Tropical convexity, mean payoff games and nonarchimedean convex programming

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Based on work with Akian and Guterman (tropical geometry and games) and Allamigeon, Benchimol, Joswig (tropical linear programming) Allamigeon, Skomra (tropical semidefinite programming / nonarchimedean spectraedra).

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- Consequences: log-barrier interior point methods are not strongly polynomial (obstruction to one of the approaches to Smale Problem # 9); game algorithms to solve a class of nonarchimedean semialgebraic convex programming problems.
- Main tool: tropical convexity, and in particular tropical polyhedra.

Part I.
Some complexity issues in convex programming and games

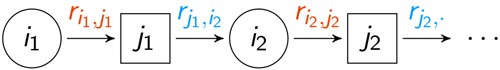
The mean payoff problem

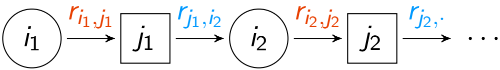
Mean payoff games

$G = (V, E)$ bipartite graph. $r_{ij} \in \mathbb{Z}$ price of the arc $(i, j) \in E$.

MAX and MIN move a token, alternatively (square states: MAX plays; circle states: MIN plays). n MIN nodes, m MAX nodes.

MIN always pays to MAX the price of the arc (having a negative fortune is allowed)

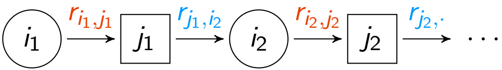

i_1
r_{i_1,j_1}
j_1
r_{j_1,i_2}
i_2
r_{i_2,j_2}
j_2
$r_{j_2,\cdot}$
$\cdots$



Initial position $i_1 := \imath$ given. Player Max wants to maximize his *mean payoff*, lim inf of:

$$\frac{r_{i_1,j_1} + r_{j_1,i_2} + r_{i_2,j_2} + \cdots + r_{j_N,i_{N+1}}}{N} \qquad \text{when } N \to +\infty$$

while Player Min wants to minimize her *mean loss*, the lim sup.



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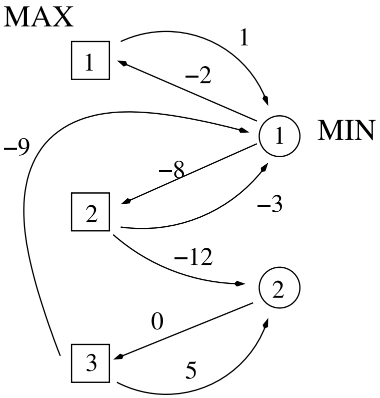
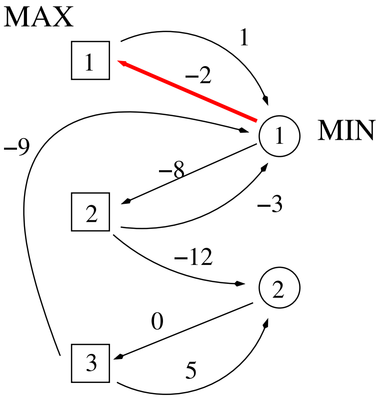
Theorem (Ehrenfeucht and Mycielski, 1979)

There exists a value $\chi_\imath \in \mathbb{R}$, and positional strategies σ and τ of Players Max and Min such that:

- *with strategy σ, the mean payoff of Player Max is at least equal to $\chi_\imath$,*
- *with strategy τ, the mean loss of Player Min does not exceed $\chi_\imath$.*



$$(\chi_1, \chi_2) = (-1, 5)$$



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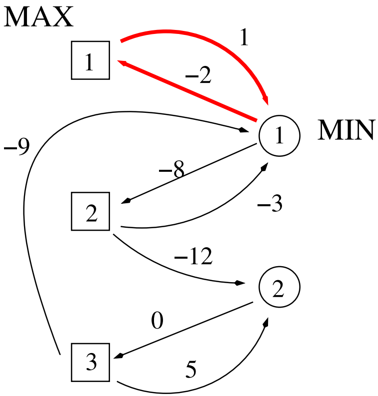

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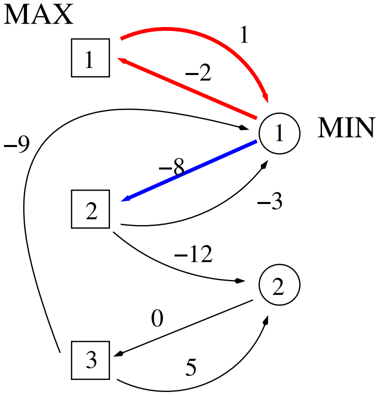

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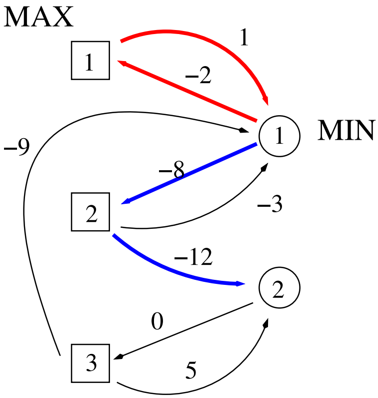

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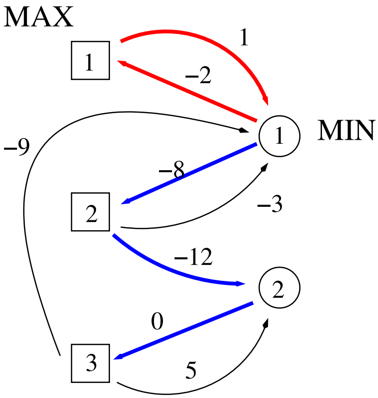

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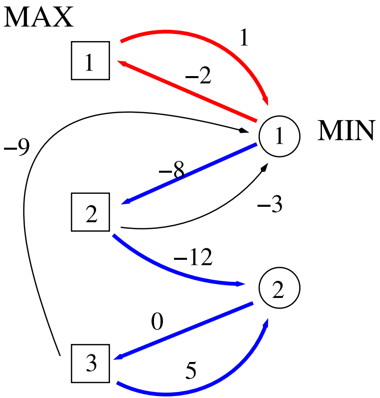

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Problem (Gurvich, Karzanov, Khachyan 88)

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Can we solve mean payoff games in polynomial time?

I.e., time $\leqslant$ poly(L)? where L is the bitlength of the input

$$L = \sum_{ij} \log_2(1 + |r_{ij}|)$$

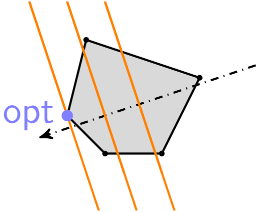
Mean payoff games in NP $\cap$ coNP Zwick and Paterson [1996], not known to be in P.

Linear programming

A linear program is an optimization problem:

$$\min c \cdot x;\ Ax \leqslant b,\ x \in \mathbb{R}^n \ ,$$

where $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$.



Question (9th problem of Smale)

Can we decide whether $\{x \mid Ax \leqslant b\}$ *is empty in strongly polynomial time?*

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$\neq$ strongly polynomial (arithmetic model): number of arithmetic operations bounded by poly(m, n), **and** the size of operands of arithmetic operations is bounded by poly(L).

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What do classical approaches of LP say about this problem?

The simplex method (Dantzig, 1947)

Iterate over adjacent vertices (basic points) of the polyhedron while improving the objective function

$$c^\top v^1 \geqslant c^\top v^2 \geqslant \ldots \geqslant c^\top v^N$$

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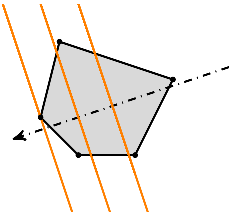

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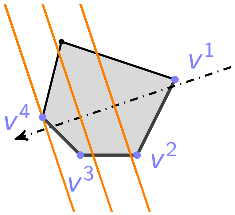

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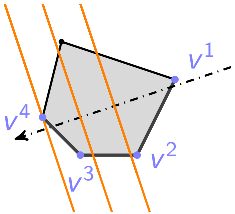

the algorithm is parametrized by a *pivoting rule*, which selects the next edge to be followed.

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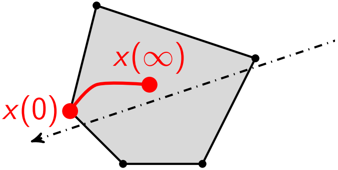
- It is not even known that the graph of the polyhedron has polynomial diameter (polynomial Hirsch conjecture), ie that the perfectly lucid pivoting rule taking the shortest path to the optimum makes a polynomial number of steps.

Interior points

For all $\mu > 0$, consider the *barrier problem*

$$\min c \cdot x - \mu\Big(\sum_{i=1}^{m} \log(b_i - A_i x)\Big), \quad b_i - A_i x > 0 \ i \in [m]$$

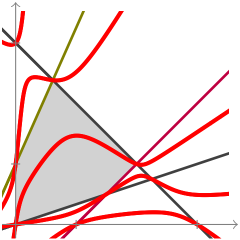
$\mu \mapsto x(\mu)$ optimal solution, is the central path. branch of an algebraic curve. $x(0)$ is the solution of the LP.



"the good convergence properties of Karmarkar's algorithm arise from good geometric properties of the set of trajectories", Bayer, Lagarias 89.

- Interior point methods make a homotopy: move in a suitable neighborhood of the central path, alternating Newton steps and decreasing μ. This leads to weakly polynomial bounds for LP. Is there a strongly polynomial interior point method?
- The total curvature of the central path provides a geometric complexity measure (independent of the technicalities of interior point methods). Dedieu and Shub (2005) conjectured that it is $O(n)$ (n number of variables).

This was motivated by a theorem of Dedieu-Malajovich-Shub (2005): total curvature is $O(n)$, averaged over all 2^{n+m} LP's (cells of the arrangement of hyperplanes), $\epsilon_i A_i x \leqslant b_i$, $\eta_j x_j \geqslant 0$, $\epsilon_i, \eta_j = \pm 1$.



Deza, Terlaky and Zinchenko (2008) constructed a redundant Klee-Minty cube, showing that a total curvature exponential in n is possible, and revised the conjecture of Dedieu and Shub:

Conjecture (Continuous analogue of Hirsch conjecture, [Deza, Terlaky, and Zinchenko, 2008])

The total curvature of the central path is $O(m)$, where m is the number of constraints.

Theorem (Allamigeon, Benchimol, SG, Joswig, MPG is “not more difficult” than LP)

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This leads to a polynomiality result in an average sense. Adler, Karp and Shamir showed that a pivoting algorithm (shadow vertex) takes a polynomial time in average.

Probabilistic model: replace each inequality $A_i x \leqslant b_i$ and $x_j \geqslant 0$ by $\epsilon_i A_i x \leqslant \epsilon_i b_i$ and $\epsilon_j x_j \geqslant 0$ with $\epsilon_i, \epsilon_j \in \{\pm 1\}$, probability $1/2$. This entails that the shadow vertex rule solves mean payoff games in polynomial time in average.

Allamigeon, Benchimol, SG, ICALP 2014)

Theorem (Allamigeon, Benchimol, SG, Joswig, SIAM. J. Appl. Alg. Geom. 2018)

- *There is a LP with $2r+2$ variables and $3r+4$ inequalities such that the central path has a total curvature in $\Omega(2^r)$. (Disproves DTZ).*

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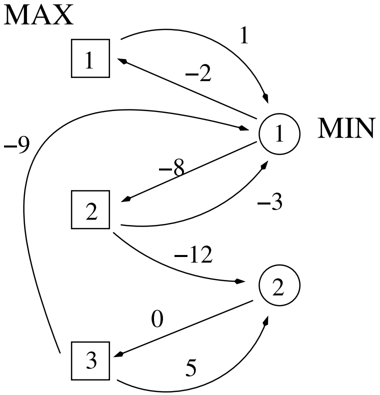
- *There is a LP with* $2r + 2$ *variables and* $3r + 4$ *inequalities such that the central path has a total curvature in* $\Omega(2^r)$. *(Disproves DTZ).*
- *Moreover, on this LP, log-barrier interior point methods make* $\Omega(2^r)$ *iterations.*

Although the word “tropical” appears in none of these statements, the proofs rely on tropical geometry in an essential way, through linear programming over non-archimedean fields, and tropical modules / convex cones.

Part II.
Operator approach to mean payoff games

v_i^k value of the game in horizon k and initial state (i, MIN).

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



$$v^1 = (0, 0)$$
$$v^2 = (-11, 5)$$
$$v^3 = (-15, 10)$$
$$v^4 = (-16, 15)$$
$$\chi = \lim_{k\to\infty} v^k/k = (-1, 5)$$

Theorem (Shapley)

$$v^k = T(v^{k-1}), \qquad v^0 = 0 \ .$$

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$$[T(x)]_j = \min_{i\in[m],\ j\to i} \left(r_{ji} + \max_{k\in[n],\ i\to k} (r_{ik} + x_k) \right)$$

Definition

An abstract Shapley operator is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that T is monotone (or order preserving)

$$(M) : \qquad x \leqslant y \implies T(x) \leqslant T(y)$$

and additively homogeneous

$$(AH) : \qquad T(se + x) = se + T(x), \qquad \forall s \in \mathbb{R}$$

where $e = (1, \ldots, 1)$ is the n-dimensional unit vector.

This entails that T is sup-norm nonexpansive:

$$\|T(x) - T(y)\|_\infty \leqslant \|x - y\|_\infty$$

Known axioms in non-linear potential theory / game theory / PDE viscosity solutions theory, e.g. Crandall and Tartar, PAMS 80.

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General example of Shapley operator $T:\mathbb{R}^n\to\mathbb{R}^n$,

$$T_i(x)=\inf_{a\in A}\sup_{b\in B}\big(r_i^{ab}+\sum_{j\in[n]}P_{ij}^{ab}x_j\big)$$

where $P_{ij}^{ab}\geqslant 0$, $\sum_j P_{ij}^{ab}=1$.

T is the one day operator of a repeated game, in which MIN selects a, MAX selects b, MIN pays r_i^{ab} in state i, and next state becomes j with probability P_{ij}^{ab}.

$[T^k(0)]_i$ is the value of the standard game in horizon k, starting from state i.

$[T^k(u)]_i$ is the value of a modified game, in which MAX receives an additional payment of u_j in the terminal state j.

We allow the inf and sup not to commute, this is the 'turn based' situation, MIN plays first, MAX plays next, and each player is informed of the previous action of the other player. In the original example of Shapley (1953), $T_i(x)=\inf_{\mu\in\Delta(A)}\sup_{\nu\in\Delta(B)}\int d\mu(a)d\nu(b)(r_i^{ab}+\sum_{j\in[n]}P_{ij}^{ab}x_j)$, where $\Delta(\cdot)$ denotes the set of probability measures on a space, i.e. players choose measures on actions rather than actions. This models the situations in which MAX and MIN play simultaneously. This reduces to the general example, replacing A by $\Delta(A)$ and B by $\Delta(B)$. More generally, every Shapley operator can be written as in the general example (Kolokoltsov 92), even with deterministic transitions, allowing infinite A (Rubinov, Singer 01, Sparrow, and Gunawardena 04).

Theorem (Bewley, Kohlerg 76, Neyman 03)

The mean payoff vector

$$\lim_{k\to\infty} T^k(0)/k$$

does exist if $T:\mathbb{R}^n\to\mathbb{R}^n$ *is semi-algebraic and nonexpansive in any norm.*

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Finite action space and perfect information implies T piecewise linear.

Semi-algebraic is needed to deal with finite action space, player playing simultaneously (incomplete information) - Shapley's original example.

This result still holds if T is definable in a o-minimal structure, and nonexpansive, e.g., log-exp type, entropy games... Bolte, SG, Vigeral, MOR 14.

Winning certificates

Theorem ("subharmonic vectors" Akian, SG, Guterman, IJAC 2012)

Let T be the Shapley operator of a deterministic mean payoff game. The following are equivalent.

A Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ always extends continuously $(\mathbb{R} \cup \{-\infty\})^n \to (\mathbb{R} \cup \{-\infty\})^n$, the topology of $\mathbb{R} \cup \{-\infty\}$ being given by the metric $d(x, y) = |e^x - e^y|$ (Burbanks, Nussbaum, Sparrow)

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- *there exists $u \in (\mathbb{R} \cup \{-\infty\})^n$, $u_j \neq -\infty$, and*

$$u \leqslant T(u)$$

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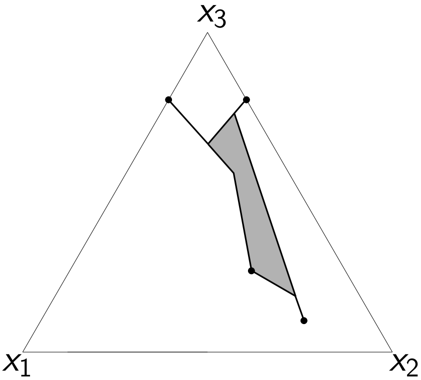
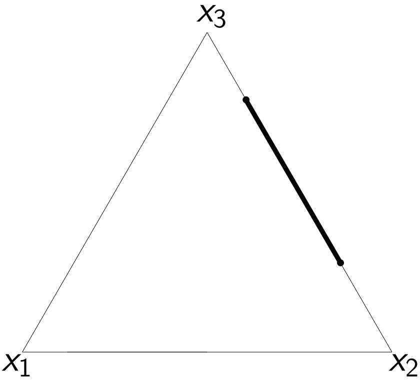
Space of subharmonic vectors



states 1,2,3 winning



states 2,3 winning

Theorem (subharmonic vectors, cont.)

For an arbitrary Shapley operator T (infinite action space and stochastic transitions allowed), the game has one initial state winning for MAX, i.e., $\exists j, \liminf_{k\to\infty}[T^k(0)]_j/k \geqslant 0$, iff there exists $v \in (\mathbb{R}\cup\{-\infty\})^n, v \not\equiv (-\infty,\dots,-\infty)$, such that

$$v \leqslant T(v)$$

For stochastic games, the support of $v \in V$ is a winning dominion of MAX (set of winning states of MAX a.s. invariant under a strategy of MAX). Not all winning states are in winning dominions. Allamigeon, SG, Skomra, JSC 2017

Part III.
Tropical modules / convex cones

Tropical semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

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For some duality results, embed $\mathbb{R}_{\max}$ in the complete semiring $\overline{\mathbb{R}}_{\max} := \mathbb{R} \cup \{\pm\infty\}$ (set $-\infty + (+\infty) = -\infty$ for “0” $:= -\infty$ to be aborbing).

We shall also need $\mathbb{R}_{\min} := \mathbb{R} \cup \{+\infty\}$, and $\overline{\mathbb{R}}_{\min} := \mathbb{R} \cup \{\pm\infty\}$, equipped with min as addition, instead of max.

Exemples of tropical modules over $\mathbb{R}_{\max}$

Scalars act on vectors by "λx" $= \lambda e + x$.

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$\mathbb{R}^n_{\max}$: free $\mathbb{R}_{\max}$-module, $V \subset \mathbb{R}^n_{\max}$ is a submodule, aka tropical convex cone, if for all $x, y \in V, \lambda, \mu \in \mathbb{R}_{\max}$,

$$\text{"}\lambda x + \mu y\text{"} = \sup(\lambda e + x, \mu e + y) \in V \quad .$$

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V is a tropical convex set if the same is true conditionnally to "$\lambda + \mu = 1$", i.e., $\max(\lambda, \mu) = 0$.

Proposition ("subharmonic vectors")

$V \subset \mathbb{R}_{\max}^n$ is a closed $\mathbb{R}_{\max}$-submodule iff there is a Shapley operator $T : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ such that $V = \{v \in \mathbb{R}_{\max}^n \mid v \leqslant T(v)\}$.

"Only if", take $T = P_V$, where P_V is the operator of best approximation:

$$P_V(x) = \max\{v \in V \mid v \leqslant x\} \ .$$

The max belongs to the set since V stable by the sup of two vectors, and closed

Tropical adjoints

Let $A \in \mathbb{R}_{\max}^{m\times n}, x \in \mathbb{R}_{\max}^{n},\ y \in \mathbb{R}_{\max}^{n}$

$$(Ax)_i = \max_{j\in[n]}(A_{ij} + x_j), \qquad i \in [m]$$

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$$Ax \leqslant y \iff x \leqslant A^\sharp y$$

$$(A^\sharp y)_j = \min_{i\in[m]}(-A_{ij} + y_i), \quad j \in [n]$$

The adjoint $A^\sharp$ is a priori defined as a self-map of the order completion $\overline{\mathbb{R}}_{\max} := (\mathbb{R} \cup \{\pm\infty\})^n$ of $\mathbb{R}_{\max}^n$, but it does preserve $\mathbb{R}^n$ as soon as the game has no states without actions. More on adjoints: Cohen, SG, Quadrat, LAA 04

The Shapley operator of a MPG can be written as

$$[T(v)]_j = \min_{i\in[m],\ j\to i} \left(-A_{ij} + \max_{k\in[n],\ i\to k} (B_{ik} + v_k) \right)$$

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The sets of subharmonic certificates $\{v \mid Av \leqslant Bv\}$ is a tropical polyhedral cone, i.e. a set defined as the intersection of finitely many linear inequalities.

Let's see how tropical polyhedral cones look like...

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

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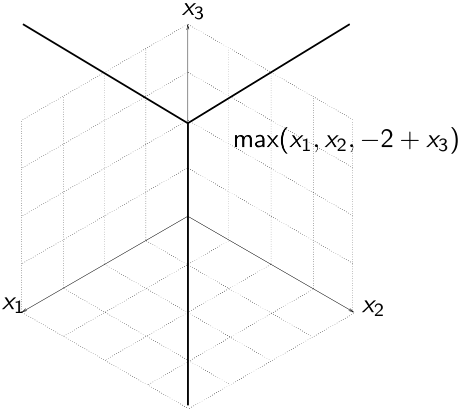

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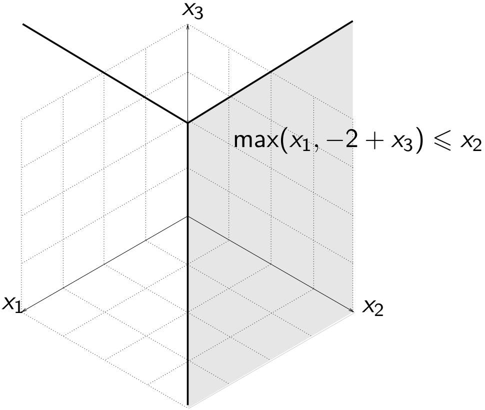

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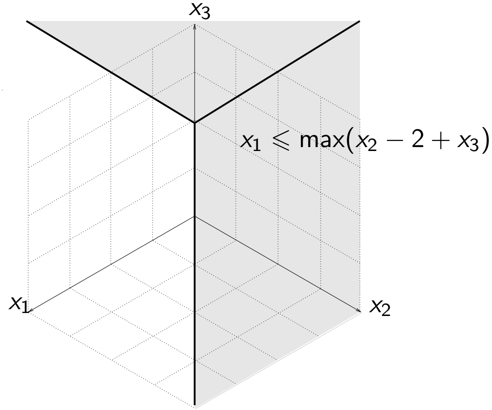

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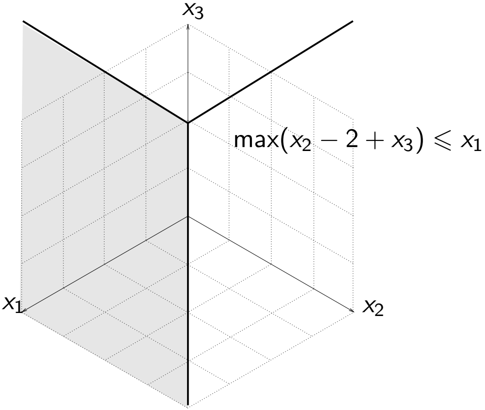
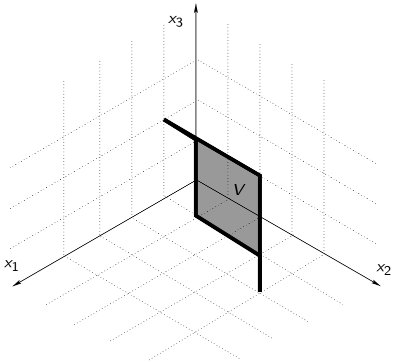


Tropical polyhedral cones

can be defined equivalently either as intersections of finitely many half-spaces or as finitely generated submodules of $\mathbb{R}^n_{\max}$.



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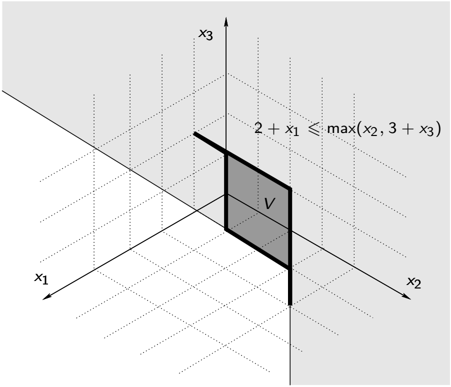

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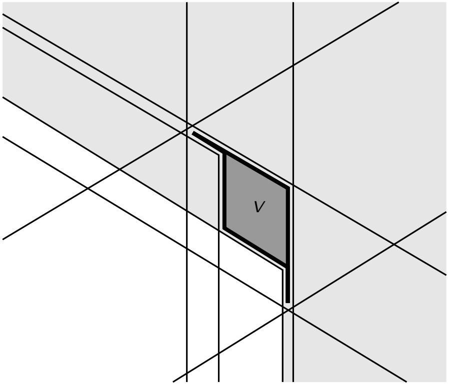
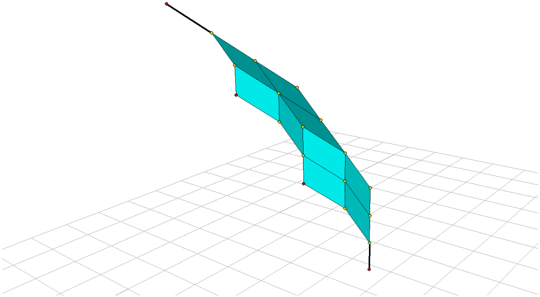


A tropical polytope with four vertices



Structure of a polyhedral complex (Develin, Sturmfels) whose cells C are alcoved polyhedra (Postnikov):

$$C := \{x \in \mathbb{R}^n \mid x_i - x_j \leqslant a_{ij}, \forall i, j\} \ , \text{ for some } a_{ij} \in \mathbb{R} \cup \{-\infty\}$$

Part IV.
Link between nonarchimedean and tropical convexity

Let $\mathbb{K}$ be a real closed field with a nonarchimedean valuation having $\mathbb{R}$ has the value group.

E.g., generalized Puiseux series:

$$\boldsymbol{x} = \boldsymbol{x}(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} ,$$

where the sequence $(\alpha_i)_i \subset \mathbb{R}$ is strictly decreasing and either finite or unbounded and c_i are real.

Can take either formal series (Markwig), or rather the subfield series absolutely converging for t large enough (van den Dries and Speissegger), then:

$$\mathrm{val}(\boldsymbol{x}) = \lim_{t\to\infty} \frac{\log|\boldsymbol{x}(t)|}{\log t} = \alpha_1 \qquad (\text{and } \mathrm{val}(0) = -\infty) .$$

A $\mathcal{S} \subset \mathbb{K}^n$ is basic semialgebraic if

$$\mathcal{S} = \{(x_1, \ldots, x_n) \in \mathbb{K}^n \colon P_i(x_1, \ldots, x_n) \diamond 0, \ \diamond \in \{>, =\}, \forall i \in [q]\}$$

where $P_1, \ldots, P_q \in \mathbb{K}[x_1, \ldots, x_n]$. A semialgebraic set is a finite union of basic semialgebraic sets.

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A set $S \subset \mathbb{R}^n$ is basic semilinear if it is of the form

$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon \ell_i(x_1, \ldots, x_n) \diamond h^{(i)}, \ \diamond \in \{>, =\}, \forall i \in [q]\}$$

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Theorem (Alessandrini, Adv. in Geom. 2013)

If $\mathcal{S} \subset \mathbb{K}^n_{>0}$ is semi-algebraic, then $\mathrm{val}(\mathcal{S}) \subset \mathbb{R}^n$ is semilinear and it is closed.

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A constructive version follows from Denef-Pas quantifier elimination in valued fields, see Allamigeon, SG, Skomra, DCG 2020. See also Jell, Scheiderer and Yu arXiv:1810:05132.

Theorem (Semi-algebraic version of "Kapranov theorem")

Consider a collection of m regions delimited by hypersurfaces:

$$\boldsymbol{\mathcal{S}}_i := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} \mid \boldsymbol{P}_i^-(\boldsymbol{x}) \leqslant \boldsymbol{P}_i^+(\boldsymbol{x})\}, \qquad i \in [m]$$

where $\boldsymbol{P}_i^{\pm} = \sum_\alpha \boldsymbol{p}_{i,\alpha}^{\pm} \boldsymbol{x}^\alpha \in \mathbb{K}_{\geqslant 0}[x]$, *and let*

$$S_i := \{x \in \mathbb{R}^n \mid \max_\alpha(\operatorname{val} p_{i,\alpha}^- + \langle \alpha, x\rangle) \leqslant \max_\alpha(\operatorname{val} p_{i,\alpha}^+ + \langle \alpha, x\rangle)\}$$

Then

$$\operatorname{val}(\bigcap_{i\in[m]} \boldsymbol{\mathcal{S}}_i) \subset \bigcap_{i\in[m]} \operatorname{val}(\boldsymbol{\mathcal{S}}_i) \subset \bigcap_{i\in[m]} S_i$$

and *the equality holds if* $\bigcap_{i\in[m]} S_i$ *is the closure of its interior; in particular if the valuations* val $\boldsymbol{p}_{i,\alpha}^{\pm}$ *are generic.*

See Allamigeon, SG, Skomra DCG2020

Example 1.

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{K}^3_{>0} \mid \boldsymbol{x}_1^2 \leqslant t\boldsymbol{x}_2 + t^4\boldsymbol{x}_2\boldsymbol{x}_3\}$$

$$\operatorname{val}\mathcal{S} = \{x \in \mathbb{R}^3 \mid 2x_1 \leqslant \max(1 + x_2, 4 + x_2 + x_3)\}$$

Example 2.

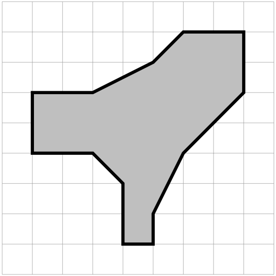

Figure: This set is the closure of its interior.

Correspondence: convex semialgebraic sets $\to$ stochastic mean payoff games

Theorem (Allamigeon, SG, Skomra, coro of JSC 2018 + DCG 2020)

Let $C \subset \mathbb{R}^n$. TFAE:

- *C is the image by* val *of a convex semialgebraic cone in $\mathbb{K}^n_{>0}$;*
- *C is a closed tropical convex cone and it is semilinear;*
- *$C = \{v \in \mathbb{R}^n \mid v \leqslant T(v)\}$, where T is a Shapley operator of a stochastic turn based zero-sum game with finite action spaces and rational transition probabilities*

$$T_i(x) = \inf_{a \in A} \sup_{b \in B} \big(r_i^{ab} + \sum_{j \in [n]} P_{ij}^{ab} x_j \big)$$

(A, B finite, $P_{ij}^{ab} \in \mathbb{Q}$).

A spectrahedron in $\mathbb{K}^n$ is a set of the form

$$\mathcal{S} = \{x \in \mathbb{K}^n \colon Q^{(0)} + x_1 Q^{(1)} + \cdots + x_n Q^{(n)} \text{ is positive semidefinite}\} .$$

where $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{K}^{m \times m}$ are given symmetric matrices.
A tropical spectrahedron is defined as the image by the valuation of a spectrahedron included in $\mathbb{K}^n_{\geqslant 0}$.
The heart of the correspondence lies in the following special case.
Recall a matrix is Metzler if its off diagonal entries are nonpositive.

Theorem (tropical Metzler spectrahedra)

For Metzler matrices $(\boldsymbol{Q}^{(k)})_k$ with generic valuations, the set $\operatorname{val}(\boldsymbol{\mathcal{S}})$ is described by the tropical minor inequalities of order 1 and 2,

$$\forall i,\ \max_{\boldsymbol{Q}_{ii}^{(k)} > 0} (x_k + \operatorname{val}(\boldsymbol{Q}_{ii}^{(k)})) \geqslant \max_{\boldsymbol{Q}_{jj}^{(l)} < 0} (x_l + \operatorname{val}(\boldsymbol{Q}_{jj}^{(l)}))$$

$$\forall i \neq j,\ \max_{\boldsymbol{Q}_{ii}^{(k)} > 0} (x_k + \operatorname{val}(\boldsymbol{Q}_{ii}^{(k)})) + \max_{\boldsymbol{Q}_{jj}^{(k)} > 0} (x_k + \operatorname{val}(\boldsymbol{Q}_{jj}^{(k)})) \geqslant 2 \max_{\boldsymbol{Q}_{ij}^{(l)} < 0} (x_l + \operatorname{val}(\boldsymbol{Q}_{ij}^{(l)})) .$$

- Nemirovski asked whether / Helton-Nie conjectured that every convex semi-algebraic subset of $\mathbb{R}^n$ is the projection of a spectrahedron in some higher dimensional space $\mathbb{R}^p$.
- Scheiderer (SIAG 2018) disproved the conjecture, by showing that the cone of nonnegative forms of degree $2d$ in n variables is not a projection a spectrahedron, unless $n \leqslant 2$, $2d = 2$, or $(n, 2d) = (3, 4)$. The same is true over $\mathbb{K}$ (by completeness of real closed fields).
- The correspondence shows that the tropicalizations (images by the valuation) of convex semialgebraic sets and of spectrahedra coincide.

Special case of polyhedra

Theorem

1. *Every tropical polyhedron P can be written as $P = \operatorname{val}\mathcal{P}$ where $\mathcal{P}$ is a polyhedron in $\mathbb{K}^n_{\geqslant 0}$.*
2. *Moreover, P is the uniform (Hausdorff) limit of*

$$\log_t \mathcal{P} := \{\frac{\log z}{\log t} \mid z \in \mathcal{P}\}$$

as $t \to \infty$.

Part 1 was proved by Develin and Yu. Part 2 in Allamigeon, Benchimol, SG, Joswig, SIAGA 2018. Related result in Briec and Horvath.

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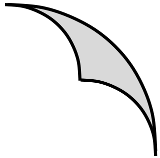

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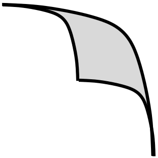

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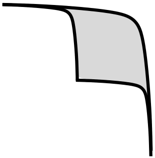

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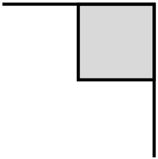

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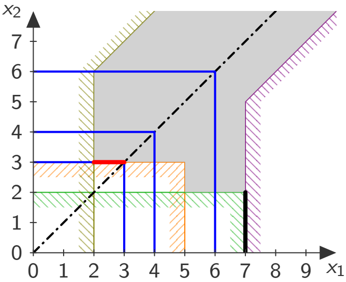
Tropical linear program

$$\min \text{``}c^\top x\text{''}; \qquad \text{``}A^+x + b^+ \geqslant A^-x + b^-\text{''}$$

$$\min \max_j c_j + x_j$$

$$\max(\max_j(A^+_{ij} + x_j), b^+_i) \geqslant \max(\max_j(A^-_{ij} + x_j), b^-_i) \ .$$



Correspondence classical $\leftrightarrow$ tropical LP

Theorem (Allamigeon, Benchimol, SG, Joswig, SIAM J. Disc. Math)

Suppose that $\mathcal{P} = \{x \in \mathbb{K}^n \mid \boldsymbol{A}x + \boldsymbol{b} \geqslant 0\}$ *is included in the positive orthant of* $\mathbb{K}^n$ *and that the tropicalization of* $(\boldsymbol{A}, \boldsymbol{b})$ *is sign generic (to be defined soon). Then,*

$$\mathrm{val}(\mathcal{P}) = \{x \in \mathbb{R}^n_{\max} \mid \text{“}A^+x + b^+ \geqslant A^-x + b^-\text{”}\} \ ,$$

where $(A^+ \ b^+) = \mathrm{val}(\boldsymbol{A}^+\boldsymbol{b}^+)$ *and* $(A^- \ b^-) = \mathrm{val}(\boldsymbol{A}^- \ \boldsymbol{b}^-)$. *Moreover the classical and tropical polyhedron have the same combinatorics: valuation sends basic points to basic points, edges to edges, etc.*

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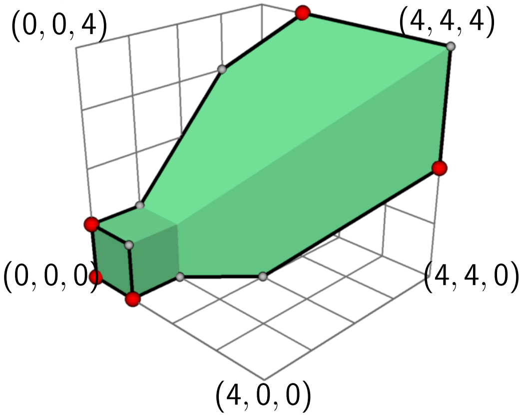
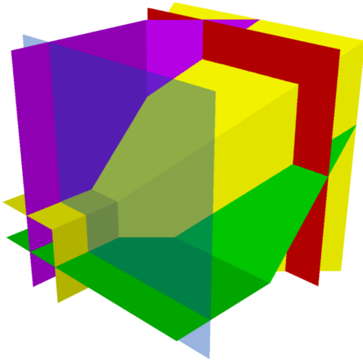
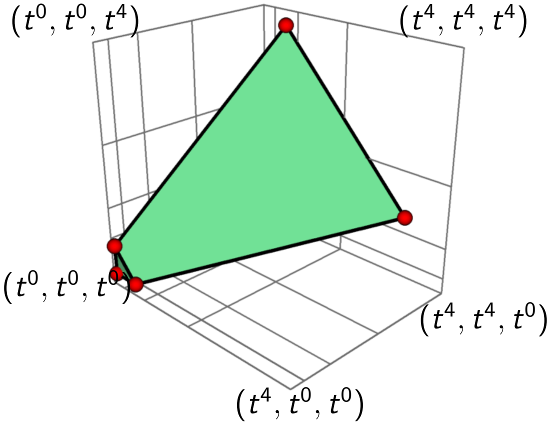
tropically extreme point implies basic, but not vice versa


(0, 0, 4)
(4, 4, 4)
(0, 0, 0)
(4, 4, 0)
(4, 0, 0)




(t^0, t^0, t^4)
(t^4, t^4, t^4)
(t^0, t^0, t^0)
(t^4, t^4, t^0)
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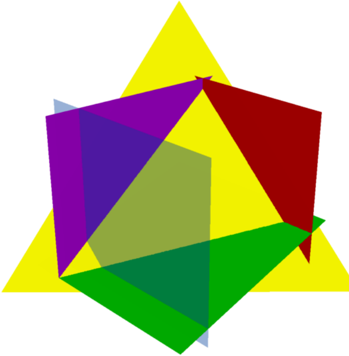


Sign genericity is controlled by tropical determinants

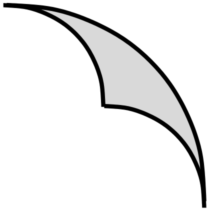
$A \in \mathbb{R}_{\max}^{n\times n}$

$$\det A = \text{“}\sum_{\sigma\in S_n} \epsilon(\sigma)\prod_i A_{i\sigma(i)}\text{”}$$
$$= \max_{\sigma\in S_n} \sum_i A_{i\sigma(i)}$$

A is sign generic if all the maximizing permutations have the same parity.

Sign-genericity is related to the even cycle problem and Polya's permanent problem. Checkable in polynomial time Robertson, Seymour, Thomas

sign generic condition not satisfied, valuation does not commute with the external representation.



$$x_1 + x_2 \leqslant 1, \qquad t^1 x_1 + x_2 \geqslant 1, \qquad x_1 + t^1 x_2 \geqslant 1$$

$$\max(X_1, X_2) \leqslant 0,\ \max(1 + X_1, X_2) \geqslant 0,\ \max(X_1, 1 + X_2) \geqslant 0 \ .$$

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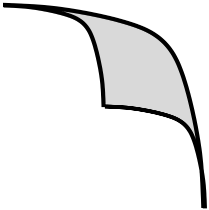

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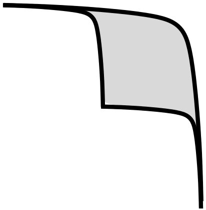

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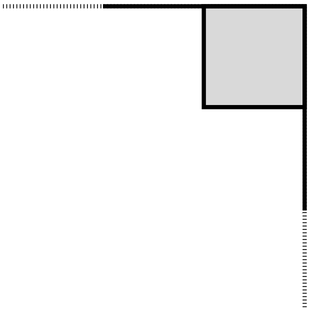

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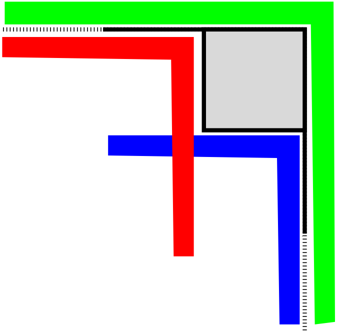

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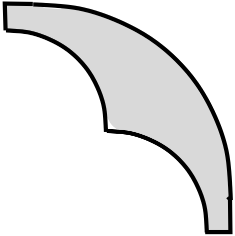

$$x_1 + x_2 \leqslant t^{\epsilon},\ t^1 x_1 + x_2 \geqslant 1,\ x_1 + t^1 x_2 \geqslant 1,\ t^2 x_1 \geqslant 1,\ t^2 x_2 \geqslant 1$$

$X_i = \log x_i / \log t, \qquad t \to \infty.$

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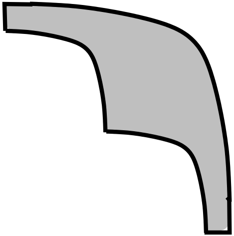

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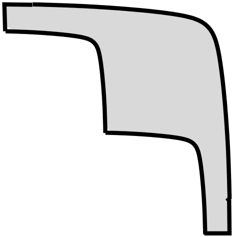

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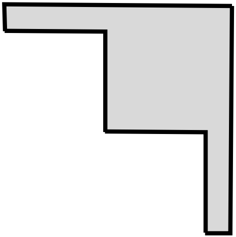

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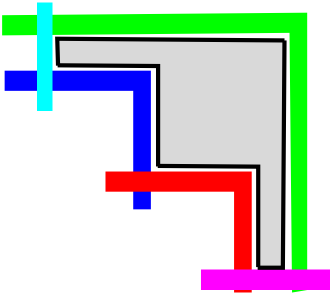

Mean payoff games are easier than combinatorial non-archimedean linear programming

Corollary (Allamigeon, Benchimol, SG, Joswig, SIAM J. Opt)

If any combinatorial (or even "semialgebraic") rule in classical linear programming would run in strongly polynomial time, then, mean payoff games could be solved in (strongly) polynomial time.

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- This pivoting rule applied to LP($\mathbb{K}$) is implemented tropically (e.g., solve assignment problems). QED

Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

$$M = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^\top & 0 \end{pmatrix} .$$

(eg signs of reduced costs).
Most known pivoting rules are combinatorial.

Part V.
Tropicalization of the central path

Primal-dual central path

$$\begin{aligned}
\text{minimize} \quad & \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\
\text{subject to} \quad & Ax + w = b,\ x > 0, w > 0.
\end{aligned} \tag{1}$$

$$\begin{aligned}
Ax + w &= b \\
-A^\top y + s &= c \\
w_i y_i &= \mu \quad \text{for all } i \in [m] \\
x_j s_j &= \mu \quad \text{for all } j \in [n] \\
x, w, y, s &> 0 \ .
\end{aligned} \tag{2}$$

For any $\mu > 0$, $\exists!$ $(x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. The central path is the image of the map $\mathcal{C} : \mathbb{R}_{>0} \to \mathbb{R}^{2m+2n}$ which sends $\mu > 0$ to the vector $(x^\mu, w^\mu, y^\mu, s^\mu)$.

The tropical central path

Assume now that $\boldsymbol{A}(t), \boldsymbol{b}(t), \boldsymbol{c}(t)$ have entries in $\mathbb{K}$ (absolutely converging Puiseux series with real exponents, $t \to \infty$).

The tropical central path is the image by the valuation of the central path. It is the log-limit, taking the parameter $\mu := t^\lambda$,

$$\mathcal{C}^{\mathrm{trop}} : \lambda \mapsto \lim_{t \to \infty} \frac{\log \mathcal{C}(t^\lambda)}{\log t} \ . \tag{3}$$

$\mathcal{C}^{\mathrm{trop}}$ can be computed by combinatorial means.

barycenter of a (compact) tropical polyhedron $\mathcal{P}$

$=$ greatest point of the set $\mathcal{P}$ w.r.t. the coordinate-wise order $\leqslant$

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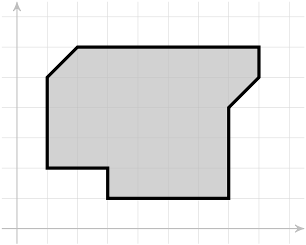

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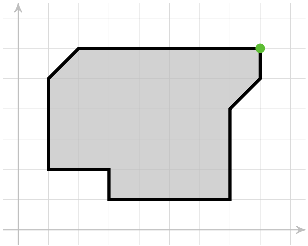

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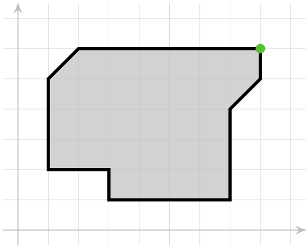

Let $\boldsymbol{\mathcal{P}}$ be the feasible set of $\mathbf{LP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$:

$$\boldsymbol{\mathcal{P}} \coloneqq \{(\boldsymbol{x}, \boldsymbol{w}) \in \mathbb{K}^{n+m}_{\geqslant 0} \colon \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w} = \boldsymbol{b}\} \, .$$

Assume, for simplicity, $\boldsymbol{b}, \boldsymbol{c} \geqslant 0$.

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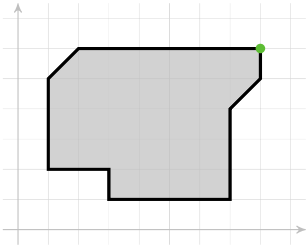

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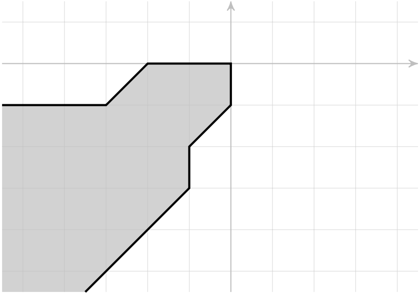
Theorem

The image under val *of the point* $(\boldsymbol{x}^{\boldsymbol{\mu}}, \boldsymbol{w}^{\boldsymbol{\mu}})$ *of the primal central path is given by the barycenter of the tropical polyhedron:*

$$\mathrm{val}(\boldsymbol{\mathcal{P}}) \cap \{(x, w) \in \mathbb{R}_{\max}^{n+m} \colon \mathrm{val}(\boldsymbol{c})^{\top} \odot x \leqslant \mathrm{val}(\boldsymbol{\mu})\} .$$

$$\mathcal{P} : \begin{cases} \boldsymbol{x}_1 + \boldsymbol{x}_2 \leqslant 2 \\ t\boldsymbol{x}_1 \leqslant 1 + t^2 \boldsymbol{x}_2 \\ t\boldsymbol{x}_2 \leqslant 1 + t^3 \boldsymbol{x}_1 \\ \boldsymbol{x}_1 \leqslant t^2 \boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 \geqslant 0 \end{cases} \qquad \mathrm{val}(\mathcal{P}) : \begin{cases} \max(x_1, x_2) \leqslant 0 \\ 1 + x_1 \leqslant \max(0, 2 + x_2) \\ 1 + x_2 \leqslant \max(0, 3 + x_1) \\ x_1 \leqslant 2 + x_2 \end{cases}$$



$$\text{minimize} \quad \boldsymbol{x}_1 + t^3 \boldsymbol{x}_2 \qquad\qquad \max(x_1, 3 + x_2)$$

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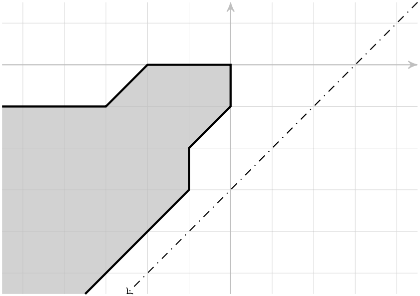

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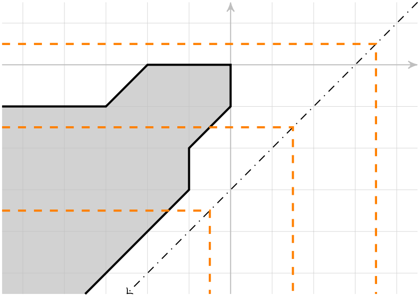

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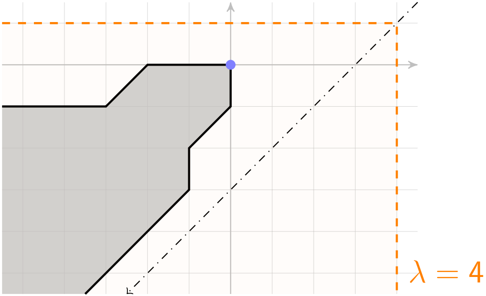

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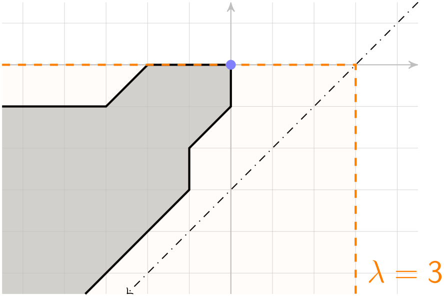

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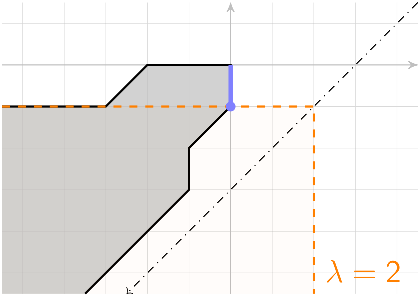

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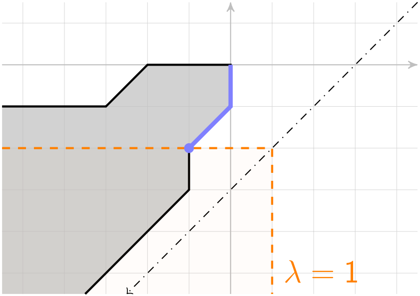

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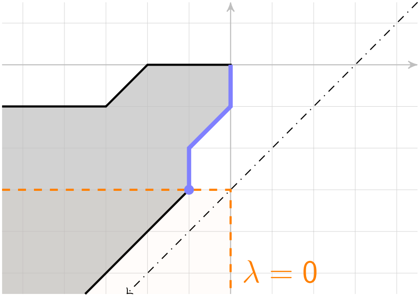

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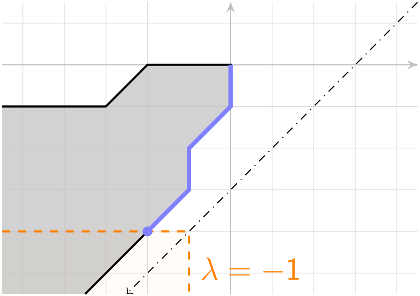
$\max(x_1, 3 + x_2) \leqslant \lambda$

$$\mathrm{val}(\mathcal{P}): \begin{cases} \max(x_1, x_2) \leqslant 0 \\ 1 + x_1 \leqslant \max(0, 2 + x_2) \\ 1 + x_2 \leqslant \max(0, 3 + x_1) \\ x_1 \leqslant 2 + x_2 \end{cases}$$



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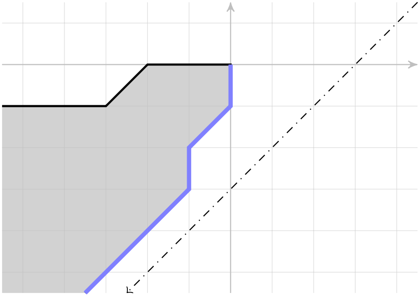


The counter example . . .

$$
\begin{aligned}
\min \quad & \boldsymbol{v}_0 \\
\text{s.t.} \quad & \boldsymbol{u}_0 \leqslant t^1 \\
& \boldsymbol{v}_0 \leqslant t^2 \\
& \boldsymbol{v}_i \leqslant t^{(1-\frac{1}{2^i})}(\boldsymbol{u}_{i-1} + \boldsymbol{v}_{i-1}) && \text{for } 1 \leqslant i \leqslant r \qquad \textbf{LW}_r \\
& \boldsymbol{u}_i \leqslant t^1 \boldsymbol{u}_{i-1} && \text{for } 1 \leqslant i \leqslant r \\
& \boldsymbol{u}_i \leqslant t^1 \boldsymbol{v}_{i-1} && \text{for } 1 \leqslant i \leqslant r \\
& \boldsymbol{u}_r \geqslant 0, \; \boldsymbol{v}_r \geqslant 0
\end{aligned}
$$

Theorem (Allamigeon, Benchimol, SG, Joswig SIAGA 2018)

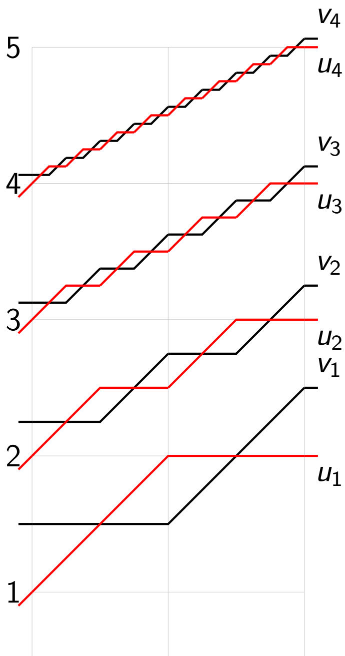
For t large enough, the total curvature of the central path is $\geqslant (2^{r-1}-1)\pi/2$.

Large enough: $\log_2 t = \Omega(2^r)$.

$$\begin{aligned}
\boldsymbol{u}_0 &\leqslant t^1 & u_0 &\leqslant 1\\
\boldsymbol{v}_0 &\leqslant t^2 & v_0 &\leqslant 2\\
\boldsymbol{v}_i &\leqslant t^{(1-\frac{1}{2^i})}(\boldsymbol{u}_{i-1}+\boldsymbol{v}_{i-1}) & v_i &\leqslant 1-\frac{1}{2^i}+\max(u_{i-1},v_{i-1})\\
\boldsymbol{u}_i &\leqslant t^1\boldsymbol{u}_{i-1} & u_i &\leqslant 1+u_{i-1}\\
\boldsymbol{u}_i &\leqslant t^1\boldsymbol{v}_{i-1} & u_i &\leqslant 1+v_{i-1}\\
\boldsymbol{u}_r &\geqslant 0,\ \boldsymbol{v}_r \geqslant 0 & c^\top x = v_0 &\leqslant \lambda
\end{aligned}$$

The tropical central path is given by

$$\begin{aligned}
u_0 &= 1\\
v_0 &= \min(2,\lambda)\\
v_i &= 1-\frac{1}{2^i}+\max(u_{i-1},v_{i-1})\\
u_i &= 1+\min(u_{i-1},v_{i-1})
\end{aligned}$$


v_4
5
u_4
v_3
4
u_3
v_2
3
u_2
v_1
2
u_1
1

Corollary (Interior point methods are not strongly polynomial, Allamigeon, Benchimol, SG, Joswig SIAGA 2018)

Suppose that

$$t > \left(\frac{\left((10r-1)!\right)^8}{1-\theta} \right)^{2^{r+2}} .$$

Then, any log-barrier interior point method which stays in a wide neighborhood of the primal-dual central path of $\mathbf{LW}_r(t)$ *needs to perform at least* 2^{r-1} *iterations to reduce the duality measure from* t^2 *to* 1.

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Wide neighborhood:

$$\mathcal{N}_t^{-\infty}(\mu) := \left\{(x,w,s,y) \in \boldsymbol{\mathcal{P}}(t) \times \boldsymbol{\mathcal{Q}}(t) \colon \bar{\mu}(x,w,s,y) = \mu \text{ and } \left(\begin{smallmatrix} xs \\ wy \end{smallmatrix}\right) \geqslant (1-\theta)\mu e\right\}$$

$$\bar{\mu}(x,w,s,y) := \frac{1}{m+n}\left(x^\top s + w^\top y\right)$$

and $\theta \in \,]0,1[$ parametrizes the size of the neighborhood.

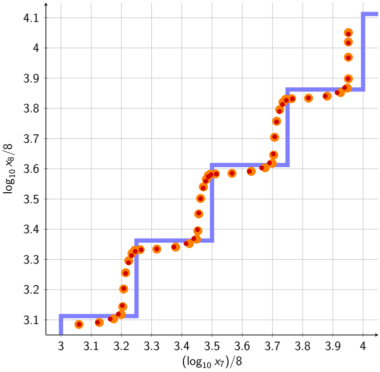

Figure: The successive iterations performed by the predictor-corrector interior point method of Mizuno et al. [1993] on the linear program $\mathbf{LW}_4(t)$ to reduce the duality measure μ from t^2 to 1, when t is equal to 10^8 (left). The points depict the projection of the iterations on the last two coordinates (u_4, v_4) in logarithmic scale (where the logarithm is taken in base t). Orange and red points respectively correspond to prediction and corrections steps. The tropical central path is depicted in blue.

Concluding remarks

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- Game algorithms can be used to solve non-archimedean instances, and real instances with a large parameter.

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