

Weight monodromy conjecture

X sm proper / \mathbb{F}_p non-arch local field.

$\ell \neq$ residual char of \mathbb{F}_p .

$G_{\mathbb{F}_p} \curvearrowright H^i(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is

quasi-pure of weight i .

§1. Quasi-purity

$$G_{\mathbb{F}_p} \curvearrowright V / \mathbb{Q}_\ell^\rho$$

$\exists \mathbb{F}_{p'}/\mathbb{F}_p$ finite ext s.t. $I_{\mathbb{F}_{p'}}^\rho \curvearrowright V$ unipotent.

$$t \in \prod_{\ell \neq p} \mathbb{Z}_\ell^{(1)} \cong I_{\mathbb{F}_{p'}}^t$$

a top gen

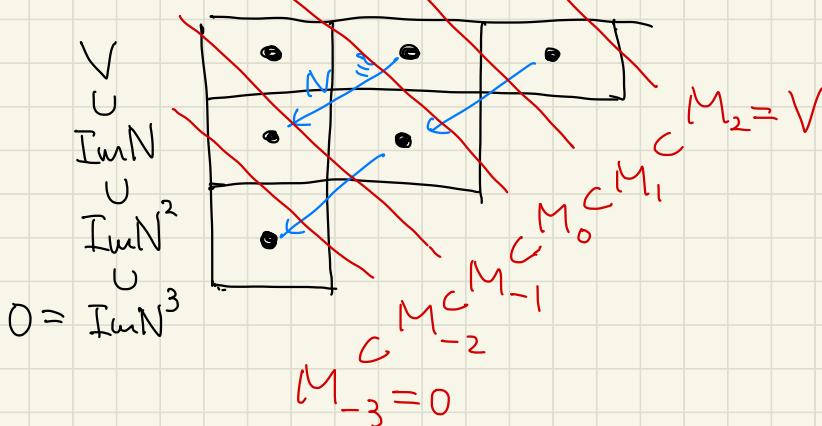
$\rho :=$ res. char.

$$N := \log(\rho(t)) : V \rightarrow V \text{ nilpotent.}$$

\rightsquigarrow monodromy filtr M on V .

Ex. $N^3 = 0$.

$$\text{Ker } N \subset \text{Ker } N^2 \subset \text{Ker } N^3 = V$$



Def. $G_{\mathbb{R}} \cap V$ q.pure of wt w

$$\xrightarrow[\text{def}]{\Leftrightarrow} M_{\bullet} = W_{\bullet+w}$$

$\xrightarrow{\Leftrightarrow} G_{\mathbb{R}}^M V$ pure of wt $w+i$)

Rem WMC is not sensitive to finite ext.

~ By deJong, reduced to X semistable.

Known cases char p, $\dim X \leq 2$. X complete
(Deligne, T.Ito) (Rapoport-Zink) ~~intersection~~
(Scholze).

§2. Formulation for families $\bar{\eta}$

$$f: X \rightarrow Y, X, Y \xleftarrow{ft / \mathbb{Z}_p} \mathcal{O}_B$$

proper, gen sm.
(henselian)

WMC is "local". Y should be local.

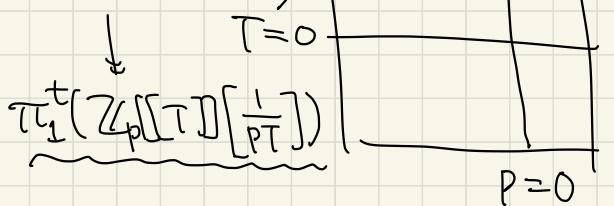
$$Y = \text{Spec } \underline{\mathbb{Z}_p[[T]]}$$

Typical: f is smooth outside $pT=0$.

semistable along $pT=0$

$$\rightsquigarrow H^i(X_{\bar{\eta}}, \mathbb{Q}_l) \cap \pi_1(\underline{\mathbb{Z}_p[[T]]}\left[\frac{1}{pT}\right])$$

unipotent



Abhyankar lemma

$$0 \rightarrow \prod_{\substack{P \\ \ell \neq P}} \mathbb{Z}_{\ell}((1)) \xrightarrow{\oplus 2} \pi_1^t(\mathbb{Z}_p[[T]]\left[\frac{1}{pT}\right]) \rightarrow G_{\mathbb{F}_p} \rightarrow 0.$$

t_p, t_T

Frob

relations?

$$N_p = \log(p(t_p)), N_T = \log(p(t_T))$$

~~Conj~~

$$\begin{array}{c} M^{P=0} \\ \text{Gr}_j \end{array} \quad \begin{array}{c} M^{T=0} \\ \text{Gr}_i \end{array} \quad H^w(X_j, \mathbb{Q}_\ell)$$

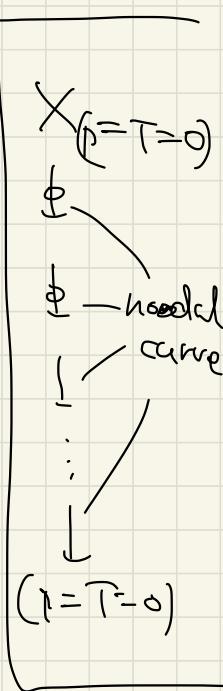
monod filtr
for N_T

$$C N_p$$

pure of wt $w + i + j$.

$$\begin{array}{c} M^{T=0} \\ \text{Gr}_i \end{array} \quad \begin{array}{c} M^{P=0} \\ \text{Gr} \end{array} \quad H^w$$

$$\text{wt } w + i + j.$$



(Rem)

$$H^w \quad M^{P=0}$$

$$\begin{array}{c} M^{T=0} \\ \text{Gr} \end{array} H^w$$

$$\hookrightarrow N_p$$

$$\{$$

induced filtr \neq monod filtr.

$$V \supset F_i$$

$$U \supset U$$

$$W \supset \underline{F_i \cap W} \text{ induced.}$$

Easy observations

over $\mathbb{F}_p[[t]][[T]]$

- i) If $X \rightarrow Y$ comes from varieties/ \mathbb{F}_q .
 then Conj follows Weil II.

$$\exists X_0 \rightarrow Y_0 \text{ sm outside}$$

$$\begin{matrix} U \\ Z_0 \\ SNC \text{ div} \end{matrix}$$

- ii) $X \rightarrow Y$ rel curve,
 (by analyzing the weight spectral seq)

- iii) If $X \rightarrow Y$ rel surf.
 (or complete intersection)

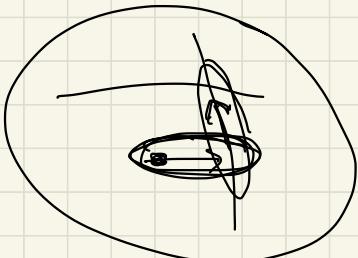
then Conj for $Gr^{T=0} Gr^{P=0}$ holds.

Kosikawa's observation

The WMC follows.

• Conj. for $Gr^{P=0} Gr^{T=0}$ of

families of hyp. surf's.



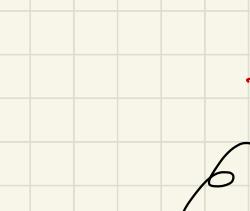
$$\begin{matrix} \mathcal{X} \\ \downarrow \\ A^1_{\mathbb{Q}_p} \end{matrix}$$

$$\begin{matrix} G_{\mathbb{Q}_p} \cong Gr^{T=0} & X \subset \mathbb{P}^N_Y \\ \text{birat.} \\ X' \sim X/\mathbb{Q}_p & \text{sing hypers} \end{matrix}$$

§3. Formulation for general families.

$$\underline{\pi_* (\mathbb{Z}_p[[\tau]] \left[\frac{1}{p\tau} \right]) \supseteq V}$$

does NOT satisfy the Grothendieck monodromy.



$\mathbb{Z}_p[[\tau]] \left[\frac{1}{p\tau} \right]$ not local.

$$\underline{\mathbb{Q}_p((\tau))}$$

ht-2-valuation.

henselian

satisfies

the monodromy thm.

Def. $G_{\mathbb{Q}_p((\tau))} \supseteq V$

lexi.

K

K'

hens val

w val $q_p \cong \mathbb{Z}^2$

is g. pure of wt w

$\iff \exists K'/K \text{ fin. } \xrightarrow{\quad \downarrow \quad} I_K, QV$

unip.

τ, ∂

s.t. top nilp

$$\mathbb{Z}^2 \cong \mathbb{Z}[\tau][\partial]$$

$$Gr_i(Gr_j V)$$

pure of wt

$w+i+j$

Gr V changes by blow-up

Gr Gr V not

$$\begin{array}{ccc} & X & \\ \exists & \nearrow & \downarrow \text{blow-up} \\ \text{Spec } R & \longrightarrow & \text{Spec } \mathbb{Z}_p[[\tau]] \\ | & & \\ \text{rk 2 val ring} & & \\ \text{of } \mathbb{Q}_p((\tau)) & & \end{array}$$

$$\begin{array}{ccc} t & & \\ \text{Gr}_{\mathbb{Q}_p((\tau))} & \longrightarrow & \pi_1(\mathbb{Z}_p[[\tau]][\frac{1}{p\tau}]) \\ \downarrow & & \downarrow \\ \text{Gr}_{\mathbb{Q}_p} & \xrightarrow{\quad \bullet \quad} & \pi_1(\mathbb{Z}_p[[\tau]][\frac{1}{p}]) \\ t_p & & t_p \end{array}$$

