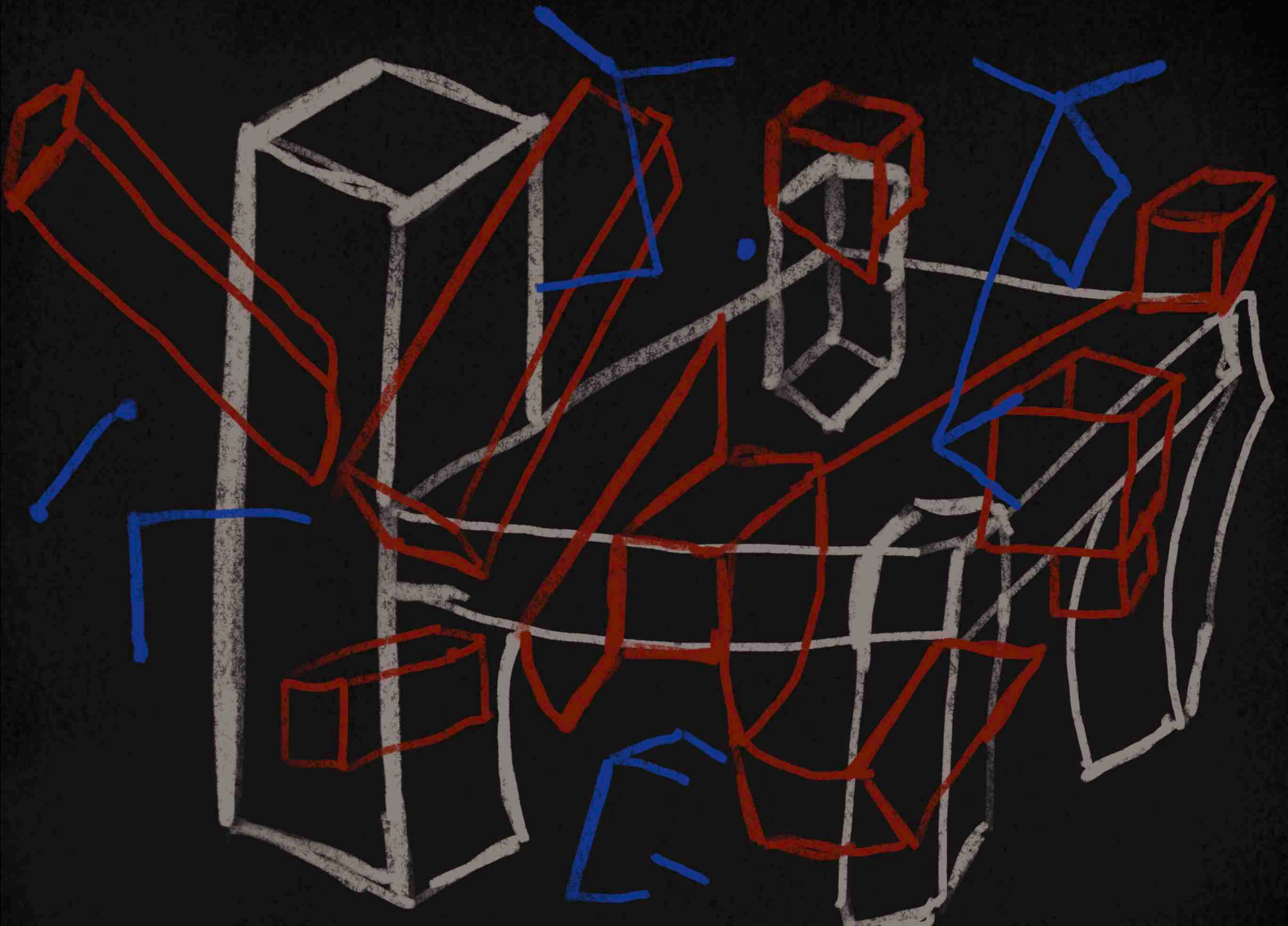


QUANTUM ERGODICITY:

Let M be a compact [neg. curved] manifold, normalized volume μ.

The Laplacian has a discrete spectrum:

$$0 \le \lambda_0 \le \lambda_1 \le \dots, \quad b_n \in L^2(M)$$

$$\Delta b_n = \lambda_n b_n \qquad \|b_n\|_2 = 1$$

Let $\mu_n \in P(M)$ be defined by $\mu_n(A) = \int_A b_n^2 \, d\mu$

"square measure"

Conjecture [QUE] $\mu_n \xrightarrow{*} \mu$.

STILL OPEN

Meaning of QUE: high energy eigenfunctions equidistribute: $\forall$ test function $a : M \to \mathbb{R}$

$$\int a \, d\mu_n \to \int a \, d\mu$$

Quantum Ergodicity: results

T [Shnirelman, Zelditch, de Verdière]

$$\lim_{N\to\infty} \frac{1}{N}\sum_{1}^{N} \left| \int a \, d\mu_n - \int a \, d\mu \right|^2 = 0$$

Corollary: $\mu_n \xrightarrow{*} \mu$ for a density 1 subsequence

T [Lindenstrauss] $\mu_n \to \mu$ if M is arithmetic
and b_n are joint for Hecke + Δ.

T [Anantharaman] lower bound on the entropy
for any $\lim^* \mu_{n_i}$

T [Dyatlov-Jin] $\lim^* \mu_{n_i}$ is fully supported

PROBLEM: b_n are "frozen". No room to move.
We know b_n exists and thats it.
"Physics" would give a mixture of eigenfunctions

Berry's conjecture: "b_n behaves like a Gaussian eigenwave"

NO precise formulation!

This is what prompted the QUE question [Sarnak]

Best guess: look at the value distribution of b_n.

Conj [Hejhal - Rackner] $V_{b_n} \xrightarrow{*}$ normal [Gaussian]

WHAT IS NORMAL?

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

mean μ, variance σ^2

"Unique" attractor of sums of independent things.

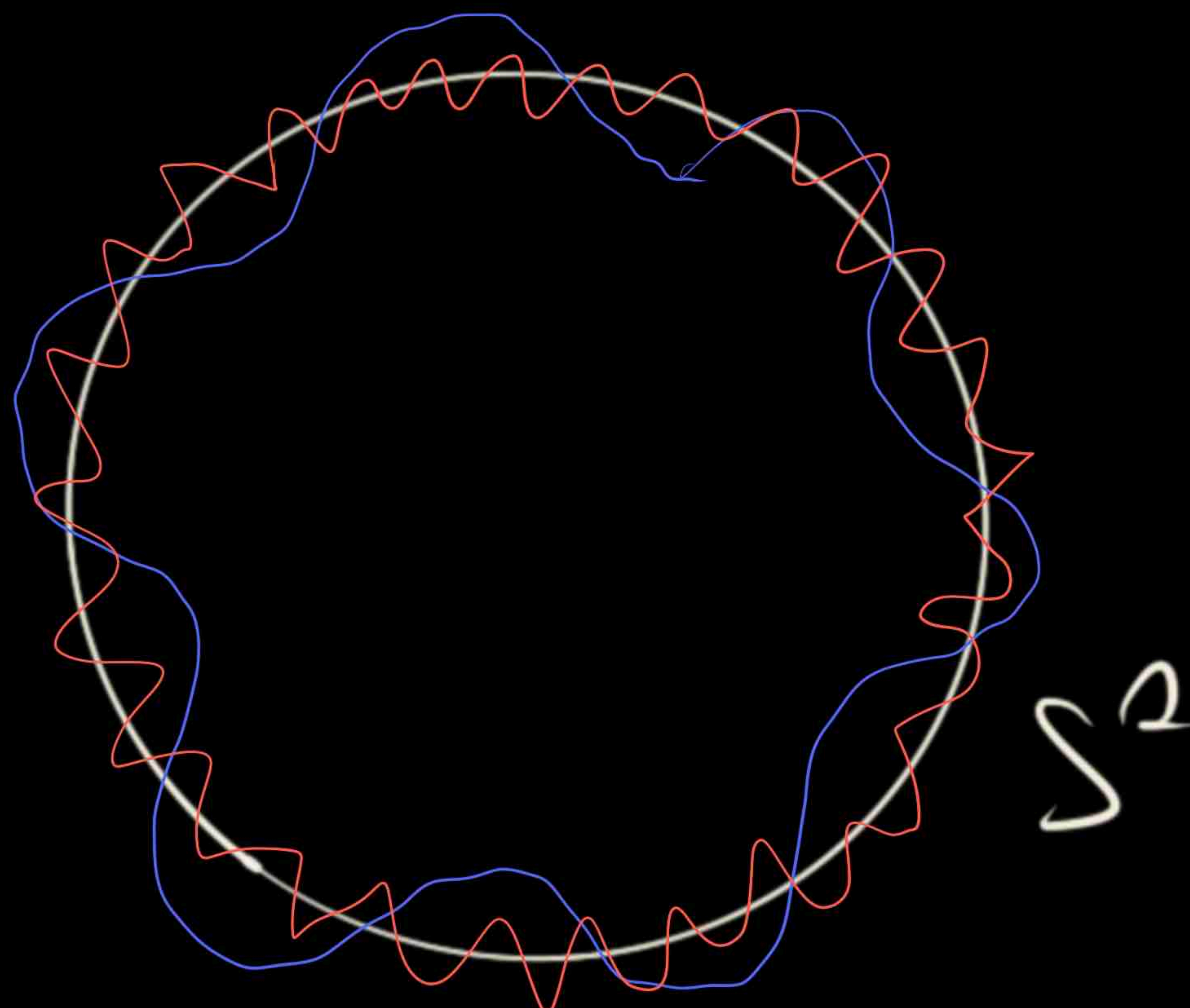
PROBLEM: k-th moment $\int x^k \, dV_{b_n}$ MAY NOT CONVERGE!

IDEA: Take the limit of b_n themselves!

How? $M = S^1$

b_n are cosine waves

They dont converge to a meaningful object!



Hence the square measure idea of Sarnak-Rudnick

SOLUTION: rescale with the wavelength

then take a local sampling

for S^1 this gives the SIN wave.

$\boxed{\text{FRAMEWORK:}}$ Benjamini-Schramm convergence of manifolds [Abert - Biringer]

Space of rooted manifolds $\mathcal{M}$: rooted GH metric

rooted limit can do bad things

M finite volume manifold:

pick the root at random

+ weak* convergence

SAME for (M_n, F_n) where $F_n : M_n \to \mathbb{R}$

What is the limit of manifolds?

It's a URM (unimodular) means, T' is invariant under geodesic flow.

What if $M_n \to X$ X fixed?

seems boring.

Example: M fixed, $M_n = n \cdot M \xrightarrow{BS} \mathbb{R}^d$!

BUT: $(M_n, F_n) \to (X, F)$ then F is NOT fixed but invariant random!

(under the isometry group of X)

So (M_n, F_n) becomes a stationary object!

[Furstenberg for Szemerédi]

CHANGE OF PERSPECTIVE

We had: (M, b_n) b_n is λ_n-eigenfunction

Let $M_n = \lambda_n M$ rescale the Riemannian metric.

Let $F_n = b_n$ same, but of eigenvalue 1!

$\lim (M, b_n)$ makes no sense

$\lim^{BS} (M_n, F_n)$ is an invariant random eigenwave on $\mathbb{R}^d$!

$\overset{\shortparallel}{(\mathbb{R}^d, F)}$

If exists, as lots of things may go wrong.

BERRY'S CONJECTURE IN MATH. FORM

C: Let M be compact, neg. curved, λ_n, b_n as before. Then (M_n, F_n) BS converges to the Gaussian random eigenwave.

Now one can prove or disprove this.

T: Berry implies QUE.

Honest question: QUE is already f. hard. WHY make an even harder conjecture?

I.	II.	III.
JUST CAUSE ITS FUN	MAKE ROOM makes a "frozen" object into a rich dynamical one	DIVIDE AND CONQUER! opens much simpler problems!

III. Let M be a compact neg. curved surface.

A Wigner wave is a BS limit of eigenfunctions

The farthest eigenwave from Gaussian is

SIN : $\sin(x)$ translated and rotated

Problem: Prove that SIN is <u>not</u> a Wigner wave!

Meaning: you cant locally copy and patch sin on M to turn it to an eigenfunction.

STILL HARD!

WHAT IS JOINT GAUSSIAN?

F random function, $x_1, \dots, x_d \in X$

$c_1, \dots, c_n \in \mathbb{R}$

$\sum c_i F(x_i)$ has normal distribution

WHY CARE IF YOU DO DISCRETE GROUPS?

ℓ^2 eigenfunctions don't exist

Gaussians always do!!!

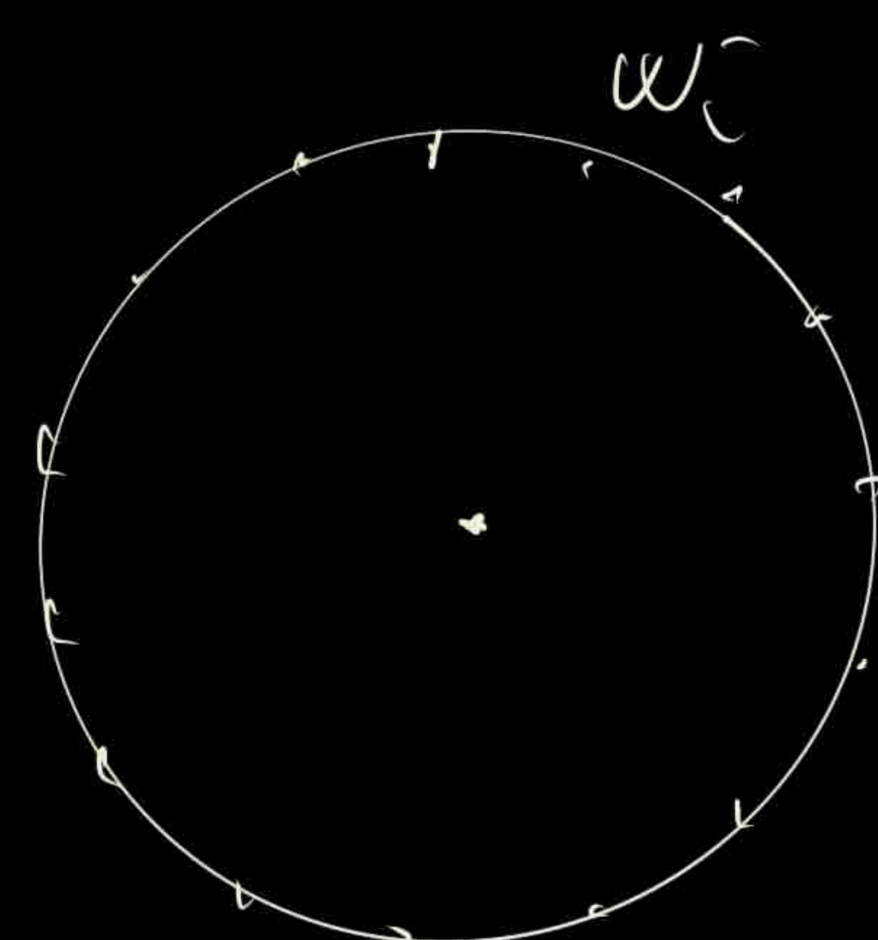
TAKE A MOMENT:

- already 6-th moment can go to hell

  reason: one bad point, maximum norm is too big.

- can already be seen on standard tori

  $\mathbb{R}^d/\mathbb{Z}^d$:



$f_w(z) = \cos\langle z, w\rangle$

all 1 at 0