On Multiple zeta values and their q-analogues

Based on joint work with J. Castillo-Medina, K. Ebrahimi-Fard, S. Paycha, J. Singer, J. Zhao

Dominique Manchon
LMBP, CNRS-Université Clermont-Auvergne

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Multiple zeta values are given by the following iterated series:

$$\zeta(n_1, \ldots, n_k) = \sum_{m_1 > \cdots > m_k > 0} \frac{1}{m_1^{n_1} \cdots m_k^{n_k}}. \tag{1}$$

- The n_j's are positive integers.
- The series converges provided $n_1 \geq 2$. It makes also sense for $n_1, \ldots, n_k \in \mathbb{Z}$ provided:

$$n_1 + \cdots + n_j > j \text{ for any } j \in \{1, \ldots, k\}. \tag{2}$$

- The integer k is the **depth**, the sum $w := n_1 + \cdots + n_k$ is the **weight**.

- Appear in the work of **L. Euler** in depths 1 and 2 (18th century),
- Some examples in higher depths by **N. Nielsen** (1904),
- Appear first as a whole in **J. Ecalle**, Les fonctions résurgentes vol. 2 (1981).

The **Multiple zeta function** is given by the same iterated series:

$$\zeta(z_1,\ldots,z_k) = \sum_{m_1>\cdots>m_k>0} \frac{1}{m_1^{z_1}\cdots m_k^{z_k}}, \tag{3}$$

where the z_j's are complex numbers.

Theorem (S. Akiyama, S. Egami, Y. Tanigawa, 2001)

The series (3) *converges provided:*

$$\mathrm{Re}(z_1+\cdots+z_j) > j \text{ for any } j\in\{1,\ldots,k\}. \tag{4}$$

It defines a holomorphic function of k complex variables in this domain, which can be meromorphically extended to $\mathbb{C}^k$. The subvariety of singularities in given by:

$$\begin{aligned} \mathcal{S}_k = \Big\{(z_1,\ldots,z_k)\in\mathbb{C}^k,\ z_1 &= 1 \text{ or} \\ z_1+z_2 &\in \{2,1,0,-2,-4,\ldots\} \text{ or} \\ \exists j\in\{3,\ldots,k\},\ z_1+\cdots+z_j &\in \mathbb{Z}_{\le j}\Big\}. \end{aligned}$$

Quasi-shuffle relations

The product of two MZVs is a linear combination of MZVs!
For example:

$$\begin{aligned}\zeta(n_1)\zeta(n_2) &= \sum_{m_1>m_2>0}\frac{1}{m_1^{n_1}m_2^{n_2}}+\sum_{m_2>m_1>0}\frac{1}{m_1^{n_1}m_2^{n_2}}+\sum_{m_1=m_2>0}\frac{1}{m_1^{n_1}m_2^{n_2}}\\ &= \zeta(n_1,n_2)+\zeta(n_2,n_1)+\zeta(n_1+n_2).\end{aligned}$$

The most general quasi-shuffle relation displays as follows:

$$\zeta(n_1,\ldots,n_p)\zeta(n_{p+1},\ldots,n_{p+q})=\sum_{r\geq 0}\sum_{\sigma\in \mathrm{qsh}(p,q;r)}\zeta(n_1^\sigma,\ldots,n_{p+q-r}^\sigma).$$

- Here $\mathrm{qsh}(p,q;r)$ stands for (p,q)**-quasi-shuffles of type** r. They are surjections

$$\sigma:\{1,\ldots,p+q\}\twoheadrightarrow\{1,\ldots,p+q-r\}$$

  subject to $\sigma_1<\cdots<\sigma_p$ and $\sigma_{p+1}<\cdots<\sigma_{p+q}$.
- n_j^σ stands for the **sum** of the n_r's for $\sigma(r)=j$.
- The sum above contains only one or two terms.

Integral representation and shuffle relations

MZVs have an iterated integral representation (**M. Kontsevich, D. Zagier**):

$$\zeta(n_1,\ldots,n_k)=\int_{0\le t_w\le\cdots\le t_1\le 1}\frac{dt_1}{t_1}\cdots\frac{dt_{n_1-1}}{t_{n_1-1}}\frac{dt_{n_1}}{1-t_{n_1}}\cdots\frac{dt_{n_1+\cdots+n_{k-1}+1}}{t_{n_1+\cdots+n_{k-1}+1}}\cdots\frac{dt_{w-1}}{t_{w-1}}\frac{dt_w}{1-t_w}$$

As a consequence, there is a second way to express the product of two MZVs as a linear combination of MZVs: the **shuffle relations**.

Example:

$$\begin{aligned}\zeta(2)\zeta(2) &= \int_{\substack{0\le t_2\le t_1\le 1\\ 0\le t_4\le t_3\le 1}}\frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\\ &= 4\zeta(3,1)+2\zeta(2,2).\end{aligned}$$

Regularization relations

A third group of relations can be deduced from a natural extension of the preceding ones: the **regularization relations**. The simplest one is:

$$\zeta(2,1) = \zeta(3),$$

obtained as follows:

$$\begin{aligned}\zeta(1)\zeta(2) &= \zeta(1,2) + \zeta(2,1) + \zeta(3) \\ &= \zeta(1,2) + 2\zeta(2,1).\end{aligned}$$

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These three groups of relations constitute the so-called **double shuffle relations**.

It is conjectured that no other relations occur among multiple zeta values. For example the **duality relations**, coming from $t_j \mapsto 1 - t_j$ in the integral representation, should be deducible from DS relations.

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$$\zeta(2,1) = \zeta(3), \quad \zeta(2,1,1) = \zeta(4), \ldots, \quad \zeta(2,1^{\{n-2\}}) = \zeta(n) \ldots$$

Multiple polylogarithms (in one variable)

For any $t \in [0,1]$,

$$\mathrm{Li}_{n_1,\ldots,n_k}(t) := \int_{0\le t_w \le \cdots \le t_1 \le t} \frac{dt_1}{t_1} \cdots \frac{dt_{n_1-1}}{t_{n_1-1}} \frac{dt_{n_1}}{1-t_{n_1}} \cdots \frac{dt_{n_1+\cdots+n_{k-1}+1}}{t_{n_1+\cdots+n_{k-1}+1}} \cdots \frac{dt_{w-1}}{t_{w-1}} \frac{dt_w}{1-t_w}$$

$$= \sum_{m_1>\cdots>m_k>0} \frac{t^{m_1}}{m_1^{n_1}\cdots m_k^{n_k}}.$$

$$x(t) := \frac{1}{t}, \qquad y(t) := \frac{1}{1-t}.$$

Three operators on the space of continuous maps $f : [0,1] \to \mathbb{R}$:

$$\begin{aligned} X[f](t) &:= x(t)f(t), \\ Y[f](t) &:= y(t)f(t), \\ R[f](t) &:= \int_0^t f(u)\,du. \end{aligned}$$

$\Rightarrow$ **Concise expression** of the multiple polylogarithm:

$$\mathrm{Li}_{n_1,\ldots,n_k} = (R \circ X)^{n_1-1} \circ (R \circ Y) \circ \cdots \circ (R \circ X)^{n_k-1} \circ (R \circ Y)[\mathbf{1}].$$

R is a **weight zero Rota-Baxter operator**:

$$R[f]R[g] = R\big[R[f]g + fR[g]\big].$$

We have of course for any positive integers $n_1, \ldots n_k$ with $n_1 \geq 2$:

$$\mathrm{Li}_{n_1,\ldots,n_k}(1) = \zeta(n_1, \ldots, n_k).$$

Word description of the quasi-shuffle relations

- Introduce the infinite alphabet $Y := \{y_1, y_2, y_3, \ldots\}$.
- Y^* is the set of words with letters in Y.
- $\mathbb{Q}\langle Y\rangle$ is the linear span of Y^* on $\mathbb{Q}$.
- **Quasi-shuffle product** on $\mathbb{Q}\langle Y\rangle$:

$$u_1 \cdots u_p \,⧢\!\!\!\!\!+\, u_{p+1} \cdots u_{p+q} := \sum_{r\geq 0} \sum_{\sigma\in \mathrm{qsh}(p,q;r)} u_1^\sigma \cdots u_{p+q-r}^\sigma,$$

where u_j^σ is the **internal product** of the u_r's with $\sigma(r) = j$. The internal product is given by $y_i \diamond y_j = y_{i+j}$. For later use, the **shuffle product** is defined by:

$$u_1 \cdots u_p ⧢ u_{p+1} \cdots u_{p+q} := \sum_{\sigma\in \mathrm{qsh}(p,q;0)} u_1^\sigma \cdots u_{p+q}^\sigma.$$

Example

$$\begin{aligned} y_2 \,\text{⧢}\!\!\!\!\!\!+\; y_3y_1 &= y_2y_3y_1 + y_3y_2y_1 + y_3y_1y_2 + y_5y_1 + y_3y_3, \\ y_2 \,\text{⧢}\, y_3y_1 &= y_2y_3y_1 + y_3y_2y_1 + y_3y_1y_2. \end{aligned}$$

- Notation: $Y^*_{\mathrm{conv}} := Y^* \setminus y_1 Y^*$.
- For any word $y_{n_1} \cdots y_{n_k}$ in Y^*_{conv} we set:

$$\zeta_{⧢}(y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).$$

- Extend $\zeta_{⧢}$ linearly.

- Notation: $Y^*_{\mathrm{conv}} := Y^* \setminus y_1 Y^*$.
- For any word $y_{n_1} \cdots y_{n_k}$ in Y^*_{conv} we set:

$$\zeta_{\uplus}(y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).$$

- Extend $\zeta_{\uplus}$ linearly.
- The quasi-shuffle relations are rewritten as follows: for any $u, v \in Y^*_{\mathrm{conv}}$,

$$\zeta_{\uplus}(u)\zeta_{\uplus}(v) = \zeta_{\uplus}(u \uplus v). \tag{5}$$

- Notation: $Y^*_{\mathrm{conv}} := Y^* \setminus y_1 Y^*$.
- For any word $y_{n_1} \cdots y_{n_k}$ in Y^*_{conv} we set:

$$\zeta_{\ast}(y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).$$

- Extend $\zeta_{\ast}$ linearly.
- The quasi-shuffle relations are rewritten as follows: for any $u, v \in Y^*_{\mathrm{conv}}$,

$$\zeta_{\ast}(u)\zeta_{\ast}(v) = \zeta_{\ast}(u \ast v). \tag{5}$$

- **example:**

$$\begin{aligned}\zeta_{\ast}(y_2)\zeta_{\ast}(y_3y_1) &= \zeta_{\ast}(y_2 \ast y_3y_1)\\ &= \zeta_{\ast}(y_2y_3y_1 + y_3y_2y_1 + y_3y_1y_2 + y_5y_1 + y_3y_3),\end{aligned}$$

hence:

$$\zeta(2)\zeta(3,1) = \zeta(2,3,1) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(5,1) + \zeta(3,3).$$

Extension to arguments of any sign

The quasi-shuffle product obviously extends to $\mathbb{Q}\langle Z\rangle$, where Z is the infinite alphabet $\{y_j, j \in \mathbb{Z}\}$.

Theorem (S. Paycha-DM, 2010)

There exists a character

$$\varphi : (\mathbb{Q}\langle Z\rangle, ⧢_{\!+}) \longrightarrow \mathbb{C} \tag{6}$$

such that

- $\varphi(v) = \zeta_{⧢_{+}}(v)$ *for any* $v \in Y^*_{\mathrm{conv}}$.
- *For any* $v = y_{n_1} \cdots y_{n_k} \in Z^*$ *such that* $\zeta(n_1, \ldots n_k)$ *can be defined by analytic continuation, then* $\varphi(v) = \zeta(n_1, \ldots n_k)$*. In particular,*
  - $\varphi(-n) = \zeta(-n) = -\dfrac{B_{n+1}}{n+1}$ *for any* $n \in \mathbb{Z}_+$.
  - $\varphi(-n, -n') = \zeta(-n, -n') = \dfrac{1}{2}(1 + \delta_0^{n'})\dfrac{B_{n+n'+1}}{n+n'+1}$ *for any* $n, n' \in \mathbb{Z}_+$ *with* $n + n'$ *odd.*

$\zeta(-a,-b)$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=0$	$\frac{3}{8}$	$\frac{1}{12}$	$\frac{7}{720}$	$-\frac{1}{120}$	$-\frac{11}{2\,520}$	$\frac{1}{252}$	$\frac{1}{224}$
$b=1$	$\frac{1}{24}$	$\frac{1}{288}$	$-\frac{1}{240}$	$-\frac{19}{10\,080}$	$\frac{1}{504}$	$\frac{41}{20\,160}$	$-\frac{1}{480}$
$b=2$	$-\frac{7}{720}$	$-\frac{1}{240}$	0	$\frac{1}{504}$	$\frac{113}{151\,200}$	$-\frac{1}{480}$	$-\frac{307}{166\,320}$
$b=3$	$-\frac{1}{240}$	$\frac{1}{840}$	$\frac{1}{504}$	$\frac{1}{28\,800}$	$-\frac{1}{480}$	$-\frac{281}{332\,640}$	$\frac{1}{264}$
$b=4$	$\frac{11}{2\,520}$	$\frac{1}{504}$	$-\frac{113}{151\,200}$	$-\frac{1}{480}$	0	$\frac{1}{264}$	$\frac{117\,977}{75\,675\,600}$
$b=5$	$\frac{1}{504}$	$-\frac{103}{60\,480}$	$-\frac{1}{480}$	$\frac{1}{1232}$	$\frac{1}{264}$	$\frac{1}{127\,008}$	$-\frac{691}{65\,520}$
$b=6$	$-\frac{1}{224}$	$-\frac{1}{480}$	$\frac{307}{166\,320}$	$\frac{1}{264}$	$-\frac{117\,977}{75\,675\,600}$	$-\frac{691}{65\,520}$	0

Sketch of proof: through **regularisation** and **renormalisation.**

- $\mathcal{H} := (\mathbb{Q}\langle Z\rangle, \uplus, \Delta)$ is a connected filtered Hopf algebra, where Δ stands for deconcatenation:

$$\Delta(y_{n_1}\cdots y_{n_k}) = \sum_{j=0}^{k} y_{n_1}\cdots y_{n_j}\otimes y_{n_{j+1}}\cdots y_{n_k}.$$

- $\overline{\mathcal{H}} := (\mathbb{Q}\langle \mathcal{C}\rangle, \uplus, \Delta)$ where:

$$\mathcal{C} := \{y_t,\, t\in\mathbb{C}\}.$$

- $\overline{\mathcal{H}}_{\sqcup\!\sqcup} := (\mathbb{Q}\langle \mathcal{C}\rangle, \sqcup\!\sqcup, \Delta)$, $\qquad \mathcal{H}_{\sqcup\!\sqcup} := (\mathbb{Q}\langle Z\rangle, \sqcup\!\sqcup, \Delta)$.

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- $\mathcal{R} : \overline{\mathcal{H}}_{\sqcup\!\sqcup} \to \mathrm{Maps}(\mathbb{C}, \overline{\mathcal{H}}_{\sqcup\!\sqcup})$ defined below respects $\sqcup\!\sqcup$.

$$\mathcal{R}(y_{t_1}\cdots y_{t_k}) := y_{t_1-z}\cdots y_{t_k-z}.$$

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$$\widetilde{\mathcal{R}}(y_{t_1} \cdots y_{t_1}) := \exp_H \circ \mathcal{R} \circ \log_H$$

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respects $\sqcup\!\!\!+\!\!\!\sqcup$. The character

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is defined by $\Phi = \zeta_{\sqcup\!\!+\!\!\sqcup} \circ \widetilde{\mathcal{R}}_{|_{\mathcal{H}}}$.

Then use **Birkhoff-Connes-Kreimer decomposition**:

$$\Phi = \Phi_-^{*-1} * \Phi_+,$$

where $*$ is the convolution product: $\alpha * \beta = m \circ (\alpha \otimes \beta) \circ \Delta$.

- Φ_- and Φ_+ are still characters of $(\mathbb{Q}\langle Z\rangle, ⧢)$,
- $\Phi_-(v) \in z^{-1}\mathbb{C}[z^{-1}]$ for any nonempty word v.
- $\Phi_+(v)$ is holomorphic at $z = 0$ for any word v.
- $\Phi_-^{*-1} = \Phi_- \circ S$, i.e. the inverse is given by composition on the right with the antipode.

$\Phi_-(v)$ and $\Phi_+(v)$ are given by explicit recursive formulas wrt the length of the word v (BPHZ algorithm): the commutative algebra $\mathrm{Mero}(\mathbb{C})$ splits into two subalgebras:

$$\mathrm{Mero}(\mathbb{C}) = \mathcal{A}_- \oplus \mathcal{A}_+,$$

where $\mathcal{A}_- = z^{-1}\mathbb{C}[z^{-1}]$ and $\mathcal{A}_+$ is the subalgebra of meromorphic functions which do not have a pole at $z=0$ (**minimal subtraction scheme**). Let π be the extraction of the pole part, i.e. the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. Then:

$$\begin{aligned}\Phi_-(w) &= -\pi\Big(\Phi(w) + \sum_{(w)} \Phi_-(w')\Phi(w'')\Big),\\ \Phi_+(w) &= (I-\pi)\Big(\Phi(w) + \sum_{(w)} \Phi_-(w')\Phi(w'')\Big).\end{aligned}$$

Definition:

$$\varphi(v) := \Phi_+(v)(z)_{|z=0}. \tag{8}$$

Now we want to describe **all** solutions to the problem, i.e. describe the set of all characters of $(\mathbb{Q}\langle Z\rangle, \text{⧢}_{\pm})$ which extend multiple zeta functions in the sense described above.

The renormalisation group

Let $\mathcal{H}$ be any commutative connected filtered Hopf algebra, over some base field k. Let $\mathcal{A}$ be any commutative unital k-algebra, and let $G_{\mathcal{A}}$ be the group of characters of $\mathcal{H}$ with values in $\mathcal{A}$. The product in $G_{\mathcal{A}}$ is given by convolution. The coproduct is *conilpotent*, i.e.

$$\Delta(x) = \mathbf{1} \otimes x + x \otimes \mathbf{1} + \widetilde{\Delta}(x),$$

were $\widetilde{\Delta}(x) = \sum_{(x)} x' \otimes x''$ is the reduced coproduct, and $\widetilde{\Delta}^{(k)}(x) = 0$ for $k \geq |x|$.

Proposition

Let N be a right coideal with respect to the reduced coproduct, i.e. $\widetilde{\Delta}(N) \subset N \otimes \mathcal{H}$ *and* $\varepsilon(N) = \{0\}$*. The set*

$$T_{\mathcal{A}} := \{\alpha \in G_{\mathcal{A}}, \alpha_{\big|_N} = 0\}$$

is a subgroup of $G_{\mathcal{A}}$*.*

Proof.

The unit $e = u_{\mathcal{A}} \circ \varepsilon$ clearly belongs to $T_{\mathcal{A}}$. Now for any $\alpha, \beta \in T_{\mathcal{A}}$ and for any $w \in N$ we compute:

$$\begin{aligned}
\alpha * \beta^{*-1}(w) &= \alpha * (\beta \circ S)(w) \\
&= \alpha(w) + \beta\big(S(w)\big) + \sum_{(w)} \alpha(w')(\beta \circ S)(w'') \\
&= \alpha(w) + \beta\left(-w - \sum_{(w)} w' S(w'')\right) + \sum_{(w)} \alpha(w')(\beta \circ S)(w'') \\
&= \alpha(w) - \beta(w) + \sum_{(w)} (\alpha - \beta)(w')(\beta \circ S)(w'') \\
&= 0.
\end{aligned}$$

□

return to main thm

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Now let $\zeta : N \to \mathcal{A}$ be a **partially defined character**, i.e. a linear map such that $\zeta(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and such that $\zeta(v)\zeta(w) = \zeta(v.w)$ as long as v, w and $v.w$ belong to N. Now let:

$$X_{\zeta,\mathcal{A}} := \{\varphi \in G_{\mathcal{A}}, \varphi_{|_N} = \zeta\}.$$

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Theorem (K. Ebrahimi-Fard, DM, J. Singer, J. Zhao)

$X_{\zeta,\mathcal{A}}$ is a $T_{\mathcal{A}}$-principal homogeneous space. More precisely, the left action:

$$\begin{array}{rcl} T_{\mathcal{A}} \times X_{\zeta,\mathcal{A}} & \longrightarrow & X_{\zeta,\mathcal{A}} \\ (\alpha, \varphi) & \longmapsto & \alpha * \varphi \end{array}$$

is free and transitive.

Proof.

For any $\alpha \in T_{\mathcal{A}}$, $\varphi \in X_{\zeta,\mathcal{A}}$ and $w \in N$ we have:

$$\begin{aligned} \alpha * \varphi(w) &= \alpha(w) + \varphi(w) + \sum_{(w)} \alpha(w')\varphi(w'') \\ &= \zeta(w), \end{aligned}$$

hence $\alpha * \varphi \in X_{\zeta,\mathcal{A}}$. Freeness is obvious. For transitivity, pick two elements φ, ψ in $X_{\zeta,\mathcal{A}}$ and proceed as in the previous proof. above □

We apply this general framework to $k=\mathbb{Q}$, $\mathcal{H}=(\mathbb{Q}\langle Z^*\rangle, \sqcup\!\!\!\!+, \Delta)$ and $\mathcal{A}=\mathbb{C}$. The right coideal N is the linear span of **non-singular words**, i.e. $w=y_{n_1}\cdots y_{n_k}\in Z^*\cap N$ if and only if

1. $n_1\neq 1$,
2. $n_1+n_2\notin\{2,1,0,-2,-4,\ldots\}$,
3. $n_1+\cdots+n_j\notin\mathbb{Z}_{\leq j}$ for any $j\in\{3,\ldots,k\}$.

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N is obviously a right coideal for deconcatenation. Moreover it is stable by **contractions**, like:

$$y_{n_1}y_{n_2}y_{n_3}y_{n_4}y_{n_5}y_{n_6}y_{n_7}\mapsto y_{n_1}y_{n_2+n_3+n_4}y_{n_5}y_{n_6+n_7}.$$

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We denote by $\Sigma=Z^*\setminus(Z^*\cap N)$ the set of **singular words**, and by Σ_k the set of singular words of length k. With the notations of the Introduction we have:

$$\Sigma_k=\{y_{n_1}\cdots y_{n_k},\,(n_1,\ldots,n_k)\in\mathcal{S}_k\}.$$

The partially defined character ζ is given by

$$\zeta(y_{n_1}\cdots y_{n_k})=\zeta(n_1,\ldots,n_k),$$

for any non-singular word $y_{n_1}\cdots y_{n_k}$, the RHS being the ordinary MZV or the value obtained by analytic continuation.

Thus, the set of all solutions to our initial problem is

$$X_{\zeta,\mathbb{C}} = T_{\mathbb{C}}.\varphi,$$

where φ is one particular solution (which is known to exist).

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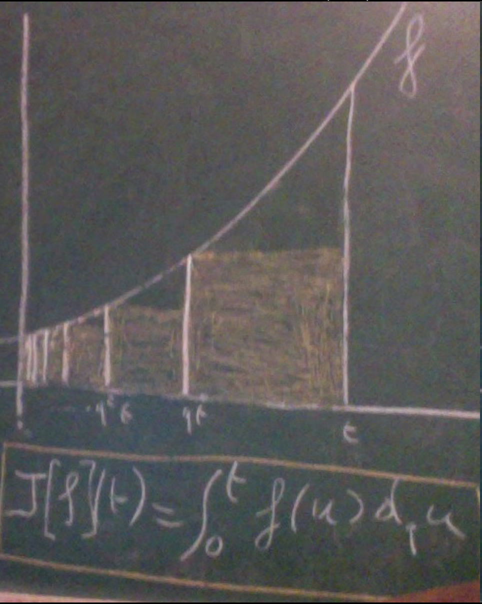
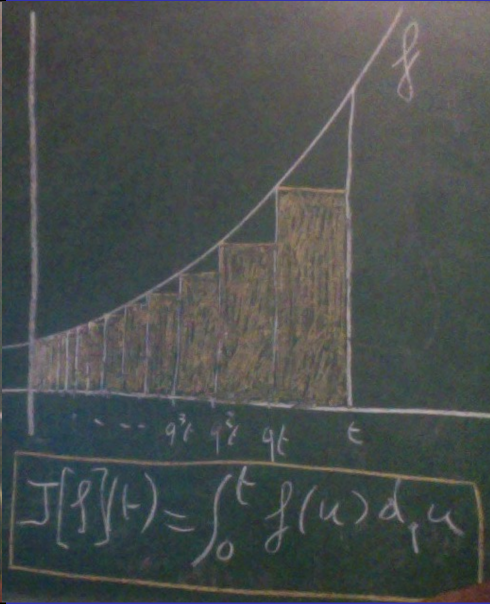
The renormalisation group $T_{\mathbb{C}}$ is big (infinite-dimensional).

q-analogues of multiple zeta values

The **Jackson integral** is defined by:

$$J[f](t) = \int_0^t f(u)\, d_q u = \sum_{n \geq 0} (q^n t - q^{n+1} t) f(q^n t).$$


f
q²t
qt
t
J[f](t) = ∫₀ᵗ f(u) d_q u


f
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- Here q is a parameter in $]0,1[$.
- When $q \nearrow 1$ the Riemann sum above converges to the ordinary integral.
- q can also be considered as an indeterminate: The Jackson integral operator J is then a $\mathbb{Q}[[q]]$-linear endomorphism of

$$\mathcal{A} := t\mathbb{Q}[[t,q]].$$

A weight -1 Rota-Baxter operator

The $\mathbb{Q}[[q]]$-linear operator $P_q : \mathcal{A} \longrightarrow \mathcal{A}$ defined by:

$$P_q[f](t) := \sum_{n \geq 0} f(q^n t) = f(t) + f(qt) + f(q^2 t) + f(q^3 t) + \cdots$$

satisfies the **weight -1 Rota-Baxter identity:**

$$P_q[f]P_q[g] = P_q\big[P_q[f]g + fP_q[g] - fg\big].$$

Operator P_q is **invertible** with inverse:

$$P_q^{-1}[f](t) = D_q[f](t) = f(t) - f(qt).$$

The q-difference operator D_q satisfies a modified Leibniz rule:

$$D_q[fg] = D_q[f]g + fD_q[g] - D_q[f]D_q[g].$$

We end up with **three identities**:

$$P_q[f]P_q[g] = P_q\Big[P_q[f]g + fP_q[g] - fg\Big], \quad (9)$$

$$D_q[f]D_q[g] = D_q[f]g + fD_q[g] - D_q[fg], \quad (10)$$

$$D_q[f]P_q[g] = D_q\Big[fP_q[g]\Big] + D_q[f]g - fg. \quad (11)$$

Note that (9), (10) and (11) are equivalent. Jump to q-shuffle

Multiple q-polylogarithms

- Introduce the functions:

$$x(t) := \frac{1}{t}, \qquad y(t) := \frac{1}{1-t}, \qquad \overline{y}(t) := \frac{t}{1-t}.$$

  Note that $\overline{y}$ is an element of $\mathcal{A}$.
- Introduce $X, Y, \overline{Y}$, multiplication operators by $x, y, \overline{y}$ resp.
- Recall:

$$\mathrm{Li}_{n_1,\ldots,n_k} = (R \circ X)^{n_1-1} \circ (R \circ Y) \circ \cdots \circ (R \circ X)^{n_k-1} \circ (R \circ Y)[\mathbf{1}].$$

- **Analogously**:

$$\mathrm{Li}^{q}_{n_1,\ldots,n_k} := (J \circ X)^{n_1-1} \circ (J \circ Y) \circ \cdots \circ (J \circ X)^{n_k-1} \circ (J \circ Y)[\mathbf{1}].$$

Ohno-Okuda-Zudilin q-multiple zeta values

(Yasuo Ohno, Jun-Ichi Okuda, Wadim Zudilin, 2012)

- Recall:

$$\zeta(n_1,\ldots,n_k)=\mathrm{Li}_{n_1,\ldots,n_k}(1).$$

- By analogy define:

$$\mathfrak{z}_q(n_1,\ldots,n_k):=\mathrm{Li}^{q}_{n_1,\ldots,n_k}(q).$$

- Some straightforward computation shows:

$$\mathfrak{z}_q(n_1,\ldots,n_k)=\sum_{m_1>\cdots>m_k}\frac{q^{m_1}}{[m_1]_q^{n_1}\cdots[m_k]_q^{n_k}},$$

with usual q-numbers:

$$[m]_q=\frac{1-q^m}{1-q}=1+q+\cdots+q^{m-1}.$$

- For any positive integers $n_1, \ldots n_k$ with $n_1 \geq 2$, the q-MZV $\mathfrak{z}_q(n_1, \ldots, n_k)$ makes sense for any complex q with $|q| \leq 1$, and we have:

$$\lim_{q \nearrow 1} \mathfrak{z}_q(n_1, \ldots, n_k) = \zeta(n_1, \ldots, n_k).$$

- Here, $q \nearrow 1$ means $q \to 1$ inside an angular sector :

$$\mathrm{Arg}(q-1) \in [\frac{\pi}{2} + \varepsilon, \frac{3\pi}{2} - \varepsilon].$$

- An alternative description in terms of the operator P_q will be very convenient:

$$\begin{aligned}\bar{\mathfrak{z}}_q(n_1,\ldots,n_k) &:= (1-q)^{-w}\mathfrak{z}_q(n_1,\ldots,n_k)\\ &= \sum_{m_1>\cdots>m_k>0}\frac{q^{m_1}}{(1-q^{m_1})^{n_1}\cdots(1-q^{m_k})^{n_k}}\\ &= P_q^{n_1}\circ\overline{Y}\circ\cdots\circ P_q^{n_k}\circ\overline{Y}[\mathbf{1}](t)\big|_{t=q}.\end{aligned}$$

where we recall that $\overline{Y}$ is the operator of multiplication by

$$\bar{y}: t\mapsto\frac{t}{1-t}.$$

Other models of qMZVs

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- Schlesinger model (2001):

$$\zeta_q^S(n_1,\ldots,n_k) := \sum_{m_1>\cdots>m_k\geq 1} \frac{1}{[m_1]_q^{n_1}\cdots[m_r]_q^{n_k}} = \mathrm{Li}^q_{n_1,\ldots,n_k}(1).$$

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- Zhao-Bradley model (2003)
  ($k=1$: Kaneko, Kurokawa and Wakayama).

$$\zeta_q(n_1,\ldots,n_k) := \sum_{m_1>\cdots>m_k\geq 1} \frac{q^{m_1(n_1-1)+\cdots+m_k(n_k-1)}}{[m_1]_q^{n_1}\cdots[m_r]_q^{n_k}}.$$

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- Multiple divisor functions (Bachmann-Kühn, 2013):

$$[n_1,\ldots,n_k] = \frac{1}{(n_1-1)!\cdots(n_k-1)!}\sum_{j>0}\left(\sum_{\substack{m_1>\cdots>m_k\geq 1\\ m_1v_1+\cdots m_kv_k=j}} v_1^{n_1-1}\cdots v_k^{n_k-1}\right)q^j.$$

Extension to arguments of any sign

- The iterated sum defining $\bar{\mathfrak{z}}_q(n_1, \ldots, n_k)$ makes perfect sense in $\mathbb{Q}[[q]]$ for any $n_1, \ldots, n_k \in \mathbb{Z}$.
- moreover it also makes sense when specializing q to a complex number of modulus < 1:

$$|\bar{\mathfrak{z}}_q(n_1, \ldots, n_k)| \le |q|^k (1 - |q|)^{-w' - k},$$

with $w' := \sum_{i=1}^{k} |n_i|$.

- **For any $n_1, \ldots, n_k \in \mathbb{Z}$ we still have (with $P_q^{-1} = D_q$):**

$$\bar{\mathfrak{z}}_q(n_1, \ldots, n_k) = P_q^{n_1} \circ \overline{Y} \circ \cdots \circ P_q^{n_1} \circ \overline{Y}[\mathbf{1}](t)\big|_{t=q}.$$

Examples

$$
\begin{aligned}
\bar{\mathfrak{z}}_q(0) &= \sum_{q>0} q^m = \frac{q}{1-q}, \\
\bar{\mathfrak{z}}_q(\underbrace{0,\ldots,0}_{k}) &= \left(\frac{q}{1-q}\right)^k, \\
\bar{\mathfrak{z}}_q(-1) &= \sum_{m>0} q^m(1-q^m) = \frac{q}{1-q} - \frac{q^2}{1-q^2}. \\
\bar{\mathfrak{z}}_q(1) &= \sum_{m>0} \frac{q^m}{1-q^m} \text{ (Lambert series)}
\end{aligned}
$$

Double q-shuffle relations

- The qMZVs described above admit both q-shuffle and q-quasi-shuffle relations.
- Double q-shuffle relations have been also settled recently (2013) by **Yoshihiro Takeyama** in the Bradley model.

q-shuffle relations

- Let $\widetilde{X}$ be the alphabet $\{d, y, p\}$.
- Let W be the set of words on the alphabet $\widetilde{X}$, ending with y and subject to
$$dp = pd = \mathbf{1},$$
where **1** is the empty word.

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  where $\mathbf{1}$ is the empty word.
- Any nonempty word in W writes uniquely $v = p^{n_1} y \cdots p^{n_k} y$, with $n_1, \ldots, n_k \in \mathbb{Z}$.
- Now define:

$$\bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(p^{n_1} y \cdots p^{n_k} y) := \bar{\mathfrak{z}}_q(n_1, \ldots, n_k)$$

  and extend linearly.

- q-shuffle product recursively given (w.r.t. length of words) by $\mathbf{1}\sqcup\!\sqcup v = v\sqcup\!\sqcup\mathbf{1} = v$ and:

$$\begin{aligned}
(yv)\sqcup\!\sqcup u = v\sqcup\!\sqcup(yu) &= y(v\sqcup\!\sqcup u),\\
pv\sqcup\!\sqcup pu &= p(v\sqcup\!\sqcup pu) + p(pv\sqcup\!\sqcup u) - p(v\sqcup\!\sqcup u),\\
dv\sqcup\!\sqcup du &= v\sqcup\!\sqcup du + dv\sqcup\!\sqcup u - d(v\sqcup\!\sqcup u),\\
dv\sqcup\!\sqcup pu = pu\sqcup\!\sqcup dv &= d(v\sqcup\!\sqcup pu) + dv\sqcup\!\sqcup u - v\sqcup\!\sqcup u.
\end{aligned}$$

for any $u, v \in W$. Explanation

- The product $\sqcup\!\sqcup$ is **commutative** and **associative**.
- The q-shuffle relations write:

$$\bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(u)\bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(v) = \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(u\sqcup\!\sqcup v).$$

return to computation

q-quasi-shuffle relations

- $\widetilde{Y}=$ alphabet $\{z_n,\, n\in\mathbb{Z}\}$, with internal product $z_i\diamond z_j=z_{i+j}$.
- Let $\widetilde{Y}^*$ be set of words with letters in $\widetilde{Y}$.
- Let $*$ be the ordinary quasi-shuffle product on $\mathbb{Q}\langle\widetilde{Y}\rangle$.
- Let T be the shift operator defined for any word u by:

$$T(z_nu):=(z_n-z_{n-1})u.$$

- The q-quasi-shuffle product $\uplus$ is (uniquely) defined by:

$$T(u\uplus v)=Tu*Tv.$$

- Define $\bar{\mathfrak{z}}_q^{\,⧢}(z_{n_1}\cdots z_{n_k}) := \bar{\mathfrak{z}}_q(n_1,\ldots,n_k)$ and extend linearly.
- the q-quasi-shuffle relations write:

$$\bar{\mathfrak{z}}_q^{\,⧢}(u)\bar{\mathfrak{z}}_q^{\,⧢}(v) = \bar{\mathfrak{z}}_q^{\,⧢}(u \,⧢\, v)$$

for any words $u, v \in \widetilde{Y}^*$.

- Define $\overline{\mathfrak{z}}_q^{\,\boxplus}(z_{n_1}\cdots z_{n_k}) := \overline{\mathfrak{z}}_q(n_1,\ldots,n_k)$ and extend linearly.
- the q-quasi-shuffle relations write:

$$\overline{\mathfrak{z}}_q^{\,\boxplus}(u)\overline{\mathfrak{z}}_q^{\,\boxplus}(v) = \overline{\mathfrak{z}}_q^{\,\boxplus}(u \boxplus v)$$

  for any words $u, v \in \widetilde{Y}^*$.
- Example of q-quasi-shuffle relation: for any $a, b \in \mathbb{Z}$,

$$\begin{aligned}\overline{\mathfrak{z}}_q(a)\overline{\mathfrak{z}}_q(b) &= \overline{\mathfrak{z}}_q(a,b) + \overline{\mathfrak{z}}_q(b,a) + \overline{\mathfrak{z}}_q(a+b)\\ &\quad -\overline{\mathfrak{z}}_q(a,b-1) - \overline{\mathfrak{z}}_q(b,a-1) - \overline{\mathfrak{z}}_q(a+b-1).\end{aligned}$$

- Note that the weight is **not** conserved, contrarily to the classical case.

- In terms on "non-modified" q-MZVs, the previous example becomes:

$$\begin{aligned}\mathfrak{z}_q(a)\mathfrak{z}_q(b) &= \mathfrak{z}_q(a,b)+\mathfrak{z}_q(b,a)+\mathfrak{z}_q(a+b)\\ &\quad -(1-q)\big[\mathfrak{z}_q(a,b-1)+\mathfrak{z}_q(b,a-1)+\mathfrak{z}_q(a+b-1)\big].\end{aligned}$$

- In the limit $q \nearrow 1$, the "weight drop term" disappears, and we recover the classical quasi-shuffle relation.

Important remark

There are *no regularization relations* in this picture. The swap

$$\mathfrak{r} : \widetilde{Y}^* \to W$$

is defined by:

$$\mathfrak{r}(z_{n_1} \cdots z_{n_k}) := p^{n_1 - 1} y \cdots p^{n_k - 1} y,$$

and the change of coding writes itself:

$$\bar{\mathfrak{z}}_q^{\uplus} = \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup} \circ \mathfrak{r}$$

in full generality.

Summing up, the double q-shuffle relations write themselves as follows:

for any $u, v \in \widetilde{Y}^*$ and for any $u', v' \in W$,

$$\begin{aligned} \bar{\mathfrak{z}}_q^{\,\uplus}(u)\bar{\mathfrak{z}}_q^{\,\uplus}(v) &= \bar{\mathfrak{z}}_q^{\,\uplus}(u \uplus v), \\ \bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup}(u')\bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup}(v') &= \bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup}(u' \sqcup\!\sqcup v'), \end{aligned}$$

and we also have:

$$\bar{\mathfrak{z}}_q^{\,\uplus} = \bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup} \circ \mathfrak{r}.$$

An example of computation using double *q*-shuffle relations

- Using *q*-quasi-shuffle:

$$\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2)=\bar{\mathfrak{z}}_q(1,2)+\bar{\mathfrak{z}}_q(2,1)+\bar{\mathfrak{z}}_q(3)-\bar{\mathfrak{z}}_q(1,1)-\bar{\mathfrak{z}}_q(2,0)-\bar{\mathfrak{z}}_q(2).$$

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- Using q-shuffle:

$$\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2) \quad = \quad \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(py)\bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(ppy)$$

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q-shuffle formulas

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▸ q-shuffle formulas

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&= \bar{\mathfrak{z}}_q(1,2) + 2\bar{\mathfrak{z}}_q(2,1) - \bar{\mathfrak{z}}_q(2,0) - \bar{\mathfrak{z}}_q(1,1).
\end{aligned}$$

q-shuffle formulas

An example of computation using double q-shuffle relations

- Using q-quasi-shuffle:

$$\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2)=\bar{\mathfrak{z}}_q(1,2)+\bar{\mathfrak{z}}_q(2,1)+\bar{\mathfrak{z}}_q(3)-\bar{\mathfrak{z}}_q(1,1)-\bar{\mathfrak{z}}_q(2,0)-\bar{\mathfrak{z}}_q(2).$$

- Using q-shuffle:

$$\begin{aligned}\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2) &= \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(py)\bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(ppy)\\ &= \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(py \sqcup\!\sqcup ppy)\\ &= \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}\big(p(y\sqcup\!\sqcup ppy + py\sqcup\!\sqcup py - y\sqcup\!\sqcup py)\big)\\ &= \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}\Big(p\big(yppy + p(2ypy - yy) - ypy\big)\Big)\\ &= \bar{\mathfrak{z}}_q^{\sqcup\!\sqcup}(pyppy + 2ppypy - ppyy - pypy)\\ &= \bar{\mathfrak{z}}_q(1,2) + 2\bar{\mathfrak{z}}_q(2,1) - \bar{\mathfrak{z}}_q(2,0) - \bar{\mathfrak{z}}_q(1,1).\end{aligned}$$

q-shuffle formulas

Hence,

$$\bar{\mathfrak{z}}_q(2,1) = \bar{\mathfrak{z}}_q(3) - \bar{\mathfrak{z}}_q(2),$$

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[**W. N. Bailey**, *An algebraic identity*, Proc. London Math. Soc. **11**, 156-160 (1936).]

Perspectives and open problems

- Are the double shuffle relations the only ones among our qMZVs?
- Combinatorial description of the q-shuffle product $\sqcup\!\sqcup$. Find a compatible coproduct.
- Parameter q yields a **regularisation** of MZVs. What about **renormalization** for $q \to 1$?

Perspectives and open problems

- Are the double shuffle relations the only ones among our qMZVs?
- Combinatorial description of the q-shuffle product $\sqcup\!\sqcup$. Find a compatible coproduct.
- Parameter q yields a **regularisation** of MZVs. What about **renormalization** for $q \to 1$?

  Behaviour of Lambert series for $q \to 1$ or $q \to e^{2i\pi/n}$: cf. Dorigoni-Kleinschmidt (2020).

References

- O. Bouillot,
  Multiple Bernoulli polynomials and numbers, preprint
  `http://www-igm.univ-mlv.fr/~bouillot/recherche.php?lang=fr`.
- L. Guo, B. Zhang,
  Renormalization of multiple zeta values,
  J. Algebra **319** No. 9, 3770-3809 (2008).
- D. Manchon, S. Paycha,
  Nested sums of symbols and renormalized multiple zeta values,
  Int. Math. Res. Not. 24, 4628-4697 (2010).
- K. Ebrahimi-Fard, D. Manchon, J. Singer,
  Renormalisation of q-regularized multiple zeta values,
  Lett. Math. Phys. **106**, 365–380 (2016).
- K. Ebrahimi-Fard, D. Manchon, J. Singer, J. Zhao,
  Renormalisation group for multiple zeta values,
  Comm. Number Theor. and phys. **12** No1, 75–96 (2018).
- Y. Ohno, J.-I. Okuda, W. Zudilin,
  Cyclic q-MZV sum,
  J. Number Theory **132** (2012), 144–155.
- Y. Takeyama,
  The algebra of a q-analogues of multiple harmonic series,
  SIGMA, Vol. 9, 15pp (2013), https://doi.org/10.3842/SIGMA.2013.061.
- J. Castillo-Medina, K. Ebrahimi-Fard, D. Manchon,
  Unfolding the double shuffle structure of q-MZVs,
  Bull. Austr. Math. Soc. **91** No3, 368-388 (2015) `arXiv:1310.1330`.

Thank you for your attention!