On Multiple zeta values and their $q$-analogues

Based on joint work with J. Castillo-Medina, K. Ebrahimi-Fard, S. Paycha, J. Singer, J. Zhao

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1 Multiple zeta values
   - Introduction
   - Multiple polylogarithms
   - Word description of the quasi-shuffle relations

2 Extension to arguments of any sign

3 The renormalisation group
   - A general framework
   - The MZV renormalisation group

4 $q$-multiple zeta values
   - The Jackson integral
   - Multiple $q$-polylogarithms
   - Ohno-Okuda-Zudilin $q$-MZVs
   - Double $q$-shuffle relations
Multiple zeta values are given by the following iterated series:

\[ \zeta(n_1, \ldots, n_k) = \sum_{m_1 > \cdots > m_k > 0} \frac{1}{m_1^{n_1} \cdots m_k^{n_k}}. \]  

The \( n_j \)'s are positive integers.

The series converges provided \( n_1 \geq 2 \). It makes also sense for \( n_1, \ldots, n_k \in \mathbb{Z} \) provided:

\[ n_1 + \cdots + n_j > j \text{ for any } j \in \{1, \ldots, k\}. \]

The integer \( k \) is the depth, the sum \( w := n_1 + \cdots + n_k \) is the weight.
• Appear in the work of **L. Euler** in depths 1 and 2 (18th century),
• Some examples in higher depths by **N. Nielsen** (1904),
The **Multiple zeta function** is given by the same iterated series:

\[
\zeta(z_1, \ldots, z_k) = \sum_{m_1 > \cdots > m_k > 0} \frac{1}{m_1^{z_1} \cdots m_k^{z_k}},
\]

where the \(z_j\)'s are complex numbers.

**Theorem (S. Akiyama, S. Egami, Y. Tanigawa, 2001)**

The series (3) converges provided:

\[
\text{Re}(z_1 + \cdots + z_j) > j \text{ for any } j \in \{1, \ldots, k\}.
\]

It defines a holomorphic function of \(k \) complex variables in this domain, which can be meromorphically extended to \(\mathbb{C}^k\). The subvariety of singularities is given by:

\[
S_k = \{(z_1, \ldots, z_k) \in \mathbb{C}^k, \ z_1 = 1 \text{ or } z_1 + z_2 \in \{2, 1, 0, -2, -4, \ldots\} \text{ or } \exists j \in \{3, \ldots, k\}, z_1 + \cdots + z_j \in \mathbb{Z}_{\leq j}\}.
\]
Quasi-shuffle relations

The product of two MZVs is a linear combination of MZVs!
For example:

\[
\zeta(n_1)\zeta(n_2) = \sum_{m_1 > m_2 > 0} \frac{1}{m_1^{n_1} m_2^{n_2}} + \sum_{m_2 > m_1 > 0} \frac{1}{m_1^{n_1} m_2^{n_2}} + \sum_{m_1 = m_2 > 0} \frac{1}{m_1^{n_1} m_2^{n_2}}
\]

\[= \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2).\]
The most general quasi-shuffle relation displays as follows:

\[ \zeta(n_1, \ldots, n_p)\zeta(n_{p+1}, \ldots, n_{p+q}) = \sum_{r \geq 0} \sum_{\sigma \in qsh(p, q; r)} \zeta(n_1^\sigma, \ldots, n_{p+q-r}^\sigma). \]

- Here \( qsh(p, q; r) \) stands for \((p, q)\)-quasi-shuffles of type \( r \). They are surjections
  \[ \sigma : \{1, \ldots, p+q\} \rightarrow \{1, \ldots, p+q-r\} \]
  subject to \( \sigma_1 < \cdots < \sigma_p \) and \( \sigma_{p+1} < \cdots < \sigma_{p+q} \).
- \( n_j^\sigma \) stands for the sum of the \( n_r \)'s for \( \sigma(r) = j \).
- The sum above contains only one or two terms.
Integral representation and shuffle relations

MZVs have an iterated integral representation (M. Kontsevich, D. Zagier):

\[ \zeta(n_1, \ldots, n_k) = \int_{0 \leq t_w \leq \cdots \leq t_1 \leq 1} \frac{dt_1}{t_1} \cdots \frac{dt_{n_1-1}}{t_{n_1-1}} \frac{dt_{n_1}}{1 - t_{n_1}} \cdots \frac{dt_{n_1+\ldots+n_{k-1}+1}}{t_{n_1+\ldots+n_{k-1}+1}} \cdots \frac{dt_{w-1}}{t_{w-1}} \frac{dt_w}{1 - t_w} \]

As a consequence, there is a second way to express the product of two MZVs as a linear combination of MZVs: the shuffle relations.

Example:

\[ \zeta(2) \zeta(2) = \int \frac{dt_1}{t_1} \frac{dt_2}{1 - t_2} \frac{dt_3}{t_3} \frac{dt_4}{1 - t_4} \]

\[ = 4 \zeta(3, 1) + 2 \zeta(2, 2). \]
Regularization relations

A third group of relations can be deduced from a natural extension of the preceding ones: the regularization relations. The simplest one is:

$$\zeta(2, 1) = \zeta(3),$$

obtained as follows:

$$\zeta(1)\zeta(2) = \zeta(1, 2) + \zeta(2, 1) + \zeta(3)$$
$$= \zeta(1, 2) + 2\zeta(2, 1).$$
Regularization relations

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Regularization relations

A third group of relations can be deduced from a natural extension of the preceding ones: the regularization relations. The simplest one is:

\[ \zeta(2, 1) = \zeta(3), \]

obtained as follows:

\[
\zeta(1)\zeta(2) = \frac{\zeta(1, 2)}{2} + \zeta(2, 1) + \zeta(3) \\
= \frac{\zeta(1, 2)}{2} + \mathcal{Q}\zeta(2, 1).
\]
These three groups of relations constitute the so-called **double shuffle relations**.

It is conjectured that no other relations occur among multiple zeta values. For example the **duality relations**, coming from \( t_j \mapsto 1 - t_j \) in the integral representation, should be deducible from DS relations.
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It is conjectured that no other relations occur among multiple zeta values. For example the **duality relations**, coming from $t_j \mapsto 1 - t_j$ in the integral representation, should be deducible from DS relations.

\[
\zeta(2, 1) = \zeta(3), \quad \zeta(2, 1, 1) = \zeta(4), \ldots, \quad \zeta(2, 1^{n-2}) = \zeta(n) \ldots
\]
Multiple polylogarithms (in one variable)

For any \( t \in [0, 1] \),

\[
\text{Li}_{n_1, \ldots, n_k}(t) := \int_{0 \leq t_w \leq \cdots \leq t_1 \leq t} \frac{dt_1}{t_1} \cdots \frac{dt_{n_1-1}}{t_{n_1-1}} \frac{dt_{n_1}}{1 - t_{n_1}} \cdots \frac{dt_{n_1+\cdots+n_{k-1}+1}}{t_{n_1+\cdots+n_{k-1}+1}} \cdots \frac{dt_{w-1}}{t_{w-1}} \frac{dt_w}{1 - t_w}
\]

\[
= \sum_{m_1 > \cdots > m_k > 0} \frac{t^{m_1}}{m_1^{n_1} \cdots m_k^{n_k}}.
\]
\[ x(t) := \frac{1}{t}, \quad y(t) := \frac{1}{1-t}. \]

**Three operators** on the space of continuous maps \( f : [0, 1] \to \mathbb{R} \):

\[
X[f](t) := x(t)f(t), \\
Y[f](t) := y(t)f(t), \\
R[f](t) := \int_0^t f(u) \, du.
\]

**⇒ Concise expression** of the multiple polylogarithm:

\[
\operatorname{Li}_{n_1, \ldots, n_k} = (R \circ X)^{n_1-1} \circ (R \circ Y) \circ \cdots \circ (R \circ X)^{n_k-1} \circ (R \circ Y)[1].
\]
$R$ is a **weight zero Rota-Baxter operator**:

\[ R[f]R[g] = R[R[f]g + fR[g]]. \]

We have of course for any positive integers $n_1, \ldots, n_k$ with $n_1 \geq 2$:

\[ \text{Li}_{n_1,\ldots,n_k}(1) = \zeta(n_1,\ldots,n_k). \]
Word description of the quasi-shuffle relations

- Introduce the infinite alphabet $Y := \{y_1, y_2, y_3, \ldots\}$.
- $Y^*$ is the set of words with letters in $Y$.
- $\mathbb{Q}\langle Y \rangle$ is the linear span of $Y^*$ on $\mathbb{Q}$.
- **Quasi-shuffle product** on $\mathbb{Q}\langle Y \rangle$:

$$u_1 \cdots u_p \mathcal{U} u_{p+1} \cdots u_{p+q} := \sum_{r \geq 0} \sum_{\sigma \in qsh(p,q;r)} u_1^\sigma \cdots u_{p+q-r}^\sigma,$$

where $u_j^\sigma$ is the **internal product** of the $u_r$’s with $\sigma(r) = j$. The internal product is given by $y_i \diamond y_j = y_{i+j}$. For later use, the **shuffle product** is defined by:

$$u_1 \cdots u_p \mathcal{U} u_{p+1} \cdots u_{p+q} := \sum_{\sigma \in qsh(p,q;0)} u_1^\sigma \cdots u_{p+q}^\sigma.$$
Example

\[ y_2 \uplus y_3 y_1 = y_2 y_3 y_1 + y_3 y_2 y_1 + y_3 y_1 y_2 + y_5 y_1 + y_3 y_3, \]
\[ y_2 \sqcup y_3 y_1 = y_2 y_3 y_1 + y_3 y_2 y_1 + y_3 y_1 y_2. \]
Notation: $Y_{\text{conv}}^* := Y^* \setminus y_1 Y^*$. 

For any word $y_{n_1} \cdots y_{n_k}$ in $Y_{\text{conv}}^*$ we set:

$$\zeta_{\uplus} (y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).$$

Extend $\zeta_{\uplus}$ linearly.
Notation: \( Y_{\text{conv}}^* := Y^* \setminus y_1 Y^*. \)

For any word \( y_{n_1} \cdots y_{n_k} \) in \( Y_{\text{conv}}^* \) we set:

\[
\zeta_{\sqcup\sqcup} (y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).
\]

Extend \( \zeta_{\sqcup\sqcup} \) linearly.

The quasi-shuffle relations are rewritten as follows: for any \( u, v \in Y_{\text{conv}}^* \),

\[
\zeta_{\sqcup\sqcup} (u) \zeta_{\sqcup\sqcup} (v) = \zeta_{\sqcup\sqcup} (u_{\sqcup\sqcup} v). \quad (5)
\]
Notation: \( Y_{\text{conv}}^* := Y^* \setminus y_1 Y^* \).

For any word \( y_{n_1} \cdots y_{n_k} \) in \( Y_{\text{conv}}^* \) we set:

\[
\zeta_{\sqcup\sqcup} (y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \ldots, n_k).
\]

Extend \( \zeta_{\sqcup\sqcup} \) linearly.

The quasi-shuffle relations are rewritten as follows: for any \( u, v \in Y_{\text{conv}}^* \),

\[
\zeta_{\sqcup\sqcup} (u) \zeta_{\sqcup\sqcup} (v) = \zeta_{\sqcup\sqcup} (u \sqcup\sqcup v). \tag{5}
\]

**example:**

\[
\zeta_{\sqcup\sqcup} (y_2) \zeta_{\sqcup\sqcup} (y_3 y_1) = \zeta_{\sqcup\sqcup} (y_2 \sqcup\sqcup y_3 y_1) \\
= \zeta_{\sqcup\sqcup} (y_2 y_3 y_1 + y_3 y_2 y_1 + y_3 y_1 y_2 + y_5 y_1 + y_3 y_3),
\]

hence:

\[
\zeta(2) \zeta(3, 1) = \zeta(2, 3, 1) + \zeta(3, 2, 1) + \zeta(3, 1, 2) + \zeta(5, 1) + \zeta(3, 3).
\]
The quasi-shuffle product obviously extends to $\mathbb{Q}\langle Z \rangle$, where $Z$ is the infinite alphabet $\{y_j, j \in \mathbb{Z}\}$.

**Theorem (S. Paycha-DM, 2010)**

There exists a character

$$\varphi : (\mathbb{Q}\langle Z \rangle, \sqcup\sqcup) \longrightarrow \mathbb{C}$$

such that

- $\varphi(v) = \zeta_{\sqcup\sqcup}(v)$ for any $v \in Y_{\text{conv}}^*$.
- For any $v = y_{n_1} \cdots y_{n_k} \in Z^*$ such that $\zeta(n_1, \ldots n_k)$ can be defined by analytic continuation, then $\varphi(v) = \zeta(n_1, \ldots n_k)$. In particular,
  - $\varphi(-n) = \zeta(-n) = -\frac{B_{n+1}}{n+1}$ for any $n \in \mathbb{Z}_+$.
  - $\varphi(-n, -n') = \zeta(-n, -n') = \frac{1}{2}(1 + \delta_0') \frac{B_{n+n'+1}}{n+n'+1}$ for any $n, n' \in \mathbb{Z}_+$ with $n + n'$ odd.
<table>
<thead>
<tr>
<th>$\zeta(-a,-b)$</th>
<th>$a = 0$</th>
<th>$a = 1$</th>
<th>$a = 2$</th>
<th>$a = 3$</th>
<th>$a = 4$</th>
<th>$a = 5$</th>
<th>$a = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{7}{720}$</td>
<td>$-\frac{1}{120}$</td>
<td>$-\frac{11}{2520}$</td>
<td>$\frac{1}{252}$</td>
<td>$\frac{1}{224}$</td>
</tr>
<tr>
<td>$b = 1$</td>
<td>$\frac{1}{24}$</td>
<td>$\frac{1}{288}$</td>
<td>$-\frac{1}{240}$</td>
<td>$-\frac{19}{10080}$</td>
<td>$\frac{1}{504}$</td>
<td>$\frac{41}{20160}$</td>
<td>$-\frac{1}{480}$</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>$-\frac{7}{720}$</td>
<td>$-\frac{1}{240}$</td>
<td>$0$</td>
<td>$\frac{1}{504}$</td>
<td>$\frac{113}{151200}$</td>
<td>$-\frac{1}{480}$</td>
<td>$-\frac{307}{166320}$</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>$-\frac{1}{240}$</td>
<td>$\frac{1}{840}$</td>
<td>$\frac{1}{504}$</td>
<td>$\frac{1}{28800}$</td>
<td>$-\frac{1}{480}$</td>
<td>$-\frac{281}{332640}$</td>
<td>$\frac{1}{264}$</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>$\frac{11}{2520}$</td>
<td>$\frac{1}{504}$</td>
<td>$-\frac{113}{151200}$</td>
<td>$-\frac{1}{480}$</td>
<td>$0$</td>
<td>$\frac{1}{264}$</td>
<td>$\frac{117977}{75675600}$</td>
</tr>
<tr>
<td>$b = 5$</td>
<td>$\frac{1}{504}$</td>
<td>$-\frac{103}{60480}$</td>
<td>$-\frac{1}{480}$</td>
<td>$\frac{1}{1232}$</td>
<td>$\frac{1}{264}$</td>
<td>$\frac{1}{127008}$</td>
<td>$-\frac{691}{65520}$</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>$-\frac{1}{224}$</td>
<td>$-\frac{1}{480}$</td>
<td>$\frac{307}{166320}$</td>
<td>$\frac{1}{264}$</td>
<td>$-\frac{117977}{75675600}$</td>
<td>$-\frac{691}{65520}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Sketch of proof: through regularisation and renormalisation.

- $\mathcal{H} := (\mathbb{Q}\langle Z \rangle, \shuffle, \Delta)$ is a connected filtered Hopf algebra, where $\Delta$ stands for deconcatenation:

$$\Delta(y_{n_1} \cdots y_{n_k}) = \sum_{j=0}^{k} y_{n_1} \cdots y_{n_j} \otimes y_{n_{j+1}} \cdots y_{n_k}.$$  

- $\overline{\mathcal{H}} := (\mathbb{Q}\langle C \rangle, \shuffle, \Delta)$ where:

$$C := \{ y_t, t \in \mathbb{C} \}.$$  

- $\overline{\mathcal{H}}_{\shuffle} := (\mathbb{Q}\langle C \rangle, \shuffle, \Delta), \quad \mathcal{H}_{\shuffle} := (\mathbb{Q}\langle Z \rangle, \shuffle, \Delta).$
Sketch of proof: through regularisation and renormalisation.

- $\mathcal{H} := (\mathbb{Q} \langle Z \rangle, \oplus, \Delta)$ is a connected filtered Hopf algebra, where $\Delta$ stands for deconcatenation:

$$\Delta(y_{n_1} \cdots y_{n_k}) = \sum_{j=0}^{k} y_{n_1} \cdots y_{n_j} \otimes y_{n_{j+1}} \cdots y_{n_k}.$$

- $\overline{\mathcal{H}} := (\mathbb{Q} \langle C \rangle, \underline{+}, \Delta)$ where:

$$C := \{ y_t, \ t \in \mathbb{C} \}.$$

- $\overline{\mathcal{H}}_{\underline{+}} := (\mathbb{Q} \langle C \rangle, \underline{+}, \Delta), \quad \mathcal{H}_{\underline{+}} := (\mathbb{Q} \langle Z \rangle, \underline{+}, \Delta)$.
- $\mathcal{R} : \overline{\mathcal{H}}_{\underline{+}} \to \text{Maps}(\mathbb{C}, \overline{\mathcal{H}}_{\underline{+}})$ defined below respects $\underline{+}$.

$$\mathcal{R}(y_{t_1} \cdots y_{t_k}) := y_{t_1-z} \cdots y_{t_k-z}.$$
• \( H \xrightarrow{\sim} \overline{H} \) is a Hopf algebra isomorphism.
\[ \mathcal{H}_\sqcup \xrightarrow{\sim} \mathcal{H} \] is a Hopf algebra isomorphism.

\[ \widetilde{\mathcal{R}} : \mathcal{H} \rightarrow \text{Maps}(\mathbb{C}, \mathcal{H}) \] defined by:

\[ \widetilde{\mathcal{R}}(y_{t_1} \cdots y_{t_1}) := \exp_H \circ \mathcal{R} \circ \log_H \]

respects \( \sqcup \). The character

\[ \Phi : (\mathbb{Q}\langle \mathbb{Z} \rangle, \sqcup) \]
\[ \bar{\mathcal{H}} \xrightarrow{\sim} \mathcal{H} \]

is a Hopf algebra isomorphism.

\[ \tilde{\mathcal{R}} : \bar{\mathcal{H}} \to \text{Maps}(\mathbb{C}, \mathcal{H}) \]

defined by:

\[ \tilde{\mathcal{R}}(y_{t_1} \cdots y_{t_1}) := \exp_H \circ \mathcal{R} \circ \log_H \]

respects \( \sqcup \sqcap \). The character

\[ \Phi : (\mathbb{Q}\langle \mathbb{Z} \rangle, \sqcup \sqcap) \longrightarrow \text{Mero}(\mathbb{C}) \quad (7) \]
\( \overline{\mathcal{H}} \xrightarrow{\sim} \overline{\mathcal{H}} \) is a Hopf algebra isomorphism.

\( \widetilde{\mathcal{R}} : \overline{\mathcal{H}} \to \text{Maps}(\mathbb{C}, \overline{\mathcal{H}}) \) defined by:

\[
\widetilde{\mathcal{R}}(y_{t_1} \cdots y_{t_1}) := \exp_H \circ \mathcal{R} \circ \log_H
\]

respects \( \boxplus \). The character

\[
\Phi : (\mathbb{Q}\langle \mathbb{Z} \rangle, \boxplus) \longrightarrow \text{Mero}(\mathbb{C})
\]  \hspace{1cm} (7)

is defined by \( \Phi = \zeta_{\boxplus} \circ \widetilde{\mathcal{R}} \big|_{\mathcal{H}} \).
Then use **Birkhoff-Connes-Kreimer decomposition**: 

$$\Phi = \Phi^{-1}_- \ast \Phi_+,$$

where $\ast$ is the convolution product: $\alpha \ast \beta = m \circ (\alpha \otimes \beta) \circ \Delta$. 
\( \Phi^+ \) and \( \Phi^- \) are still characters of \( (\mathbb{Q}(Z), +) \),

\( \Phi^- (v) \in z^{-1} \mathbb{C}[z^{-1}] \) for any nonempty word \( v \).

\( \Phi^+ (v) \) is holomorphic at \( z = 0 \) for any word \( v \).

\( \Phi_{-1}^* = \Phi_- \circ S \), i.e. the inverse is given by composition on the right with the antipode.
\( \Phi_-(v) \) and \( \Phi_+(v) \) are given by explicit recursive formulas wrt the length of the word \( v \) (BPHZ algorithm): the commutative algebra \( \text{Mero}(\mathbb{C}) \) splits into two subalgebras:

\[
\text{Mero}(\mathbb{C}) = \mathcal{A}_- \oplus \mathcal{A}_+,
\]

where \( \mathcal{A}_- = z^{-1} \mathbb{C}[z^{-1}] \) and \( \mathcal{A}_+ \) is the subalgebra of meromorphic functions which do not have a pole at \( z = 0 \) (minimal subtraction scheme). Let \( \pi \) be the extraction of the pole part, i.e. the projection onto \( \mathcal{A}_- \) parallel to \( \mathcal{A}_+ \). Then:

\[
\begin{align*}
\Phi_-(w) &= -\pi \left( \Phi(w) + \sum_{(w')} \Phi_-(w') \Phi(w'') \right), \\
\Phi_+(w) &= (I - \pi) \left( \Phi(w) + \sum_{(w')} \Phi_-(w') \Phi(w'') \right).
\end{align*}
\]
Definition:

\[ \varphi(\nu) := \Phi_+(\nu)(z) \bigg|_{z=0}. \]
Now we want to describe all solutions to the problem, i.e. describe the set of all characters of \((\mathbb{Q}\langle Z \rangle, \sqcup \sqcap)\) which extend multiple zeta functions in the sense described above.
The renormalisation group

Let $\mathcal{H}$ be any commutative connected filtered Hopf algebra, over some base field $k$. Let $\mathcal{A}$ be any commutative unital $k$-algebra, and let $G_{\mathcal{A}}$ be the group of characters of $\mathcal{H}$ with values in $\mathcal{A}$. The product in $G_{\mathcal{A}}$ is given by convolution. The coproduct is conilpotent, i.e.

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \tilde{\Delta}(x),$$

where $\tilde{\Delta}(x) = \sum_{(x)} x' \otimes x''$ is the reduced coproduct, and $\tilde{\Delta}^{(k)}(x) = 0$ for $k \geq |x|$.

Proposition

Let $N$ be a right coideal with respect to the reduced coproduct, i.e. $\tilde{\Delta}(N) \subset N \otimes \mathcal{H}$ and $\varepsilon(N) = \{0\}$. The set

$$T_{\mathcal{A}} := \{ \alpha \in G_{\mathcal{A}}, \alpha|_N = 0 \}$$

is a subgroup of $G_{\mathcal{A}}$. 
Proof.

The unit $e = u_{\mathcal{A}} \circ \varepsilon$ clearly belongs to $T_{\mathcal{A}}$. Now for any $\alpha, \beta \in T_{\mathcal{A}}$ and for any $w \in N$ we compute:

$$\alpha \ast \beta^{*-1}(w) = \alpha \ast (\beta \circ S)(w)$$

$$= \alpha(w) + \beta(S(w)) + \sum_{(w)} \alpha(w')(\beta \circ S)(w'')$$

$$= \alpha(w) + \beta \left( -w - \sum_{(w)} w'S(w'') \right) + \sum_{(w)} \alpha(w')(\beta \circ S)(w'')$$

$$= \alpha(w) - \beta(w) + \sum_{(w)} (\alpha - \beta)(w')(\beta \circ S)(w'')$$

$$= 0.$$
Definition

$\mathcal{T}_A$ is the **renormalisation group** associated to the coideal $N$. 

Now let

\[ \zeta : N \to A \]

be a partially defined character, i.e. a linear map such that $\zeta(1) = 1_A$ and such that $\zeta(v) \cdot \zeta(w) = \zeta(v \cdot w)$ as long as $v$, $w$ and $v \cdot w$ belong to $N$. Now let:

\[ X_{\zeta, A} := \{ \phi \in GA | N = \zeta \} . \]
Definition

$T_{\mathcal{A}}$ is the **renormalisation group** associated to the coideal $N$.

Now let $\zeta : N \rightarrow \mathcal{A}$ be a **partially defined character**, i.e. a linear map such that $\zeta(1) = 1_{\mathcal{A}}$ and such that $\zeta(v)\zeta(w) = \zeta(v.w)$ as long as $v, w$ and $v.w$ belong to $N$. Now let:

$$X_{\zeta, \mathcal{A}} := \{ \varphi \in G_{\mathcal{A}}, \varphi|_{N} = \zeta \}.$$
Definition

$T_{\mathcal{A}}$ is the \textbf{renormalisation group} associated to the coideal $N$.

Now let $\zeta : N \rightarrow \mathcal{A}$ be a \textbf{partially defined character}, i.e. a linear map such that $\zeta(1) = 1_{\mathcal{A}}$ and such that $\zeta(v)\zeta(w) = \zeta(v.w)$ as long as $v, w$ and $v.w$ belong to $N$. Now let:

$$X_{\zeta, \mathcal{A}} := \{ \varphi \in G_{\mathcal{A}}, \varphi|_{N} = \zeta \}.$$

Theorem (K. Ebrahimi-Fard, DM, J. Singer, J. Zhao)

$X_{\zeta, \mathcal{A}}$ is a $T_{\mathcal{A}}$-principal homogeneous space. More precisely, the left action:

$$T_{\mathcal{A}} \times X_{\zeta, \mathcal{A}} \rightarrow X_{\zeta, \mathcal{A}}$$

$$(\alpha, \varphi) \mapsto \alpha \ast \varphi$$

is free and transitive.
Proof.

For any $\alpha \in T_\mathcal{A}$, $\varphi \in X_{\zeta,\mathcal{A}}$ and $w \in N$ we have:

$$\alpha \ast \varphi(w) = \alpha(w) + \varphi(w) + \sum_{(w)} \alpha(w') \varphi(w'')$$

$$= \zeta(w),$$

hence $\alpha \ast \varphi \in X_{\zeta,\mathcal{A}}$. Freeness is obvious. For transitivity, pick two elements $\varphi, \psi$ in $X_{\zeta,\mathcal{A}}$ and proceed as in the previous proof.
We apply this general framework to $k = \mathbb{Q}$, $\mathcal{H} = (\mathbb{Q}\langle \mathbb{Z}^* \rangle, \{\pm\}, \Delta)$ and $\mathcal{A} = \mathbb{C}$. The right coideal $N$ is the linear span of non-singular words, i.e. $w = y_{n_1} \cdots y_{n_k} \in \mathbb{Z}^* \cap N$ if and only if

1. $n_1 \neq 1$,
2. $n_1 + n_2 \notin \{2, 1, 0, -2, -4, \ldots\}$,
3. $n_1 + \cdots + n_j \notin \mathbb{Z}_{\leq j}$ for any $j \in \{3, \ldots, k\}$.

$N$ is obviously a right coideal for deconcatenation.
We apply this general framework to $k = \mathbb{Q}$, $\mathcal{H} = (\mathbb{Q}[Z^*], \uplus, \Delta)$ and $\mathcal{A} = \mathbb{C}$. The right coideal $N$ is the linear span of non-singular words, i.e. $w = y_{n_1} \cdots y_{n_k} \in Z^* \cap N$ if and only if

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$N$ is obviously a right coideal for deconcatenation. Moreover it is stable by contractions, like:

$$y_{n_1} y_{n_2} y_{n_3} y_{n_4} y_{n_5} y_{n_6} y_{n_7} \mapsto y_{n_1} y_{n_2 + n_3 + n_4} y_{n_5} y_{n_6 + n_7}.$$
We apply this general framework to $k = \mathbb{Q}$, $\mathcal{H} = (\mathbb{Q}\langle Z^* \rangle, \uplus, \Delta)$ and $\mathcal{A} = \mathbb{C}$. The right coideal $N$ is the linear span of non-singular words, i.e. $w = y_{n_1} \cdots y_{n_k} \in Z^* \cap N$ if and only if

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$N$ is obviously a right coideal for deconcatenation. Moreover it is stable by contractions, like:

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We denote by $\Sigma = Z^* \setminus (Z^* \cap N)$ the set of singular words, and by $\Sigma_k$ the set of singular words of length $k$. With the notations of the Introduction we have:

$$\Sigma_k = \{ y_{n_1} \cdots y_{n_k}, (n_1, \ldots, n_k) \in S_k \}.$$
The partially defined character $\zeta$ is given by

$$\zeta(y_{n_1} \cdots y_{n_k}) = \zeta(n_1, \ldots, n_k),$$

for any non-singular word $y_{n_1} \cdots y_{n_k}$, the RHS being the ordinary MZV or the value obtained by analytic continuation.
Thus, the set of all solutions to our initial problem is

\[ X_{\xi,C} = T_C \cdot \varphi, \]

where \( \varphi \) is one particular solution (which is known to exist).
Thus, the set of all solutions to our initial problem is

$$X_{\zeta, C} = T_C \cdot \varphi,$$

where $\varphi$ is one particular solution (which is known to exist).

**The renormalisation group** $T_C$ **is big** (infinite-dimensional).
$q$-analogues of multiple zeta values
The Jackson integral is defined by:

\[ J[f](t) = \int_0^t f(u) \, d_q u = \sum_{n \geq 0} (q^n t - q^{n+1} t) f(q^n t). \]
Outline
Multiple zeta values
Extension to arguments of any sign
The renormalisation group
$q$-multiple zeta values

The Jackson integral
Multiple $q$-polylogarithms
Ohno-Okuda-Zudilin $q$-MZVs
Double $q$-shuffle relations

Dominique Manchon LMBP, CNRS-Université Clermont-Auvergne

On Multiple zeta values and their $q$-analogues
Here $q$ is a parameter in $]0, 1[$.

When $q \nearrow 1$ the Riemann sum above converges to the ordinary integral.

$q$ can also be considered as an indeterminate: The Jackson integral operator $J$ is then a $\mathbb{Q}[[q]]$-linear endomorphism of

$$\mathcal{A} := t\mathbb{Q}[[t, q]].$$
A weight $-1$ Rota-Baxter operator

The $\mathbb{Q}[[q]]$-linear operator $P_q : \mathcal{A} \longrightarrow \mathcal{A}$ defined by:

$$P_q[f](t) := \sum_{n \geq 0} f(q^n t) = f(t) + f(qt) + f(q^2 t) + f(q^3 t) + \cdots$$

satisfies the weight $-1$ Rota-Baxter identity:

$$P_q[f]P_q[g] = P_q[P_q[f]g + fP_q[g] - fg].$$

Operator $P_q$ is invertible with inverse:

$$P_q^{-1}[f](t) = D_q[f](t) = f(t) - f(qt).$$
The \( q \)-difference operator \( D_q \) satisfies a modified Leibniz rule:

\[
D_q[fg] = D_q[f]g + fD_q[g] - D_q[f]D_q[g].
\]

We end up with three identities:

\[
\begin{align*}
P_q[f]P_q[g] &= P_q[P_q[f]g + fP_q[g] - fg], \quad (9) \\
D_q[f]D_q[g] &= D_q[f]g + fD_q[g] - D_q[fg], \quad (10) \\
D_q[f]P_q[g] &= D_q[fP_q[g]] + D_q[f]g - fg. \quad (11)
\end{align*}
\]

Note that (9), (10) and (11) are equivalent.
Multiple \( q \)-polylogarithms

- Introduce the functions:

\[
  x(t) := \frac{1}{t}, \quad y(t) := \frac{1}{1-t}, \quad \overline{y}(t) := \frac{t}{1-t}.
\]

Note that \( \overline{y} \) is an element of \( \mathcal{A} \).

- Introduce \( X, Y, \overline{Y} \), multiplication operators by \( x, y, \overline{y} \) resp.

- Recall:

\[
\text{Li}_{n_1, \ldots, n_k} = (R \circ X)^{n_1-1} \circ (R \circ Y) \circ \cdots \circ (R \circ X)^{n_k-1} \circ (R \circ Y)[1].
\]

- Analogously:

\[
\text{Li}_{n_1, \ldots, n_k}^q := (J \circ X)^{n_1-1} \circ (J \circ Y) \circ \cdots \circ (J \circ X)^{n_k-1} \circ (J \circ Y)[1].
\]
Ohno-Okuda-Zudilin \( q \)-multiple zeta values

(Yasuo Ohno, Jun-Ichi Okuda, Wadim Zudilin, 2012)

- Recall:
  \[
  \zeta(n_1, \ldots, n_k) = \text{Li}_{n_1, \ldots, n_k}(1).
  \]

- By analogy define:
  \[
  \zeta_q(n_1, \ldots, n_k) := \text{Li}^{q}_{n_1, \ldots, n_k}(q).
  \]

- Some straightforward computation shows:
  \[
  \zeta_q(n_1, \ldots, n_k) = \sum_{m_1 > \cdots > m_k} \frac{q^{m_1}}{[m_1]_q^{n_1} \cdots [m_k]_q^{n_k}},
  \]
  with usual \( q \)-numbers:
  \[
  [m]_q = \frac{1 - q^m}{1 - q} = 1 + q + \cdots + q^{m-1}.
  \]
For any positive integers $n_1, \ldots n_k$ with $n_1 \geq 2$, the $q$-MZV $\zeta_q(n_1, \ldots, n_k)$ makes sense for any complex $q$ with $|q| \leq 1$, and we have:

$$\lim_{q \uparrow 1} \zeta_q(n_1, \ldots, n_k) = \zeta(n_1, \ldots, n_k).$$

Here, $q \uparrow 1$ means $q \to 1$ inside an angular sector:

$$\text{Arg}(q - 1) \in \left[\frac{\pi}{2} + \varepsilon, \frac{3\pi}{2} - \varepsilon\right].$$
An alternative description in terms of the operator $P_q$ will be very convenient:

$$
\bar{3}_q(n_1, \ldots, n_k) := (1 - q)^{-w} \bar{3}_q(n_1, \ldots, n_k)
$$

$$
= \sum_{m_1 > \cdots > m_k > 0} q^{m_1} (1 - q^{m_1})^{n_1} \cdots (1 - q^{m_k})^{n_k}
$$

$$
= P_{q}^{n_1} \circ \bar{Y} \circ \cdots \circ P_{q}^{n_k} \circ \bar{Y}[1](t) \big|_{t=q}.
$$

where we recall that $\bar{Y}$ is the operator of multiplication by

$$
\bar{y} : t \mapsto \frac{t}{1 - t}.
$$
Other models of \( q \)-MZVs

\[
\zeta_S(q)(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} [m_1]^{n_1} q \cdots [m_k]^{n_k} q = \text{Li}_{n_1, \ldots, n_k}(1).
\]

Zhao-Bradley model (2003) (\( k = 1 \): Kaneko, Kurokawa, and Wakayama).

\[
\zeta(q)(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} q^{m_1}(n_1 - 1) + \cdots + q^{m_k}(n_k - 1) [m_1]^{n_1} q \cdots [m_k]^{n_k} q.
\]

Multiple divisor functions (Bachmann-Kühn, 2013):

\[
[n_1, \ldots, n_k] = 1 \left( n_1 - 1 \right)! \cdots \left( n_k - 1 \right)! \sum_{j > 0} \left( \sum_{m_1 > \cdots > m_k \geq 1} m_1^{v_1} + \cdots + m_k^{v_k} = j \right) v_1^{n_1 - 1} \cdots v_k^{n_k - 1} q^j.
\]
Other models of $q$MZVs

- Schlesinger model (2001):

$$\zeta_S(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} \frac{1}{[m_1]_q^{n_1} \cdots [m_r]_q^{n_k}} = \operatorname{Li}_{n_1, \ldots, n_k}^q(1).$$
Other models of $q$MZVs

- Schlesinger model (2001):

$$
\zeta^S_q(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} \frac{1}{[m_1]_q^{n_1} \cdots [m_r]_q^{n_k}} = \text{Li}^q_{n_1, \ldots, n_k}(1).
$$

- Zhao-Bradley model (2003)

(k = 1: Kaneko, Kurokawa and Wakayama).

$$
\zeta_q(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} \frac{q^{m_1(n_1-1)+\cdots+m_k(n_k-1)}}{[m_1]_q^{n_1} \cdots [m_r]_q^{n_k}}.
$$
Other models of $q$MZVs

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  \[
  \zeta^S_q(n_1, \ldots, n_k) := \sum_{m_1 > \cdots > m_k \geq 1} \frac{1}{[m_1]_q^{n_1} \cdots [m_r]_q^{n_k}} = \text{Li}^q_{n_1, \ldots, n_k}(1).
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  \]

- Multiple divisor functions (Bachmann-Kühn, 2013):
  \[
  [n_1, \ldots, n_k] = \frac{1}{(n_1-1)! \cdots (n_k-1)!} \sum_{j>0} \left( \sum_{m_1 > \cdots > m_k \geq 1} \frac{v_1^{n_1-1} \cdots v_k^{n_k-1}}{m_1 v_1 + \cdots + m_k v_k = j} \right) q^j.
  \]
Extension to arguments of any sign

The iterated sum defining $\bar{\zeta}_q(n_1, \ldots, n_k)$ makes perfect sense in $\mathbb{Q}[[q]]$ for any $n_1, \ldots, n_k \in \mathbb{Z}$.

Moreover it also makes sense when specializing $q$ to a complex number of modulus $< 1$:

$$|\bar{\zeta}_q(n_1, \ldots, n_k)| \leq |q|^k (1 - |q|)^{-w' - k},$$

with $w' := \sum_{i=1}^k |n_i|$.

For any $n_1, \ldots, n_k \in \mathbb{Z}$ we still have (with $P_q^{-1} = D_q$):

$$\bar{\zeta}_q(n_1, \ldots, n_k) = P_q^{n_1} \circ \overline{Y} \circ \cdots \circ P_q^{n_1} \circ \overline{Y}[1](t)|_{t=q}.$$
Examples

\[\bar{\zeta}_q(0) = \sum_{q>0} q^m = \frac{q}{1 - q},\]

\[\bar{\zeta}_q(0, \ldots, 0)_k = \left( \frac{q}{1 - q} \right)^k,\]

\[\bar{\zeta}_q(-1) = \sum_{m>0} q^m (1 - q^m) = \frac{q}{1 - q} - \frac{q^2}{1 - q^2} .\]

\[\bar{\zeta}_q(1) = \sum_{m>0} \frac{q^m}{1 - q^m} \text{ (Lambert series)}\]
Double $q$-shuffle relations

- The $q$-MZVs described above admit both $q$-shuffle and $q$-quasi-shuffle relations.
- Double $q$-shuffle relations have been also settled recently (2013) by Yoshihiro Takeyama in the Bradley model.
\textit{q-shuffle relations}

- Let $\tilde{X}$ be the alphabet $\{d, y, p\}$.
- Let $\mathcal{W}$ be the set of words on the alphabet $\tilde{X}$, ending with $y$ and subject to
  
  $$dp = pd = 1,$$
  
  where $1$ is the empty word.
$q$-shuffle relations

- Let $\tilde{X}$ be the alphabet $\{d, y, p\}$.
- Let $W$ be the set of words on the alphabet $\tilde{X}$, ending with $y$ and subject to

$$dp = pd = 1,$$

where 1 is the empty word.
- Any nonempty word in $W$ writes uniquely $v = p^{n_1} y \cdots p^{n_k} y$, with $n_1, \ldots, n_k \in \mathbb{Z}$. 

**q-shuffle relations**

- Let $\tilde{X}$ be the alphabet $\{d, y, p\}$.
- Let $W$ be the set of words on the alphabet $\tilde{X}$, ending with $y$ and subject to
  
  \[ dp = pd = 1, \]

  where 1 is the empty word.
- Any nonempty word in $W$ writes uniquely $v = p^{n_1} y \cdots p^{n_k} y$, with $n_1, \ldots, n_k \in \mathbb{Z}$.
- Now define:
  
  \[ \tilde{\delta}_q (p^{n_1} y \cdots p^{n_k} y) := \tilde{\delta}_q (n_1, \ldots, n_k) \]

  and extend linearly.
\textbullet\  \textit{q}-shuffle product recursively given (w.r.t. length of words) by $1\shuffle v = v\shuffle 1 = v$ and:

\begin{align*}
(yv)\shuffle u &= v\shuffle (yu) = y(v\shuffle u), \\
pv\shuffle pu &= p(v\shuffle pu) + p(pv\shuffle u) - p(v\shuffle u), \\
dv\shuffle du &= v\shuffle du + dv\shuffle u - d(v\shuffle u), \\
dv\shuffle pu &= pu\shuffle dv = d(v\shuffle pu) + dv\shuffle u - v\shuffle u.
\end{align*}

for any $u, v \in W$. \hspace{1cm} \textbullet\ Explanation

\textbullet\  The product $\shuffle$ is \textbf{commutative} and \textbf{associative}.

\textbullet\  The \textit{q}-shuffle relations write:

$$\bar{\delta}_q(u)\bar{\delta}_q(v) = \bar{\delta}_q(u\shuffle v).$$

\hspace{1cm} \textbullet\ return to computation
\(q\)-quasi-shuffle relations

- \(\tilde{Y} = \) alphabet \(\{z_n, n \in \mathbb{Z}\}\), with internal product \(z_i \diamond z_j = z_{i+j}\).
- Let \(\tilde{Y}^*\) be set of words with letters in \(\tilde{Y}\).
- Let \(*\) be the ordinary quasi-shuffle product on \(\mathbb{Q}\langle \tilde{Y} \rangle\).
- Let \(T\) be the shift operator defined for any word \(u\) by:

\[
T(z_n u) := (z_n - z_{n-1}) u.
\]

- The \(q\)-quasi-shuffle product \(\uplus\uplus\) is (uniquely) defined by:

\[
T(u \uplus \uplus v) = Tu \ast Tv.
\]
Define \( \overline{\delta}_q \uplus (z_{n_1} \cdots z_{n_k}) := \overline{\delta}_q(n_1, \ldots, n_k) \) and extend linearly.

the \( q \)-quasi-shuffle relations write:

\[
\overline{\delta}_q \uplus (u) \overline{\delta}_q \uplus (v) = \overline{\delta}_q \uplus (u \uplus v)
\]

for any words \( u, v \in \tilde{Y}^* \).
Define $\bar{\bar{\zeta}}^{\uplus\uplus}_q(z_{n_1} \cdots z_{n_k}) := \bar{\bar{\zeta}}_q(n_1, \ldots, n_k)$ and extend linearly.

The $q$-quasi-shuffle relations write:

$$\bar{\bar{\zeta}}^{\uplus\uplus}_q(u) \bar{\bar{\zeta}}^{\uplus\uplus}_q(v) = \bar{\bar{\zeta}}^{\uplus\uplus}_q(u \uplus \downarrow v)$$

for any words $u, v \in \tilde{\tilde{Y}}^*$. 

Example of $q$-quasi-shuffle relation: for any $a, b \in \mathbb{Z}$,

$$\bar{\bar{\zeta}}_q(a) \bar{\bar{\zeta}}_q(b) = \bar{\bar{\zeta}}_q(a, b) + \bar{\bar{\zeta}}_q(b, a) + \bar{\bar{\zeta}}_q(a + b) - \bar{\bar{\zeta}}_q(a, b - 1) - \bar{\bar{\zeta}}_q(b, a - 1) - \bar{\bar{\zeta}}_q(a + b - 1).$$

Note that the weight is not conserved, contrarily to the classical case.
In terms on "non-modified" $q$-MZVs, the previous example becomes:

$$\zeta_q(a)\zeta_q(b) = \zeta_q(a, b) + \zeta_q(b, a) + \zeta_q(a + b) - (1 - q)\left[\zeta_q(a, b - 1) + \zeta_q(b, a - 1) + \zeta_q(a + b - 1)\right].$$

In the limit $q \uparrow 1$, the "weight drop term" disappears, and we recover the classical quasi-shuffle relation.
Important remark

There are no regularization relations in this picture. The swap

$$\tau : \tilde{Y}^* \to \mathcal{W}$$

is defined by:

$$\tau(z_{n_1} \cdots z_{n_k}) := p^{n_1 - 1}y \cdots p^{n_k - 1}y,$$

and the change of coding writes itself:

$$\tilde{\mathfrak{d}}_q^{\uparrow \downarrow} = \tilde{\mathfrak{d}}_q^{\uparrow \downarrow} \circ \tau$$

in full generality.
Summing up, the double $q$-shuffle relations write themselves as follows:

for any $u, v \in \tilde{Y}^*$ and for any $u', v' \in W$,

\[
\begin{align*}
\tilde{3}_q^{\updownarrow \downarrow} (u)\tilde{3}_q^{\updownarrow \downarrow} (v) &= \tilde{3}_q^{\updownarrow \downarrow} (u \downarrow \downarrow v), \\
\tilde{3}_q^{\downarrow \downarrow} (u')\tilde{3}_q^{\downarrow \downarrow} (v') &= \tilde{3}_q^{\downarrow \downarrow} (u' \downarrow \downarrow v'),
\end{align*}
\]

and we also have:

\[
\tilde{3}_q^{\updownarrow \downarrow} = \tilde{3}_q^{\downarrow \downarrow} \circ r.
\]
An example of computation using double $q$-shuffle relations

Using $q$-quasi-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).$$
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).$$

- Using $q$-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q^{\shuffle}(py) \bar{\zeta}_q^{\shuffle}(ppy)$$
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

$$
\bar{\vartheta}_q(1)\bar{\vartheta}_q(2) = \bar{\vartheta}_q(1, 2) + \bar{\vartheta}_q(2, 1) + \bar{\vartheta}_q(3) - \bar{\vartheta}_q(1, 1) - \bar{\vartheta}_q(2, 0) - \bar{\vartheta}_q(2).
$$

- Using $q$-shuffle:

$$
\bar{\vartheta}_q(1)\bar{\vartheta}_q(2) = \bar{\vartheta}_q(1)\downarrow\downarrow(py)\bar{\vartheta}_q(2)\downarrow\downarrow(ppy) = \bar{\vartheta}_q(1)\downarrow\downarrow(py\downarrow\downarrow ppy)
$$
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

$$\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2) = \bar{\mathfrak{z}}_q(1,2) + \bar{\mathfrak{z}}_q(2,1) + \bar{\mathfrak{z}}_q(3) - \bar{\mathfrak{z}}_q(1,1) - \bar{\mathfrak{z}}_q(2,0) - \bar{\mathfrak{z}}_q(2).$$

- Using $q$-shuffle:

$$\bar{\mathfrak{z}}_q(1)\bar{\mathfrak{z}}_q(2) = \bar{\mathfrak{z}}_q (py) \bar{\mathfrak{z}}_q (ppy) = \bar{\mathfrak{z}}_q (py \shuffle ppy) = \bar{\mathfrak{z}}_q (p(y \shuffle ppy + py \shuffle py - y \shuffle py)).$$
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:
\[
\bar{\delta}_q(1)\bar{\delta}_q(2) = \bar{\delta}_q(1,2) + \bar{\delta}_q(2,1) + \bar{\delta}_q(3) - \bar{\delta}_q(1,1) - \bar{\delta}_q(2,0) - \bar{\delta}_q(2).
\]

- Using $q$-shuffle:
\[
\begin{align*}
\bar{\delta}_q(1)\bar{\delta}_q(2) &= \bar{\delta}_q(p y)\bar{\delta}_q(p p y) \\
&= \bar{\delta}_q(p ypp y) \\
&= \bar{\delta}_q(p (ypp y + p ypp y - ypp y)) \\
&= \bar{\delta}_q(p (ypp y + p(2ypp y - yy) - ypp y)).
\end{align*}
\]
An example of computation using double $q$-shuffle relations

- **Using $q$-quasi-shuffle:**

$$
\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).
$$

- **Using $q$-shuffle:**

\[
\begin{align*}
\bar{\zeta}_q(1)\bar{\zeta}_q(2) &= \bar{\zeta}_q(p y)\bar{\zeta}_q(p y) \\
&= \bar{\zeta}_q(p y p y) \\
&= \bar{\zeta}_q(p y y y + p y y + p y - y y) \\
&= \bar{\zeta}_q(p y y y + 2 p y y - p y y - p y y)
\end{align*}
\]
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).$$

- Using $q$-shuffle:

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An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

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\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).
$$

- Using $q$-shuffle:

$$
\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q pym + ppyy - yyy - ppy.
$$
An example of computation using double $q$-shuffle relations

- **Using $q$-quasi-shuffle:**

$$
\bar{\delta}_q(1) \bar{\delta}_q(2) = \bar{\delta}_q(1, 2) + \bar{\delta}_q(2, 1) + \bar{\delta}_q(3) - \bar{\delta}_q(1, 1) - \bar{\delta}_q(2, 0) - \bar{\delta}_q(2).
$$

- **Using $q$-shuffle:**

\[
\begin{align*}
\bar{\delta}_q(1) \bar{\delta}_q(2) &= \bar{\delta}_q(\text{py}) \bar{\delta}_q(\text{ppy}) \\
&= \bar{\delta}_q(\text{py} \text{ppy}) \\
&= \bar{\delta}_q\left(p(y \text{ppy} + py \text{py} - y \text{py})\right) \\
&= \bar{\delta}_q\left(p(yppy + p(2ypy - yy) - ypy)\right) \\
&= \bar{\delta}_q\left(pyppy + 2ppy - ppyy - pppy\right) \\
&= \bar{\delta}_q(1, 2) + 2 \bar{\delta}_q(2, 1) - \bar{\delta}_q(2, 0) - \bar{\delta}_q(1, 1).
\end{align*}
\]
An example of computation using double $q$-shuffle relations

- **Using** $q$-quasi-shuffle:

  \[ \tilde{\zeta}_q(1)\tilde{\zeta}_q(2) = \tilde{\zeta}_q(1, 2) + \tilde{\zeta}_q(2, 1) + \tilde{\zeta}_q(3) - \tilde{\zeta}_q(1, 1) - \tilde{\zeta}_q(2, 0) - \tilde{\zeta}_q(2). \]

- **Using** $q$-shuffle:

  \[
  \tilde{\zeta}_q(1)\tilde{\zeta}_q(2) = \tilde{\zeta}_q\left(p y \tilde{\zeta}_q\left(pp y\right)\right) \\
  = \tilde{\zeta}_q\left(p y \tilde{\zeta}_q\left(pp y\right)\right) \\
  = \tilde{\zeta}_q\left(p(yp y + py \tilde{\zeta}_q\left(pp y\right) - y \tilde{\zeta}_q\left(pp y\right))\right) \\
  = \tilde{\zeta}_q\left(p(yp y + p(2y p y - y y) - y y)\right) \\
  = \tilde{\zeta}_q\left(p(yp y + 2pp y y - pp y - p y y)\right) \\
  = \tilde{\zeta}_q\left(1, 2\right) + 2 \tilde{\zeta}_q\left(2, 1\right) - \tilde{\zeta}_q\left(2, 0\right) - \tilde{\zeta}_q\left(1, 1\right). \]
An example of computation using double $q$-shuffle relations

- Using $q$-quasi-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(1,2) + \bar{\zeta}_q(2,1) + \bar{\zeta}_q(3) - \bar{\zeta}_q(1,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(2).$$

- Using $q$-shuffle:

$$\bar{\zeta}_q(1)\bar{\zeta}_q(2) = \bar{\zeta}_q(py)\bar{\zeta}_q(ppy)$$

$$= \bar{\zeta}_q(py\shuffle ppy)$$

$$= \bar{\zeta}_q(p(y\shuffle ppy + py\shuffle py - y\shuffle py))$$

$$= \bar{\zeta}_q(p(yppy + p(2ypy - yy) - ypy))$$

$$= \bar{\zeta}_q(pyyyyMMdd)$$

$$= \bar{\zeta}_q(1,2) + 2\bar{\zeta}_q(2,1) - \bar{\zeta}_q(2,0) - \bar{\zeta}_q(1,1).$$
Hence,

\[ \bar{\zeta}_q(2,1) = \bar{\zeta}_q(3) - \bar{\zeta}_q(2), \]

thus recovering Euler's regularization relation \[ \zeta(2,1) = \zeta(3) \] in the limit \( q \to 1 \).
Hence,

$$\bar{\zeta}_q(2,1) = \bar{\zeta}_q(3) - \bar{\zeta}_q(2),$$
or equivalently,

$$\zeta_q(2,1) = \zeta_q(3) - (1 - q)\zeta_q(2).$$
Hence,
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thus recovering Euler's regularization relation
\[ \zeta(2,1) = \zeta(3) \]
in the limit \( q \nearrow 1. \)

Perspectives and open problems

- Are the double shuffle relations the only ones among our $q$MZVs?
- Combinatorial description of the $q$-shuffle product $\shuffle$. Find a compatible coproduct.
- Parameter $q$ yields a regularisation of MZVs. What about renormalization for $q \to 1$?
Perspectives and open problems

- Are the double shuffle relations the only ones among our $q$MZVs?
- Combinatorial description of the $q$-shuffle product $⊔⊔$. Find a compatible coproduct.
- Parameter $q$ yields a regularisation of MZVs. What about renormalization for $q \to 1$?

Behaviour of Lambert series for $q \to 1$ or $q \to e^{2i\pi/n}$: cf. Dorigoni-Kleinschmidt (2020).
Outline

Multiple zeta values
Extension to arguments of any sign
The renormalisation group
q-multiple zeta values

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Thank you for your attention!