WATER WAVES - FLOATING STRUCTURE INTERACTION

EGRIN | G. Beck, D. Lannes L. Weynans | May 2021

WEC : WAVE ENERGY CONVERTER
Green washing slide

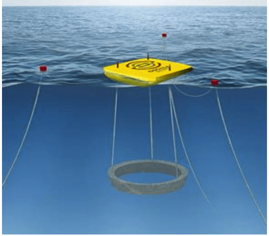
- Recovering the mouvement of a floating structure

PSL UNIVERSITÉ PARIS

WEC : WAVE ENERGY CONVERTER
Engineering washing slide : Full Euler vs. Cummins

Full Euler system
with free boundary.
+ Newton equation

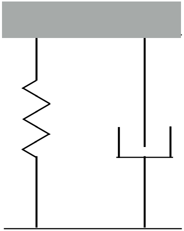
- Too costly



Cummins' Equation

$$M\ddot{\delta} + a\delta + k * \delta = 0$$

- Meaning ?



WEC : WAVE ENERGY CONVERTER

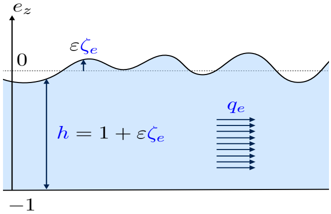
Math washing slide : Boussinesq model

- Dimensionless **Shallowness** and **Nonlinearity** parameters

$$\kappa = \frac{\text{(depth)}}{\text{(transversale scale)}} \quad \text{and} \quad \varepsilon = \frac{\text{(waves amplitude)}}{\text{(depth)}}.$$

- Weakly dispersive $\kappa \ll 1$, Weakly non-linear $\varepsilon = O(\kappa^2)$

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2)\partial_t q + \varepsilon \partial_x \left(\frac{1}{h} q^2\right) + h \partial_x \zeta = 0 \\ h = 1 + \varepsilon \zeta \end{cases}$$



Precision : $O(\varepsilon\kappa^2, \kappa^4)$

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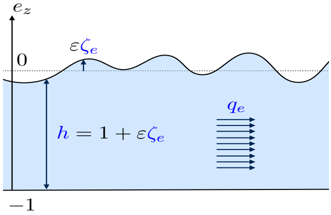

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- Non-linear Shallow Water

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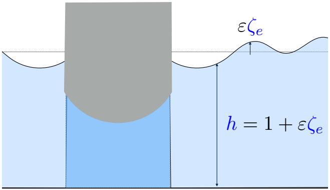

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Modeling : [John], [Lannes]
fixed body : NSW [Bocchi, He, Vergara-Hermosilla], Boussinesq **[Bresch, Lannes, Métivier]**
vertical walls, NSW with viscosity [Maity, Takahashi, Tucsnak]

non-vertical walls, NSW : [Lannes-Iguchi], [Godlewski, Parisot, Sainte-Marie, Wahl]

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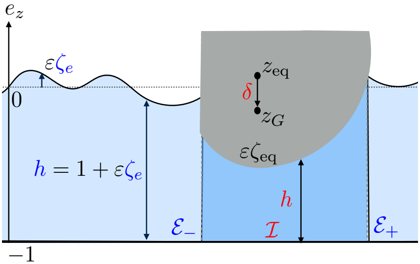
VERTICAL MOVEMENT OF A FLOATING SOLID

Coupling water-waves and solid dynamic



- Exterior domain $\mathcal{E}$

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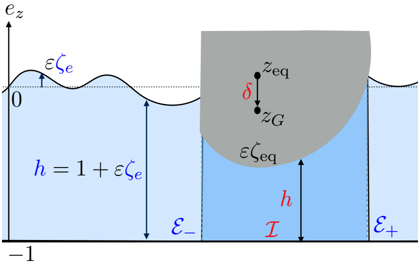

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$$P_e = P_{\text{atm}}$$

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$$\zeta(x, t) = \zeta_{\text{eq}}(x) + \delta(t)$$

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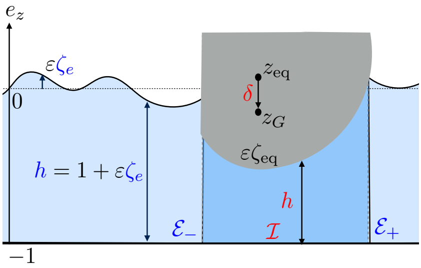

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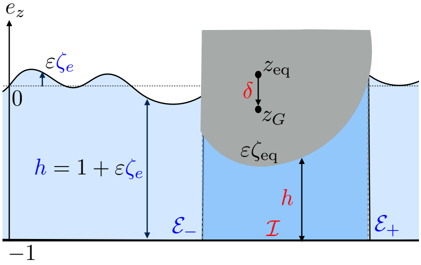

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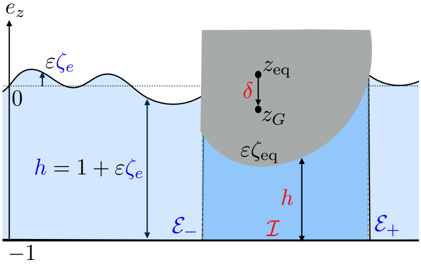

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- Vertical walls
  - **dis**continuity of ζ
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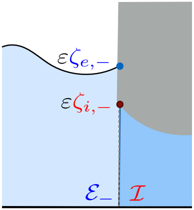
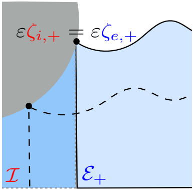


- Non-vertical walls
  - continuity of ζ
  - **free interfaces**



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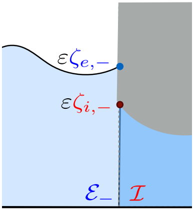
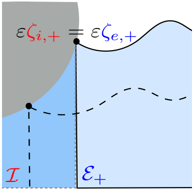


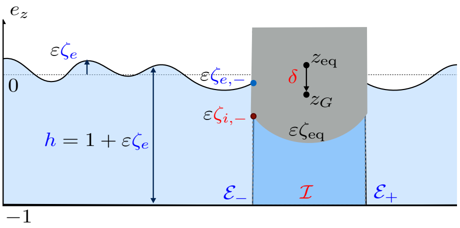


- Boundaries conditions on the pressure ($\Rightarrow$ approximate conservation of energy)

$$\frac{1}{\varepsilon}\underline{P_\pm} = \underline{(\zeta - \zeta)}_\pm + \frac{\varepsilon}{2}\underline{\left(\frac{q^2}{h^2} - \frac{q^2}{h^2}\right)}_\pm - \kappa^2\underline{\left(\frac{1}{h}\partial_x\partial_t q - \frac{1}{h}\partial_x\partial_t q\right)}_\pm$$

WAVE-STRUCTURE TRANSMISSION PROBLEM

Coupling water-waves and solid dynamic in interior domain



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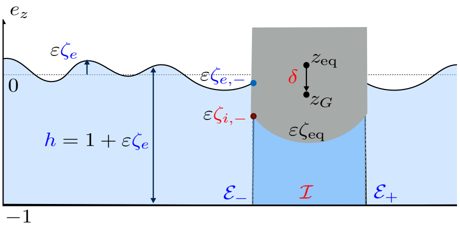

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$$\begin{cases} \varepsilon^{-1}\partial_x h \partial_x \underline{P} = \dots \\ \frac{1}{\varepsilon}\underline{P_\pm} = \underline{\Pi}_\pm - \underline{\Pi}_\pm - \kappa^2 \cdots \end{cases}$$

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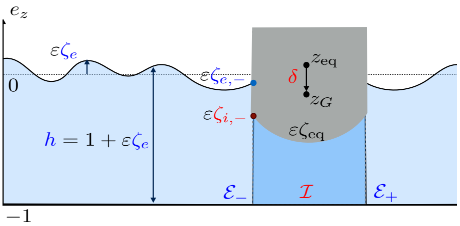

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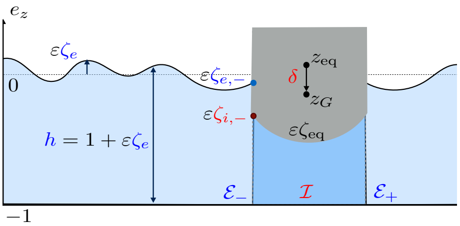
- Injecting the pressure

$\Rightarrow$ EDO's for $\delta, \langle q_i\rangle$

WAVE-STRUCTURE TRANSMISSION PROBLEM

Coupling water-waves and solid dynamic



- Added masses

$$\mathfrak{m}(\varepsilon\delta) = \tilde{m} + \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \frac{x^2}{h_i} + \langle \frac{\kappa^2}{h} \rangle$$

$$h := 1 + \varepsilon(\zeta_{eq} + \delta)$$

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$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon\mathcal{Q}_1\big(\varepsilon\delta; \dot{\delta}, \langle q\rangle\big) - \langle \underline{\Pi} - \frac{\kappa^2}{\underline{h}}\underline{\partial_x\partial_t q}\rangle = 0 \\ \alpha(\varepsilon\delta)\langle \dot{q}\rangle + \varepsilon\mathcal{Q}_2\big(\varepsilon\delta; \dot{\delta}, \langle q\rangle\big) + [\![\underline{\Pi} - \frac{\kappa^2}{\underline{h}}\underline{\partial_x\partial_t q}]\!] = 0 \end{cases}$$

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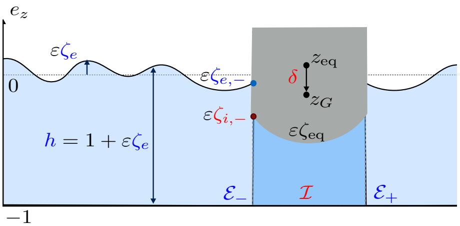

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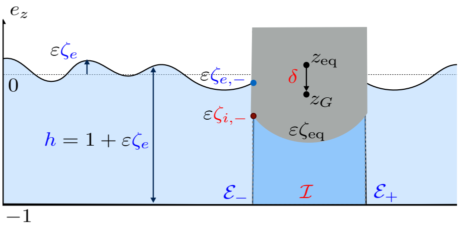

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- $\kappa = 0$ **hyperbolic system**, compatibility conditions $T^* = O(\varepsilon^{-1})$
- $\kappa \neq 0$ **dispersive perturbation** of hyperbolic system $T^* = O(1)$
- $\kappa \to 0$ compatibility conditions (inequalities) $T^* = O\big((\varepsilon + \kappa^2)^{-1}\big)$

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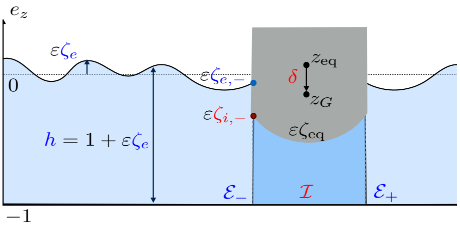

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$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2\partial_x^2)\partial_t q + \varepsilon\partial_x\left(\frac{q^2}{h}\right) + h\partial_x\zeta = 0 \end{cases}$$

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$$\underline{q}_\pm = \underline{q}_\pm \Rightarrow \begin{cases} [\![q]\!] = -\dot{\delta}[\![x]\!], \\ \langle q \rangle = \langle q \rangle, \end{cases}$$

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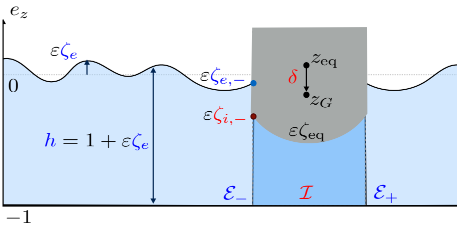
WAVE-STRUCTURE TRANSMISSION PROBLEM

Coupling water-waves and solid dynamic



- Added masses

$$\mathfrak{m}(\varepsilon\delta) = \tilde{m} + \frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \frac{x^2}{h_i} + \langle \frac{\kappa^2}{h} \rangle$$

$$h := 1 + \varepsilon(\zeta_{eq} + \delta)$$

- Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon\mathcal{Q}_1\big(\varepsilon\delta; \dot{\delta}, \langle q \rangle\big) - \langle \underline{\Pi} - \frac{\kappa^2}{\underline{h}} \boxed{\underline{\partial_x\partial_t q}} \rangle = 0 \\ \alpha(\varepsilon\delta)\dot{\langle q \rangle} + \varepsilon\mathcal{Q}_2\big(\varepsilon\delta; \dot{\delta}, \langle q \rangle\big) + [\![\underline{\Pi} - \frac{\kappa^2}{\underline{h}} \boxed{\underline{\partial_x\partial_t q}}]\!] = 0 \end{cases}$$

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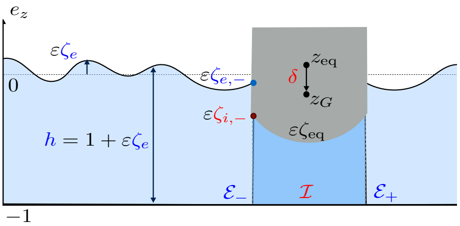

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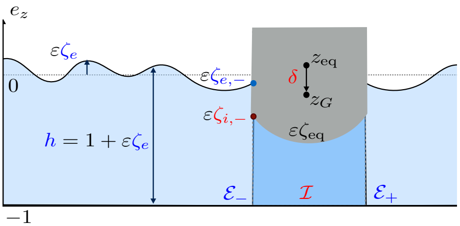
WAVE-STRUCTURE TRANSMISSION PROBLEM
Augmented formulation



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- Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon\mathcal{Q}_1\big(\varepsilon\delta; \dot{\delta}, \langle q\rangle\big) - \langle \underline{\Pi} + \frac{\kappa^2}{\underline{h}}\ddot{\underline{\zeta}} \rangle = 0 \\ \alpha(\varepsilon\delta)\langle \dot{q}\rangle + \varepsilon\mathcal{Q}_2\big(\varepsilon\delta; \dot{\delta}, \langle q\rangle\big) + [\![\underline{\Pi} + \frac{\kappa^2}{\underline{h}}\ddot{\underline{\zeta}}]\!] = 0 \\ \kappa^2 \ddot{\underline{\zeta}}_{\pm} \pm \kappa(\langle \dot{q}\rangle \pm \ddot{\delta}) + \underline{\zeta}_{\pm} + \varepsilon\mathcal{Q}_3\big(\underline{\zeta}_{\pm}, \dot{\delta}, \langle q\rangle\big) = \underline{\text{"}(1-\kappa^2\partial_x^2)^{-1}\text{"}(\Pi + \frac{\varepsilon}{2}\zeta^2)}_{\pm} \end{cases}$$

$$\underline{\Pi} := \underline{\zeta} + \frac{\varepsilon}{2}\frac{\underline{q}^2}{\underline{h}^2}$$

- Boussinesq-Abbott

$$\begin{cases} \partial_t\zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \varepsilon\partial_x\left(\frac{q^2}{h}\right) + h\partial_x\zeta = 0 \end{cases}$$

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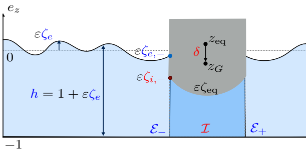
WAVE-STRUCTURE TRANSMISSION PROBLEM

Augmented added masses



- Added masses

$$\left(\begin{array}{cc|c} \mathfrak{m}(\varepsilon\delta)+\kappa\ell\langle\frac{1}{\underline{h}}\rangle & -\frac{\kappa}{2}[\![\frac{1}{\underline{h}}]\!] & \\ -\frac{\kappa}{2}[\![\frac{1}{\underline{h}}]\!] & \alpha(\varepsilon\delta)+\frac{\kappa}{\ell}\langle\frac{1}{\underline{h}}\rangle & 0_{2\times 2} \\ \hline \kappa\ell & -\kappa & \\ \kappa\ell & \kappa & \kappa^2\mathrm{Id}_{2\times 2} \end{array}\right)\begin{pmatrix}\ddot{\delta}\\ \dot{\langle q\rangle}\\ \ddot{\underline{\zeta}}_+\\ \ddot{\underline{\zeta}}_-\end{pmatrix}$$

- Newton

$$\begin{cases} \mathfrak{m}(\varepsilon\delta)\ddot{\delta}+\delta+\varepsilon\mathcal{Q}_1\big(\varepsilon\delta;\dot{\delta},\langle q\rangle\big)-\langle\underline{\Pi}+\frac{\kappa^2}{\underline{h}}\ddot{\underline{\zeta}}\rangle=0\\ \alpha(\varepsilon\delta)\dot{\langle q\rangle}+\varepsilon\mathcal{Q}_2\big(\varepsilon\delta;\dot{\delta},\langle q\rangle\big)+[\![\underline{\Pi}+\frac{\kappa^2}{\underline{h}}\ddot{\underline{\zeta}}]\!]=0\\ \kappa^2\ddot{\underline{\zeta}}_\pm\pm\kappa(\dot{\langle q\rangle}\pm\ddot{\delta})+\underline{\zeta}_\pm+\varepsilon\mathcal{Q}_3\big(\underline{\zeta}_\pm,\dot{\delta},\langle q\rangle\big)=\underline{\text{"}(1-\kappa^2\partial_x^2)^{-1}\text{"}(\Pi+\frac{\varepsilon}{2}\zeta^2)}_\pm \end{cases}$$

$$\underline{\Pi}:=\underline{\zeta}+\frac{\varepsilon}{2}\frac{\underline{q}^2}{\underline{h}^2}$$

- Boussinesq-Abbott

$$\begin{cases} \partial_t\zeta+\partial_x q=0\\ (1-\kappa^2\partial_x^2)\partial_t q+\varepsilon\partial_x\left(\frac{q^2}{h}\right)+h\partial_x\zeta=0 \end{cases}$$

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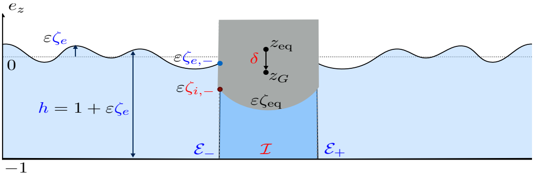
LONG TIME BEHAVIOUR

Uniform estimate

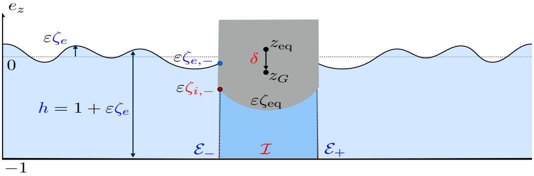


- $U(t) = (\zeta, q, \delta, \dot{\delta}, \langle q \rangle, \underline{\zeta}_\pm, \dot{\underline{\zeta}}_\pm)(t) \in H^1 \times H^2 \times \mathbb{R}^3$ **EDO** $\dot{U} = \Phi(U)$
- Augmented **energy** $E = ||\zeta, \frac{q}{\sqrt{h}}, \frac{\kappa}{\sqrt{h}}\partial_x q||_2^2 + |\delta, \dot{\delta}, \langle q \rangle|^2 + \varepsilon\kappa^3|\underline{\zeta}, \kappa\dot{\underline{\zeta}}|^2$
- **Uniform estimate**

$$||\zeta, q||_{L_t^\infty W_x^{1,\infty}} \leq M \Rightarrow \exists T_u(M, E_0), \forall t \leq \min(T, \frac{T_u}{\varepsilon})\,;\; E(t) \leq C(M, E_0)$$

LONG TIME BEHAVIOUR

Uniform estimate



- $U(t) = (\zeta, q, \delta, \dot{\delta}, \langle q \rangle, \underline{\zeta}_\pm, \dot{\underline{\zeta}}_\pm)(t) \in H^1 \times H^2 \times \mathbb{R}^3$ **EDO** $\dot{U} = \Phi(U)$
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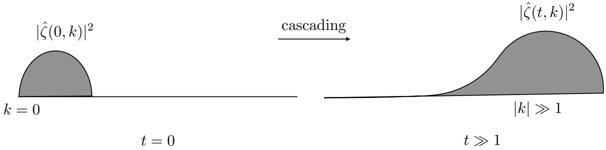
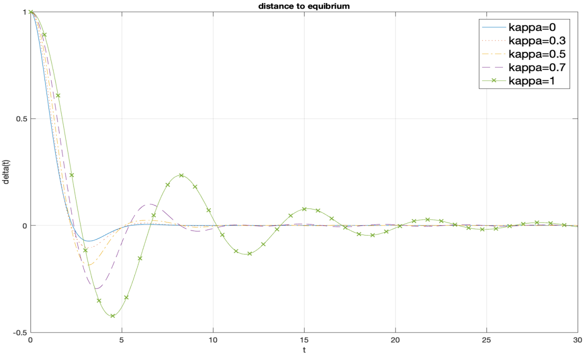
- Perspective : **wave-turbulence** (with Gallagher, Dormy and Faou)



LONG TIME BEHAVIOUR
Return to equilibrium - Linear case



- **Linear case** $\varepsilon = 0$ with $U_{|t=0} = (0, 0, \delta_0 \neq 0, 0, 0, 0)$
  Return to equilibrium : $(\delta, \dot{\delta}) \to 0$
  - $\kappa = 0$: $|\delta| \sim e^{-\alpha t}$
    $\zeta, q \in C^0_{x,t}$ admit singularities at $\{x = t\}$ (transport equation)
  - $\kappa > 0$: $|\delta| > ct^{-3/2}$
    $\zeta, q \in C^n_x H^1_{t,\mathrm{loc}}$ (**non-local** transport equation)

THANK YOU FOR ATTENTION : POEMS' POEM

POEMS= Propagation Ondes Etudes Mathématiques Simulations

La vague et le flotteur sont en couplage,
L'onde est et non-linéaire et dispersive.
Les conditions limites sont inoffensives,
Car nul ne peut parvenir jusqu'à la plage.

Sans la pression, demeurez sur le rivage,
Car elle est un multiplicateur de Lagrange
Qui représente les contraintes étranges.
Sans elle, vous ne connaitriez que naufrage.

L'expérience nomée "retour à l'équilibre"
Est de l'équation de Cummins le calibre.
Vigilance avec la pseudo-différentiallité !

Si BBM pouvait diagonaliser Boussinesq
Et l'asymptotique moins charlatanesque,
Alors le calcul de la vague serait vanité.

- Conservation of the volume

$$\Rightarrow \text{ continuity of } q : \underline{q}_{\pm} = \underline{q}_{\pm}$$

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- Conservation of energy

$$\dot{E}_{\text{tot}} = 0$$

VERTICAL MOVEMENT OF A FLOATING SOLID

Transmission conditions at the interfaces

- Conservation of the volume

$$\Rightarrow \text{ continuity of } q : \underline{q}_{\pm} = \underline{q}_{\pm}$$

- Conservation of energy

$$\dot{E}_{\text{tot}} = 0$$

- Energy $E_{\text{tot}} = E_{\text{solid}} + E_{\text{fluid}}$
  - Solid energy $E_{\text{solid}} := \frac{1}{2}\tilde{m}\ell\,\dot{\delta}^2 + \tilde{g}\ell\delta$
  - Fluid energy $E_{\text{fluid}} := \frac{1}{2}\int_{\mathbb{R}}\left(\zeta^2 + \frac{q^2}{h} + \frac{\mu}{6h}(\partial_x q)^2\right)$

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$$\dot{E}_{\text{tot}} = -[\![q(\Pi - \frac{\kappa^2}{h}\partial_x\partial_t q) - q(\Pi - \frac{\kappa^2}{h}\partial_x\partial_t q) - \frac{1}{\varepsilon} q\underline{P}]\!] + O(\varepsilon\kappa^2).$$

  - jump between $\mathcal{E}_+$ and $\mathcal{E}_-$: $[\![u]\!] = u_+ - u_-$
  - notation : $\Pi := \zeta + \frac{\varepsilon}{2}\frac{q^2}{h^2}$

VERTICAL MOVEMENT OF A FLOATING SOLID

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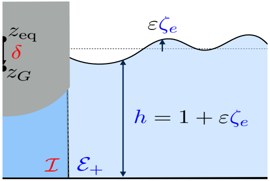
RETURN TO THE EQUILIBRIUM
BBM approximation



- $\varepsilon = 0 \Rightarrow$ **Exact** diagonalization

- **Non-local**

$$\mathfrak{m}\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_x q + \mathcal{K} *_t \partial_t q = 0 \\ \zeta = \mathcal{K} *_t q \end{cases}$$

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RETURN TO THE EQUILIBRIUM
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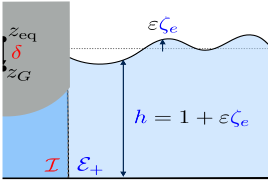

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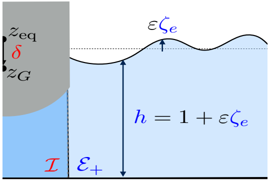

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RETURN TO THE EQUILIBRIUM

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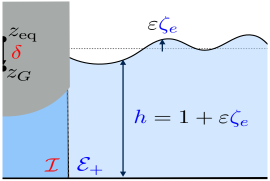

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- $\kappa = 0 \Rightarrow$ **Exact** diagonalization
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  - **Non-linear** Cummins Operator $\mathfrak{C}(\dot{\delta})$

- **Non-linear**

$$\mathfrak{m}\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathfrak{C}(\dot{\delta}) = -\underline{\zeta} - \frac{\varepsilon}{2}\frac{\underline{q}^2}{\underline{h}^2} = \frac{|\mathcal{I}|}{2}\dot{\delta} + \varepsilon NL(\dot{\delta})$$

- Wave equations

$$\begin{cases} \partial_t q + \left(1 + \varepsilon\frac{3}{2}q\right)\partial_x q = 0 \\ \zeta = q - \varepsilon\frac{3}{4}q^2 \end{cases}$$

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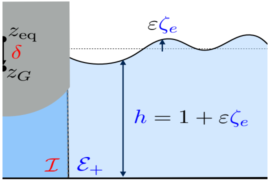
RETURN TO THE EQUILIBRIUM

BBM approximation



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$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

- $\kappa, \varepsilon \neq 0 \Rightarrow$ **Approximate** diagonalization(s)
  - **BBM** approximation(s) (order $O(\varepsilon\kappa^2, \varepsilon^2, \kappa^4)$)

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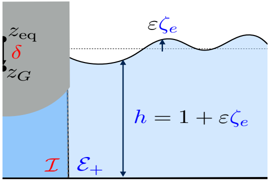
RETURN TO THE EQUILIBRIUM
BBM approximation



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$$\mathfrak{C}(\dot{\delta}) = \left(\zeta - \frac{\varepsilon}{2}q^2 - \kappa^2\partial_t\partial_x q\right)_{x=0}$$

- Wave equations

$$\begin{cases} \left(1 - \frac{\kappa^2}{2}\partial_x^2\right)\partial_t q + \left(1 + \varepsilon\frac{3}{2}q\right)\partial_x q = 0 \\ \zeta = q - \varepsilon\frac{3}{4}q^2 + \frac{\kappa^2}{2}\partial_x\partial_t q \end{cases}$$

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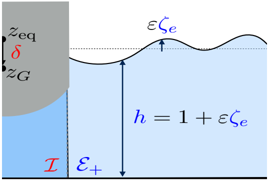
  - **BBM** approximation(s) (order $O(\varepsilon\kappa^2, \varepsilon^2, \kappa^4)$)
  - **Asymptotic expansion**
    $\mathfrak{C}(\dot{\delta}) = \frac{|\mathcal{I}|}{2} Op(\sqrt{1 + \kappa^2 s^2})\dot{\delta} + \varepsilon NL(\dot{\delta}) + O(\varepsilon\kappa^2, \varepsilon^2, \kappa^4)$

RETURN TO THE EQUILIBRIUM

BBM approximation



- $\varepsilon = 0 \Rightarrow$ **Exact** diagonalization
  - Laplace representation
  - **Non-local** Cummins Operator $\mathfrak{C}(\dot{\delta})$
- $\kappa = 0 \Rightarrow$ **Exact** diagonalization
  - Riemann invariants
  - **Non-linear** Cummins Operator $\mathfrak{C}(\dot{\delta})$

- Cummins operator

$$\mathfrak{C}(\dot{\delta}) = \left(\zeta - \frac{\varepsilon}{2}q^2 - \kappa^2\partial_t\partial_x q\right)_{x=0}$$

- **Non-linear Non-local**

$$\mathfrak{m}\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

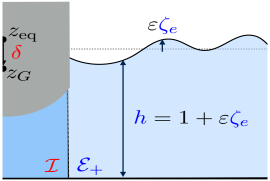
- Wave equations

$$\begin{cases} \left(1 - \frac{\kappa^2}{2}\partial_x^2\right)\partial_t q + \left(1 + \varepsilon\frac{3}{2}q\right)\partial_x q = 0 \\ \zeta = q - \varepsilon\frac{3}{4}q^2 + \frac{\kappa^2}{2}\partial_x\partial_t q \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

- $\kappa, \varepsilon \neq 0 \Rightarrow$ **Approximate** diagonalization(s)
  - **BBM** approximation(s) (order $O(\varepsilon\kappa^2, \varepsilon^2, \kappa^4)$)
  - **Asymptotic expansion**
    $\mathfrak{C}(\dot{\delta}) = \frac{|\mathcal{I}|}{2} Op(\sqrt{1 + \kappa^2 s^2})\dot{\delta} + \varepsilon NL(\dot{\delta}) + O(\varepsilon\kappa^2, \varepsilon^2, \kappa^4)$



1. Well-posedness
2. **Linear** $\varepsilon = 0$
   - non dispersive case $\kappa = 0$
   - dispersive case $\kappa > 0$
   - non-local transport equations
3. non-linear $\varepsilon > 0$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta} - \frac{\varepsilon}{2}\frac{\underline{q}^2}{\underline{h}^2} + \frac{\kappa^2}{\underline{h}}\underline{\partial_x \partial_t q}$$

- Newton

$$\mathfrak{m}(\varepsilon\delta)\ddot{\delta} + \delta + \varepsilon\beta(\varepsilon\delta)\dot{\delta}^2 + \mathfrak{C}(\dot{\delta}) = 0$$

- Boussinesq-Abbott

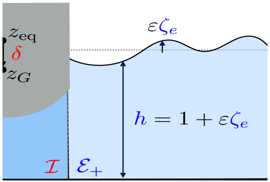
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2)\partial_t q + \varepsilon \partial_x \left(\frac{q^2}{h}\right) + h\partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



- Explicit solution

$$q = \zeta = -\frac{|\mathcal{I}|}{2}\dot{\delta}(t-x)\mathbb{I}_{t\geq x},$$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta}$$

- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Linear waves

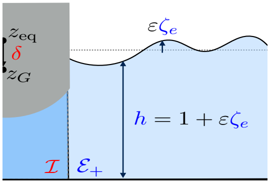
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



- Explicit solution

$$q = \zeta = -\frac{|\mathcal{I}|}{2}\dot{\delta}(t - x)\mathbb{I}_{t \geq x},$$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta} = \frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Linear waves

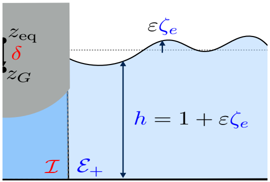
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



- Explicit solution

$$q = \zeta = -\frac{|\mathcal{I}|}{2}\dot{\delta}(t - x)\mathbb{I}_{t \geq x},$$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta} = \frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Linear waves

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \zeta = 0 \end{cases}$$

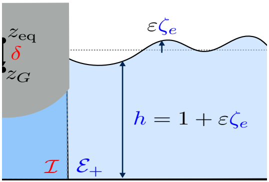
- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

- Decay test : δ exponentially decreasing

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Non-dispersive linear waves



- Explicit solution

$$q = \zeta = -\frac{|\mathcal{I}|}{2}\dot{\delta}(t-x)\mathbb{I}_{t\geq x},$$

- Regularity $C^0(\mathcal{E}_+ \times \mathbb{R}^+)$
- Singularities at $\{x = t\}$ since $\ddot{\delta}(0) = -\delta_0 \mathfrak{m}(0)^{-1}$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta} = \frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Linear waves

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \zeta = 0 \end{cases}$$

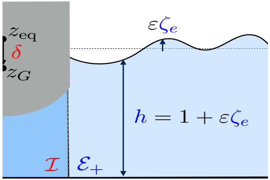
- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta},$$

- Decay test : δ exponentially decreasing

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathfrak{C} : \dot{\delta} \mapsto -\underline{\zeta} + \kappa^2 \underline{\partial_x \partial_t q}$$

- Wave equations

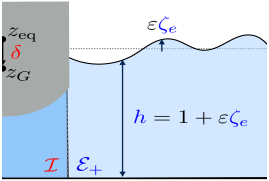
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s\partial_x \hat{q}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

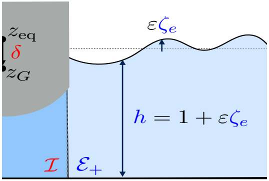
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace $\mathcal{L} : q(t) \mapsto \widehat{q}(s) := \int q(t)e^{-st}dt \quad \Re(s) > 0$

$$\hat{q}(s,x) = \hat{\underline{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

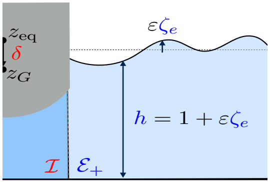
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace $\mathcal{L} : q(t) \mapsto \widehat{q}(s) := \int q(t)e^{-st}dt \quad \Re(s) > 0$

$$\hat{q}(s,x) = \hat{\underline{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- Newton

$$\hat{\delta} = \frac{\mathfrak{m}(0)s + \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}}{\mathfrak{m}(0)s^2 + \frac{|\mathcal{I}|}{2}s\sqrt{1+\kappa^2 s^2} + 1}\delta_0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s\partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

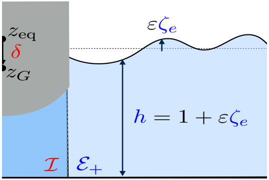
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

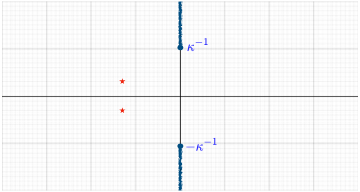
- Laplace $\mathcal{L} : q(t) \mapsto \widehat{q}(s) := \int q(t)e^{-st}dt \quad \Re(s) > 0$

$$\hat{q}(s,x) = \underline{\hat{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves





- Newton

$$\hat{\delta} = \frac{\mathfrak{m}(0)s + \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}}{\mathfrak{m}(0)s^2 + \frac{|\mathcal{I}|}{2}s\sqrt{1+\kappa^2 s^2} + 1}\delta_0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s\partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

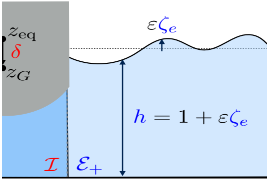
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

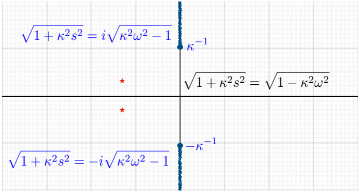
- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace $\mathcal{L} : L^2(e^{-\eta_0 t}dt) \to \mathcal{H}_{\eta_0}$ $\quad ||\hat{u}||^2_{\eta_0} := \sup_{\eta>\eta_0}\int |\hat{u}(\eta+i\omega)|^2 d\omega$

$$\hat{q}(s,x) = \hat{\underline{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$





- Newton

$$\hat{\delta} = \frac{\mathfrak{m}(0)s + \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}}{\mathfrak{m}(0)s^2 + \frac{|\mathcal{I}|}{2}s\sqrt{1+\kappa^2 s^2} + 1}\delta_0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s\partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

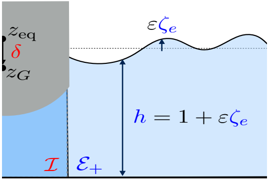
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace $\mathcal{L} : L^2(e^{-\eta_0 t}dt) \to \mathcal{H}_{\eta_0}$ $\quad ||\hat{u}||^2_{\eta_0} := \sup_{\eta>\eta_0}\int |\hat{u}(\eta+i\omega)|^2 d\omega$

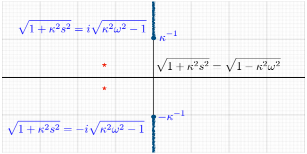
$$\hat{q}(s,x) = \underline{\hat{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



Decay test $\delta \in H^2$, $t\delta \notin L^2$, $|\delta| > ct^{-3/2-\alpha}$



- Newton

$$\hat{\delta} = \frac{\mathfrak{m}(0)s + \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}}{\mathfrak{m}(0)s^2 + \frac{|\mathcal{I}|}{2}s\sqrt{1+\kappa^2 s^2} + 1}\delta_0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s\partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

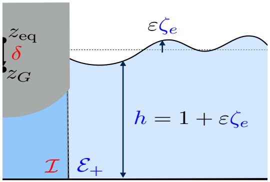
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace $\mathcal{L} : L^2(e^{-\eta_0 t}dt) \to \mathcal{H}_{\eta_0}$ $\quad ||\hat{u}||^2_{\eta_0} := \sup_{\eta>\eta_0}\int |\hat{u}(\eta+i\omega)|^2 d\omega$

$$\hat{q}(s,x) = \hat{\underline{q}}(s)e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$



- Laplace multiplier

$$\sqrt{1+\kappa^2 s^2} = \kappa s + \frac{1}{\kappa s + \sqrt{1+\kappa^2 s^2}}$$

$$Op(\sqrt{1+\kappa^2 s^2})u = \kappa \partial_t u + K_\kappa *_t u$$

- Newton

$$\mathfrak{m}(0)\ddot{\delta} + \delta + \mathfrak{C}(\dot{\delta}) = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

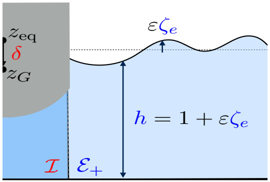
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace

$$\hat{q}(s,x) = \underline{\hat{q}}(s) e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- Laplace multiplier

$$\sqrt{1+\kappa^2 s^2} = \kappa s + \frac{1}{\kappa s + \sqrt{1+\kappa^2 s^2}}$$

$$Op(\sqrt{1+\kappa^2 s^2})u = \kappa \partial_t u + K_\kappa *_t u$$

- Newton

$$\left(\mathfrak{m}(0) + \frac{|\mathcal{I}|}{2}\kappa\right)\ddot{\delta} + \delta + \frac{|\mathcal{I}|}{2} K_\kappa *_t \dot{\delta} = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

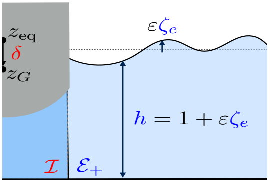
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace

$$\hat{q}(s,x) = \underline{\hat{q}}(s) e^{-\frac{s}{\sqrt{1+\kappa^2 s^2}}x} \qquad \hat{\zeta}(s,x) = \frac{\hat{q}(s,x)}{\sqrt{1+\kappa^2 s^2}}$$



- Laplace multiplier

$$\sqrt{1+\kappa^2 s^2} = \kappa s + \frac{1}{\kappa s + \sqrt{1+\kappa^2 s^2}}$$
$$Op(\sqrt{1+\kappa^2 s^2})u = \kappa \partial_t u + K_\kappa *_t u$$

- Bessel kernel $\hat{\mathcal{K}}(s) := 1/\sqrt{1+\kappa^2 s^2}$

- Newton

$$\left(\mathfrak{m}(0) + \frac{|\mathcal{I}|}{2}\kappa\right)\ddot{\delta} + \delta + \frac{|\mathcal{I}|}{2} K_\kappa *_t \dot{\delta} = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1-\kappa^2\partial_x^2)\partial_t q + \partial_x \zeta = 0 \end{cases}$$

- Boundary condition

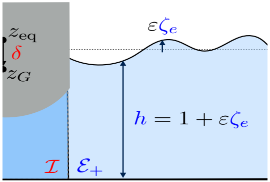
$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace

$$\partial_x \hat{q} = -\frac{s}{\sqrt{1+\kappa^2 s^2}}\underline{\hat{q}} e^{-s\hat{\mathcal{K}}(s)x} = -\frac{s}{\sqrt{1+\kappa^2 s^2}}\hat{q} \qquad \hat{\zeta} = \frac{\hat{q}}{\sqrt{1+\kappa^2 s^2}}$$

RETURN TO THE EQUILIBRIUM - LINEAR CASE

Dispersive linear waves



- **Laplace multiplier**

$$\sqrt{1+\kappa^2 s^2} = \kappa s + \frac{1}{\kappa s+\sqrt{1+\kappa^2 s^2}}$$

$$Op(\sqrt{1+\kappa^2 s^2})u = \kappa \partial_t u + K_\kappa *_t u$$

- **Bessel kernel** $\hat{\mathcal{K}}(s) := 1/\sqrt{1+\kappa^2 s^2}$

- **Newton**

$$\left(\mathfrak{m}(0) + \frac{|\mathcal{I}|}{2}\kappa\right)\ddot{\delta} + \delta + \frac{|\mathcal{I}|}{2} K_\kappa *_t \dot{\delta} = 0$$

- **Cummins operator**

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- **Wave equations**

Caputo : $\partial_x q + \mathcal{K} *_t \partial_t q = 0$

R.L : $\partial_x q + \partial_t(\mathcal{K} *_t q) = 0$

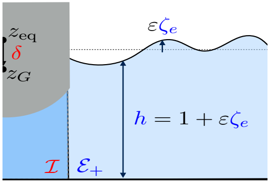
Link : $\partial_t(\mathcal{K} *_t q) = \mathcal{K} *_t \partial_t q + (q|_{t=0})\mathcal{K}$

- **Laplace**

$$\partial_x \hat{q} = -\frac{s}{\sqrt{1+\kappa^2 s^2}}\underline{\hat{q}}e^{-s\hat{\mathcal{K}}(s)x} = -\frac{s}{\sqrt{1+\kappa^2 s^2}}\hat{q} \qquad \hat{\zeta} = \frac{\hat{q}}{\sqrt{1+\kappa^2 s^2}}$$

- **Boundary condition**

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$



- Laplace multiplier

$$\sqrt{1+\kappa^2 s^2} = \kappa s + \frac{1}{\kappa s + \sqrt{1+\kappa^2 s^2}}$$
$$Op(\sqrt{1+\kappa^2 s^2})u = \kappa \partial_t u + K_\kappa *_t u$$

- Bessel kernel $\hat{\mathcal{K}}(s) := 1/\sqrt{1+\kappa^2 s^2}$

- Newton

$$\left(\mathfrak{m}(0) + \frac{|\mathcal{I}|}{2}\kappa\right)\ddot{\delta} + \delta + \frac{|\mathcal{I}|}{2} K_\kappa *_t \dot{\delta} = 0$$

- Cummins operator

$$\mathcal{L}\left[\mathfrak{C}(\dot{\delta})\right] = -\hat{\zeta} + \kappa^2 s \partial_x \hat{q} = \frac{|\mathcal{I}|}{2}\sqrt{1+\kappa^2 s^2}\,\hat{\dot{\delta}}$$

- Wave equations

Caputo : $\partial_x q + \mathcal{K} *_t \partial_t q = 0$
R.L : $\partial_x q + \partial_t(\mathcal{K} *_t q) = 0$
Link : $\partial_t(\mathcal{K} *_t q) = \mathcal{K} *_t \partial_t q + (q|_{t=0})\mathcal{K}$

- Boundary condition

$$\underline{q} = -\frac{|\mathcal{I}|}{2}\dot{\delta}$$

- Laplace

$q \in C^0_x H^1_t \cap C^n_x H^1_{t,\alpha}$ where $L^2_{t,\alpha} := L^2(e^{-\alpha t}dt)$ but $\partial_x q \notin L^2_t$

$$\begin{cases} \partial_t u + \partial_x u = f \\ u(t=0) = u^{\text{in}}(x) \\ u(x=0) = \underline{u}(t) \end{cases}$$

- $\underline{u}(t=0) = u^{\text{in}}(x=0)$

NON-LOCAL TRANSPORT EQUATION

Caputo / Riemann-Liouville $\partial_x(\mathcal{K} *_x u) = \mathcal{K} *_x \partial_x u + (u|_{x=0})\mathcal{K}$

$$\begin{cases} \partial_t u + \partial_x u = f \\ u(t=0) = u^{\text{in}}(x) \\ u(x=0) = \underline{u}(t) \end{cases}$$

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$$\begin{cases} \partial_t u + \mathcal{K}(x,t) *_x \partial_x u = f \\ u(t=0) = u^{\text{in}}(x) \\ u(x=0) = \underline{u}(t) \end{cases}$$

- C.C 2 : $\partial_t \underline{u} = f_{|x=0}$

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NON-LOCAL TRANSPORT EQUATION

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$$\partial_t \tilde{u} + \tilde{\mathcal{K}}(p\tilde{u} - \underline{u}) = \tilde{f}$$

- C.C 2 : $\partial_t \underline{u} = f_{|x=0}$

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$$\Pi_t : \quad L^2_{x,\alpha} \to L^2_{x,\alpha} := L^2_x(e^{-\alpha} dx)$$
$$u^{\text{in}} \mapsto \mathcal{L}_x^{-1}\left[exp(-\int_0^t p\tilde{\mathcal{K}}(p,t')dt')\tilde{u}^{\text{in}}\right]$$

$$\Re[p\tilde{\mathcal{K}}] \geq c_\alpha \geq 0 \Rightarrow |||\Pi_t||| \leq e^{-c_\alpha t}$$

$$\partial_t \tilde{u} + \tilde{\mathcal{K}}(p\tilde{u} - \underline{u}) = \tilde{f}$$

$$\partial_t \tilde{u} + p\tilde{\mathcal{K}}\tilde{u} = \tilde{f}$$

NON-LOCAL TRANSPORT EQUATION

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$$\Re[p\tilde{\mathcal{K}}] \geq c_\alpha \geq 0 \Rightarrow |||\Pi_t||| \leq e^{-c_\alpha t}$$

$$\partial_t \tilde{u} + \tilde{\mathcal{K}}(p\tilde{u} - \underline{u}) = \tilde{f}$$

$$u = \underbrace{\underbrace{\Pi_t u^{\text{in}}}_{L^2_{x,\alpha}} + \int_0^t \underbrace{\Pi_{t-t'} f}_{L^1_{loc,t} L^2_{x,\alpha}} dt'}_{u} + \int_0^t \underbrace{\underline{u}(t')}_{L^1_{loc,t}} \Pi_{t-t'} \underbrace{\mathcal{K}}_{L^2_{x,\alpha}} dt'$$

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NON-LOCAL TRANSPORT EQUATION

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$$\Re[p\tilde{\mathcal{K}}] \geq c_\alpha \geq 0 \Rightarrow |||\Pi_t||| \leq e^{-c_\alpha t}$$

$$\partial_t \tilde{u} + \tilde{\mathcal{K}}(p\tilde{u} - \underline{u}) = \tilde{f}$$

$$u = \underbrace{\underbrace{\Pi_t u^{\text{in}}}_{L^2_{x,\alpha}} + \int_0^t \underbrace{\Pi_{t-t'} f}_{L^1_{loc,t} L^2_{x,\alpha}} \, dt'}_{u} + \int_0^t \underbrace{\underline{u}(t')}_{L^1_{loc,t}} \Pi_{t-t'} \underbrace{\mathcal{K}}_{L^2_{x,\alpha}} dt'$$

$$\partial_t \tilde{u} + p\tilde{\mathcal{K}}\tilde{u} = \tilde{f}$$

$$|p\tilde{\mathcal{K}}| = O(1) \Rightarrow u \in C^n_t L^2_{x,\alpha}$$

NON-LOCAL TRANSPORT EQUATION

Caputo / Riemann-Liouville $\partial_x(\mathcal{K} *_x u) = \mathcal{K} *_x \partial_x u + (u|_{x=0})\mathcal{K}$

$$\begin{cases} \partial_t u + \partial_x u = f \\ u(t=0) = u^{\text{in}}(x) \\ u(x=0) = \underline{u}(t) \end{cases}$$

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$$\begin{cases} \partial_t u + \mathcal{K}(x,t) *_x \partial_x u = f \\ u(t=0) = u^{\text{in}}(x) \\ u(x=0) = \underline{u}(t) \end{cases}$$

$$\partial_t \tilde{u} + \tilde{\mathcal{K}}(p\tilde{u} - \underline{u}) = \tilde{f}$$

$$u = \underbrace{\underbrace{\Pi_t u^{\text{in}}}_{L^2_{x,\alpha}} + \int_0^t \underbrace{\Pi_{t-t'} f}_{L^1_{loc,t} L^2_{x,\alpha}} dt'}_{u} + \int_0^t \underbrace{\underline{u}(t')}_{L^1_{loc,t}} \Pi_{t-t'} \underbrace{\mathcal{K}}_{L^2_{x,\alpha}} dt'$$

- C.C 2 : $\partial_t \underline{u} = f_{|x=0}$

$$\begin{cases} \partial_t u' + \partial_x(\mathcal{K}(x,t) *_x u') = f' \\ u'(t=0) = u^{\text{in}'}(x) \end{cases}$$

$$\partial_t \tilde{u}' + p\tilde{\mathcal{K}}\tilde{u}' = \tilde{f}'$$

$$|p\tilde{\mathcal{K}}|_{\Re(p)=\alpha} = O(1) \Rightarrow u \in C^n_t L^2_{x,\alpha}$$

NON-LOCAL TRANSPORT EQUATION

Bessel case $\mathcal{K} = 1/\sqrt{1+\kappa^2 p^2}$

$$\begin{cases} \partial_t q + \partial_x q = 0 \\ u(t=0) = q^{\text{in}}(x) \\ q(x=0) = 0 \end{cases}$$

$$\Pi_t : \quad L^2_{x,\alpha} \to L^2_{x,\alpha} := L^2_x(e^{-\alpha} dx)$$
$$u^{\text{in}} \mapsto \mathcal{L}_x^{-1}\left[exp(-\int_0^t p\tilde{\mathcal{K}}(p,t')dt')\tilde{u}^{\text{in}}\right]$$

- $q^{\text{in}}(x=0) = 0$

$$\Re[p\tilde{\mathcal{K}}] \geq c_\alpha \geq 0 \Rightarrow |||\Pi_t||| \leq e^{-c_\alpha t}$$

$$\begin{cases} \partial_t q + \mathcal{K}(x,t) *_x \partial_x q = 0 \\ q(t=0) = q^{\text{in}}(x) \\ q(x=0) = 0 \end{cases}$$

$$\partial_t \tilde{q} + \tilde{\mathcal{K}}(p\tilde{q} - 0) = 0$$

$$q = \Pi_t q^{\text{in}}$$

$$|p\tilde{\mathcal{K}}|_{\Re(p)=0} = O(\infty) \Rightarrow q \notin C^n_t L^2_{x,0}$$

$$\begin{cases} \partial_t q' + \partial_x(\mathcal{K}(x,t) *_x q') = 0 \\ q'(t=0) = u^{\text{in}'}(x) \end{cases}$$

$$\partial_t \tilde{u'} + p\tilde{\mathcal{K}}\tilde{u'} = 0$$

$$q' = \Pi_t q^{\text{in}'}$$

$$|p\tilde{\mathcal{K}}|_{\Re(p)=\alpha} = O(1) \Rightarrow q \in C^n_t L^2_{x,\alpha}$$