Omega)$.

We use reference functions $U^R, \pi^R = \pi(h\overline{q}^R, 0)$ for reduction to hom. problem.

Wellposedness of ‘Projection solution’

$$U = \Pi_h(U^* - U^R - \delta_t \pi^R) + U^R \qquad (\text{and } \delta_t \pi = U^* - U)$$

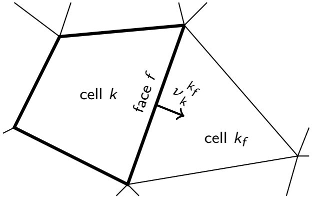
form a (unique) solution of the correction step, are independent of the reference functions and satisfy

$$\begin{aligned} \frac{1}{\delta_t}\left(\|U\|_h^2 - \|U^*\|_h^2\right) &= -\delta_t \|\pi(\overline{q}, q_B)\|_h^2 - 2\langle U, \pi(\overline{q}, q_B)\rangle_h \\ &= -\delta_t \|\pi(\overline{q}, q_B)\|_h^2 - 2\langle \overline{u} \cdot \nu, h\overline{q}\rangle_{\partial\Omega}. \end{aligned}$$

Fully discrete correction step

Aim: Preserve (discrete) *projection property* / *orthogonality* at the discrete level

$\mathbb{T}$ tesselation • $a_\star = (a_k)_{k\in\mathbb{T}}$ discrete function • given $h_\star > 0$



Design of a simple scheme

1. discrete scalar product $\langle f_\star, g_\star\rangle^\delta_{h_\star} = \sum_{k\in\mathbb{T}} |k| f_k \cdot g_k\, h_k$
2. $\mathbb{A}^\delta_{h_\star}$ (and projection $\Pi^\delta_{h_\star}$) with central difference operator

$$\nabla^\delta_k \cdot \varphi_\star = \frac{1}{|k|} \sum_{f\in\mathbb{F}_k} \frac{|f|}{2} (\varphi_k + \varphi_{k_f}) \cdot \nu_f^{k_f}$$

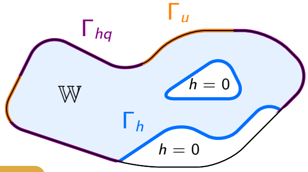
3. (reconstruction of hydrodynamic pressure $\overline{q}_\star, (\overline{q}_B)_\star$ from $\delta_t \pi_\star = U^*_\star - U_\star$)

⇝ solve e.g. as system of equations on $\overline{u}_\star$

Orthogonality $\langle U_\star, \pi_\star\rangle^\delta_{h_\star} = -\sum_{k\in\mathbb{T}} \sum_{f\in\mathbb{F}_k} |f| \frac{h_{k_f}\overline{q}_{k_f}\overline{u}_k + h_k \overline{q}_k \overline{u}_{k_f}}{2} \cdot \nu_k^{k_f} = 0$

⇝ entropy-stable scheme [Parisot '19]

Boundary conditions for simple scheme



- If $h_\star \geq 0$: project only on the *wet area* $\mathbb{W} \subset \mathbb{T}$ with $h_k > 0$ for all $k \in \mathbb{W}$
- decompose the boundary faces $\partial\mathbb{W} = \Gamma_u \cup \Gamma_{hq} \cup \Gamma_h$
- k_i/k_g interior/ghost cell for boundary face f, ν_f outward normal

Projection condition (homogeneous)

$$\left(h_{k_g}\overline{q}_{k_g}\overline{u}_{k_i} + h_{k_i}\overline{q}_{k_i}\overline{u}_{k_g}\right)\cdot\nu_f = 0 \qquad \text{for any } f \in \partial\mathbb{W}$$

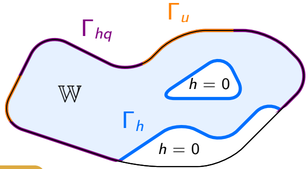
This is satisfied setting

Homogeneous boundary conditions

$$\overline{u}_{k_g}\cdot\nu_f = \begin{cases} -\overline{u}_{k_i}\cdot\nu_f \\ \overline{u}_{k_i}\cdot\nu_f \\ 0 \end{cases} \qquad h_{k_g}\overline{q}_{k_g} = \begin{cases} h_{k_i}\overline{q}_{k_i} & f \in \Gamma_u \\ -h_{k_i}\overline{q}_{k_i} & f \in \Gamma_{hq} \\ 0 & f \in \Gamma_h \end{cases}$$

$\rightsquigarrow$ well-posedness and entropy stability of fully discrete correction step

Boundary conditions for simple scheme



- If $h_\star \geq 0$: project only on the *wet area* $\mathbb{W} \subset \mathbb{T}$ with $h_k > 0$ for all $k \in \mathbb{W}$
- decompose the boundary faces $\partial\mathbb{W} = \Gamma_u \cup \Gamma_{hq} \cup \Gamma_h$
- k_i / k_g interior/ghost cell for boundary face f, ν_f outward normal

Projection condition (homogeneous)

$$\left(h_{k_g}\overline{q}_{k_g}\overline{u}_{k_i} + h_{k_i}\overline{q}_{k_i}\overline{u}_{k_g}\right) \cdot \nu_f = 0 \qquad \text{for any } f \in \partial\mathbb{W}$$

Given boundary data: $\widetilde{u}_f$ for $f \in \Gamma_u$ and $\widetilde{hq}_f$ for $f \in \Gamma_{hq}$

Inhomogeneous boundary conditions

$$\overline{u}_{k_g} \cdot \nu_f = \begin{cases} 2\widetilde{u}_f - \overline{u}_{k_i} \cdot \nu_f \\ \overline{u}_{k_i} \cdot \nu_f \\ 0 \end{cases} \qquad h_{k_g}\overline{q}_{k_g} = \begin{cases} h_{k_i}\overline{q}_{k_i} & f \in \Gamma_u \\ 2\widetilde{hq}_f - h_{k_i}\overline{q}_{k_i} & f \in \Gamma_{hq} \\ 0 & f \in \Gamma_h \end{cases}$$

⇝ well-posedness and entropy stability of fully discrete correction step

UNIVERSITÄT BIELEFELD

Faculty of Mathematics

Boundary conditions of full system

Given at time step t_n: $h_\star^n,\ \overline{u}_\star^n,\ (\overline{w}_\star^n,\ \sigma_\star^n)$ on $\mathbb{T}$

Step	solving for...	Required
(Ia) Shallow water step	$\begin{pmatrix} h_\star^* \\ h_\star^* \overline{u}_\star^* \end{pmatrix}$	$h_{k_g}^n,\ \overline{u}_{k_g}^n \cdot \nu_f$ $[\overline{u}_{k_g}^n \cdot \tau_f$ if inflow$]$
(Ib) Advection step	$\begin{pmatrix} \overline{w}_\star^* \\ \sigma_\star^* \end{pmatrix}$	$[\overline{w}_{k_g}^n$ if inflow$]$
(II) Correction step	$\begin{pmatrix} \overline{u}_\star^{n+1} \\ \overline{w}_\star^{n+1} \\ \sigma_\star^{n+1} \end{pmatrix} \Big/ \begin{pmatrix} \overline{q}^{n+1} \\ q_B^{n+1} \end{pmatrix}$	$\overline{u}_{k_g}^{n+1} \cdot \nu_f$ on Γ_u $h_{k_g}^{n+1} \overline{q}_{k_g}^{n+1}$ on Γ_{hq}

$\rightsquigarrow$ fixes $\delta_t^n > 0$ and $\mathbb{W}$

$\rightsquigarrow$ periodic, wall bc, dry front [Parisot '19]

Wave generation inlet

$\rightsquigarrow$ prescribing the water depth and velocity ($\partial\mathbb{T} = \Gamma_u$) as wave profile

(I) $\widetilde{h}_f^n, \widetilde{u}_f^n, \widetilde{v}_f^n, \widetilde{w}_f^n$

(II) $\widetilde{u}_f^{n+1}$

Travelling
Soliton
$\Omega = (0, 1)$

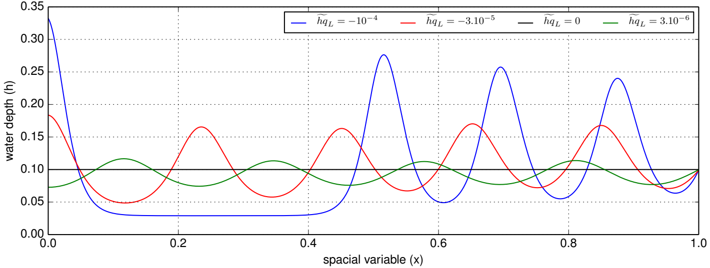
UNIVERSITÄT BIELEFELD
Faculty of Mathematics

Imposed discharge and pressure

Treat boundary as Γ_{hq}

$$h^n_{k_g}\overline{u}^n_{k_g}\cdot\nu^{k_g}_{k_i} = 2\widetilde{hu}^n_f - h^n_{k_i}\overline{u}^n_{k_i}\cdot\nu^{k_g}_{k_i} \qquad \text{and} \qquad h^n_{k_g}\overline{q}^n_{k_g} = 2\widetilde{hq}^n_f - h^n_{k_i}\overline{q}^n_{k_i},$$

⇝ recover $\overline{u}^n_{k_g}\cdot\nu_f$ and $h^n_{k_g}$ if $\overline{u}^n_{k_i}\cdot\nu_f \neq 0$



⇝ similarities to stationary cnoidal waves

Summary and outlook

Summary

- 'projection solution' for correction step of the semi-discrete GN equations
- well-posedness of the correction step for a class of boundary conditions
- design of scheme with discrete projection property
- ⇝ entropy stability by construction
- simulation for bc of practical interest: wave generation and imposed discharge

Next steps

- adaptive strategy
- justify semi-discretization
- ...

Thank you – Merci