MRS[1] factorizations and applications

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INTRODUCTION

Zeta functions in several variables and polyzetas

Let $\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r | \forall m = 1,\ldots,r, \Re(s_1)+\ldots+\Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1,\ldots,s_r) \in \mathcal{H}_r$

$$\zeta(s_1,\ldots,s_r) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}.$$

By a Abel's theorem, for $n \in \mathbb{N}, z \in \mathbb{C}, |z|< 1$, this value can be obtained as

$$\zeta(s_1,\ldots,s_r) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z),$$

where the following functions are well defined for $(s_1,\ldots,s_r) \in \mathbb{C}^r, |z|< 1$

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \text{ and } \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n.$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1,\ldots,s_r}(1)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1, r>0}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1,\ldots,s_r}(+\infty)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1, r>0}}.$$

We use the correspondence between words over $X = \{x_0, x_1\}$, $Y = \{y_k\}_{k\geq 1}$:

$$(s_1,\ldots,s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 \in X^*x_1.$$

$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_{z_0}^{z}(x_0^{s_1-1}x_1\ldots x_0^{s_k-1}x_1)$, where $z_0 \rightsquigarrow z$ is a path in a simply connected domain, $\widetilde{\mathbb{C}\setminus\{0,1\}}$, with a subdivision $(z_0, z_1, \ldots, z_k, z)$ and

$$\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \quad \text{with} \quad \begin{cases}\omega_0(z) = dz/z,\\ \omega_1(z) = dz/(1-z).\end{cases}$$

Words and noncommutative formal series

- Let $(X^*, 1_{X^*})$ (resp. $(Y^*, 1_{Y^*})$) be the free monoid generated by X (resp. Y) equipped with the order $x_1 \succ x_0$ (resp. $y_1 \succ y_2 \succ \ldots$).
$\mathcal{L}yn\mathcal{X}$: set of Lyndon words over $\mathcal{X}$ which denotes X or Y.
$A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$): set of polynomials (resp. formal series) over $\mathcal{X}$ with coefficients in the commutative ring A containing $\mathbb{Q}$.
- On $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$ (resp. $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{⧢}, \varepsilon)$), for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), one defines
  - $\Delta_{\sqcup\!\sqcup}\, x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x$, or equivalently $u \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup u = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v)$,
  - $\Delta_{⧢}\, y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l$, or equivalently $u ⧢ 1_{Y^*} = 1_{Y^*} ⧢ u = u$ and
  $x_i u ⧢ y_j v = y_i(u ⧢ y_j v) + y_j(y_i u ⧢ v) + y_{i+j}(u ⧢ v)$.
- Considering A as the differential ring of holomorphic functions on a simply connected domain Ω, denoted by $(\mathcal{H}(\Omega), \partial)$ and equipped with 1_Ω as neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, as follows

$$\mathbf{d}S = \sum_{w \in \mathcal{X}^*} (\partial\langle S|w\rangle) w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

$\mathrm{Const}(\mathcal{H}(\Omega)) = \mathbb{C}.1_\Omega$ and $\mathrm{Const}(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle) = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

First structures of polylogarithms and harmonic sums

1. $\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective[2]

$$\begin{array}{rcl} \mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \to & (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \mapsto & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} \ (\textit{i.e.}\ \mathrm{Li}_{s_1,\ldots,s_r}) \\ x_0^k & \mapsto & \log^k(z)/k!. \end{array}$$

Thus, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ is $\mathbb{C}$-algebraically independent.

2. The following morphism of algebras is injective

$$\begin{array}{rcl} \mathrm{P}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) & \to & (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot, 1), \\ w & \mapsto & \mathrm{P}_w(z) := \dfrac{\mathrm{Li}_{\pi_X w}(z)}{1-z} = \displaystyle\sum_{n\geq 0}\mathrm{H}_w(n)z^n. \end{array}$$

Hence, $\{\mathrm{P}_w\}_{w\in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-algebraically independent, for[3] $\odot$.

3. The following morphism of algebras is injective

$$\begin{array}{rcl} \mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) & \to & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \\ y_{s_1}\ldots y_{s_r} & \mapsto & \mathrm{H}_{y_{s_1}\ldots y_{s_r}} \ (\textit{i.e.}\ \mathrm{H}_{s_1,\ldots,s_r}). \end{array}$$

Hence, $\{\mathrm{H}_w\}_{w\in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ is $\mathbb{C}$-algebraically independent.

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[2] For $(s_1,\ldots,s_r)\in\mathbb{N}_+^r, r\geq 1, k\geq 0$.

[3] For any $u,v\in Y, \mathrm{P}_u\odot\mathrm{P}_v = \mathrm{P}_{u ⧢ v}$.

Towards more about structure of $\mathcal{Z}$

4. The following polymorphism of algebras is surjective[4]

$$\zeta : \begin{array}{c}(\mathbb{C}[\mathcal{L}ynX - X], ш, 1_{X^*})\\ (\mathbb{C}[\mathcal{L}ynY - \{y_1\}], \text{ш}_-, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{c} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \\ y_{s_1}\ldots y_{s_r}\end{array} \mapsto \zeta(s_1, \ldots, s_r).$$

$$\forall l_1, l_2 \in \mathcal{L}ynX - X, \zeta(l_1)\zeta(l_2) = \zeta(l_1 ш l_2) = \zeta((\pi_Y l_1) \text{ш}_- (\pi_Y l_2)).$$

ζ can be extended as characters:

$$\zeta_ш : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \to (\mathcal{Z}, ., 1), \quad \zeta_{\text{ш}_-} : (\mathbb{C}\langle Y\rangle, \text{ш}_-, 1_{Y^*}) \to (\mathcal{Z}, ., 1),$$

$$\zeta_ш(x_0) = 0 = \log(1),$$

$$\zeta_ш(x_1) = 0 = \mathrm{f.p.}_{z\to 1}\log(1-z), \quad \{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}},$$

$$\zeta_{\text{ш}_-}(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), \quad \{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}.$$

Conjecture 1 (Zagier's dimension conjecture)

$\forall k \ge 1, \mathcal{A}_k := \mathrm{span}_{\mathbb{Z}}\{\zeta(s_1, \ldots, s_r), s_1 + \ldots + s_r = k\}_{s_1+\ldots+s_r \in \mathcal{H}_r \cap \mathbb{N}^r, r\ge 0}$, and $d_k := \dim_{\mathbb{Z}} \mathcal{A}_k$. Then $d_k = d_{k-2} + d_{k-3}$ with $d_0 = 1, d_1 = 0, d_2 = 1$.

$\mathcal{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\zeta(w), |w| = k\}_{w\in x_0X^*x_1} = \mathrm{span}_{\mathbb{Q}}\{\zeta(w), (w) = k\}_{w\in(Y-\{y_1\})Y^*}$.

Then[5] $\mathcal{Z}_k = \mathbb{Q}\otimes_{\mathbb{Z}} \mathcal{A}_k$. $\bigoplus_{k\ge 0}\mathcal{A}_k \to \mathcal{Z}$ is injective ? $\mathcal{Z} = \bigoplus_{k\ge 0}\mathcal{Z}_k$?

[4] For $(s_1, \ldots, s_r) \in \mathbb{N}_+^r, r \ge 1$.

[5] For any $w = x_{s_1}\ldots x_{s_r} \in \mathcal{X}^*, |w| = r$. If $\mathcal{X} = X$ then $(w) = |w|$ and if $\mathcal{X} = Y$ then $(w) = |\pi_X w| = s_1 + \ldots + s_r$ being weight of $(s_1, \ldots, s_r)$.

ABEL LIKE THEOREMS VIA BIALGEBRAS

Comb. of noncommutative co-commutative bialgebras

Concatenation-shuffle bialgebra[6], $(A\langle\mathcal{X}\rangle, \mathtt{conc}, \Delta_{\sqcup\!\sqcup}, 1_{\mathcal{X}^*}, \varepsilon)$:

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} S_w\otimes P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l} \quad \text{(MRS-factorization)},$$

where $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ is a basis of $\mathcal{L}ie_A\langle\mathcal{X}\rangle$, defined by

$P_l = l$ if $l\in\mathcal{X}$ and $P_l = [P_u, P_v]$ if $l\in\mathcal{L}yn\mathcal{X}, \mathrm{st}(l) = (u, v)$,

$\{P_w\}_{w\in\mathcal{X}^*}$ is the PBW basis of $\mathcal{U}(\mathcal{L}ie_A\langle\mathcal{X}\rangle)$ and $\{S_w\}_{w\in\mathcal{X}^*}$ is its dual basis, containing the pure transcendence basis, $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, of $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$.

Concatenation-stuffle bialgebra[7], $(A\langle Y\rangle, \mathtt{conc}, \Delta_{\uplus}, 1_{Y^*}, \varepsilon)$:

$$\mathcal{D}_{Y} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w\otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes \Pi_l} \quad (\uplus - \text{modified MRS-factorization}),$$

where $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ is a basis of $\mathrm{Prim}(Y)$, defined by[8]

$\Pi_l = \pi_1(l)$ if $l\in Y$ and $\Pi_l = [\Pi_u, \Pi_v]$ if $l\in\mathcal{L}ynY, \mathrm{st}(l) = (u, v)$,

$\{\Pi_w\}_{w\in X^*}$ is the PBW basis of $\mathcal{U}(\mathrm{Prim}(Y))$ and $\{\Sigma_w\}_{w\in Y^*}$ is its dual basis, containing the pure transcendence basis, $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$, of $(A\langle Y\rangle, \uplus, 1_{Y^*})$.

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[6] Letters are primitive, for $\Delta_{\sqcup\!\sqcup}$.

[7] Only letter y_1 is primitive, for $\Delta_{\uplus}$.

[8] π_1 denotes a $\uplus$-modified eulerian projector.

Noncommutative generating series

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z) w = (\mathrm{Li}_\bullet \otimes \mathtt{Id})\mathcal{D}_X = e^{-\log(1-z)x_1}\mathrm{L}_{\mathtt{reg}}(z) e^{\log(z)x_0},$$

$$\mathrm{H}(n) := \sum_{w\in Y^*} \mathrm{H}_w(n) w = (\mathrm{H}_\bullet \otimes \mathtt{Id})\mathcal{D}_Y = e^{\mathrm{H}_{y_1}(n) y_1}\mathrm{H}_{\mathtt{reg}}(n),$$

where $\mathrm{L}_{\mathtt{reg}} := \prod^{\searrow}_{l\in\mathcal{L}ynX\setminus X} e^{\mathrm{Li}_{S_l} P_l}$ and $\mathrm{H}_{\mathtt{reg}} := \prod^{\searrow}_{l\in\mathcal{L}ynY\setminus\{y_1\}} e^{\mathrm{H}_{\Sigma_l}\Pi_l}$.

We put also[9]

$$Z_{ш} := \mathrm{L}_{\mathtt{reg}}(1) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX \\ l\neq x_0,x_1}} e^{\zeta(S_l)P_l} \quad \text{and} \quad Z_{ш\!\!\!+} := \mathrm{H}_{\mathtt{reg}}(+\infty) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynY \\ l\neq y_1}} e^{\zeta(\Sigma_l)\Pi_l}.$$

L satisfies[10]

$$(DE) \quad \mathbf{d}S = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)S \quad \text{and } \mathrm{L}(z) \sim_0 e^{x_0\log(z)}.$$

L and $Z_{ш}$ (resp. H and $Z_{ш\!\!\!+}$) are group-like, for $\Delta_{ш}$ (resp. $\Delta_{ш\!\!\!+}$).

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[9] The polynomials S_l and P_l (resp. Σ_l and Π_l) are homogenous in weight and $\zeta(S_l)$ (resp. $\zeta(\Sigma_l)$) is convergent, for $l \in \mathcal{L}ynX \setminus X$ (resp. $\mathcal{L}ynY \setminus \{y_1\}$).

[10] For $x_0 = A/2i\pi$ and $x_1 = -B/2i\pi$, (DE) is nothing else (KZ_3) and $Z_{ш}$ corresponds to the Drindfel'd associator, Φ_{KZ}.

More about generating series

Let $\gamma_\bullet$ be the character on $(\mathbb{C}\langle Y\rangle, \,⧢\,, 1_{Y^*})$ defined by $\gamma_{1_{Y^*}} = 1$ and[11]

$$\forall l \in \mathcal{L}ynY, \quad \gamma_{\Sigma_l} := \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\Sigma_l}(n) = \zeta(\Sigma_l), \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}.$$

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\Sigma_l}\Pi_l} = e^{\gamma y_1} Z_{⧢}.$$

Let us consider

$$\mathtt{Mono}(z) := \sum_{n\ge 0} \mathrm{P}_{y_1^n} y_1^n \in \mathcal{H}(\Omega)\langle\langle y_1\rangle\rangle \quad \text{and} \quad \mathtt{Const} := \sum_{k\ge 0} \mathrm{H}_{y_1^k} y_1^k.$$

Then[12]

$$\mathtt{Mono}(z) = (1-z)^{-1} e^{-\log(1-z)y_1} \quad \text{and}^{13} \quad \mathtt{Const} = \exp\Big(-\sum_{k\ge 0} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\Big).$$

Let us also consider[14]

$$B'(y_1) := \exp\Big(\sum_{k\ge 2} \zeta(k) \frac{(-y_1)^k}{k}\Big) \text{ and } B(y_1) := \exp\Big(\gamma y_1 - \sum_{k\ge 2} \zeta(k) \frac{(-y_1)^k}{k}\Big).$$

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[11] In particular, $\gamma_{\Sigma_{y_1}} = \gamma_{y_1} = \gamma$.

[12] Because $\mathrm{P}_{y_1^k}(z) = (1-z)^{-1}\,\mathrm{Li}_{x_1^k}(z)$ with $\mathrm{Li}_{x_1^k}(z) = (-\log(1-z))^k/k!, k \ge 1$.

[13] By Newton-Girard identity, or by $(ty_k)^* = \exp_{⧢}(-\sum_{n\ge 0} y_{nk}(-t)^n/n), k \ge 1$.

Note also that $\mathtt{Const}^{-1} = \sum_{n\ge 0} \mathrm{H}_{y_1^n}(-y_1)^n = \exp(\sum_{k\ge 0} \mathrm{H}_{y_k}(-y_1)^k/k)$.

[14] $B'(y_1)$ corresponds to the Ecalle's mould $\mathtt{Mono}$. $\mathbb{C}\langle\langle y_1\rangle\rangle \ni B(y_1) = \Gamma^{-1}(1+y_1)$.

Chen generating series of ω_0 and ω_1 along a path $z_0 \rightsquigarrow z$

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w \quad \text{with} \quad \begin{cases} \omega_0(z) = z^{-1} dz, \\ \omega_1(z) = (1-z)^{-1} dz. \end{cases}$$

By a Ree's theorem, $C_{z_0 \rightsquigarrow z}$ is group-like, for $\Delta_{\sqcup\!\sqcup}$, and is solution[15] of (DE).

Let g be the transformation $z \mapsto 1 - z$. Then $g^* \omega_0 = -\omega_1$ and $g^* \omega_1 = -\omega_0$. Hence,

$$C_{g(z_0) \rightsquigarrow g(z)} = \sum_{w \in X^*} \alpha_{g(z_0)}^{g(z)}(w) w = \sum_{w \in X^*} \alpha_{z_0}^{z}(w) \sigma(w) = \sigma(C_{z_0 \rightsquigarrow z}),$$

where σ is the morphism defined by $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$.

On the other hand, one has

$$\mathrm{L}(z) = C_{z_0 \rightsquigarrow z} \mathrm{L}(z_0) \quad \text{and} \quad \mathrm{L}(g(z)) = C_{g(z_0) \rightsquigarrow g(z)} \mathrm{L}(g(z_0)).$$

Since $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$ then

$$C_{g(z_0) \rightsquigarrow g(z)} = \sigma(\mathrm{L}(z) \mathrm{L}^{-1}(z_0)) \sim_{z_0 \to 0} \sigma(\mathrm{L}(z)) e^{x_1 \log(z_0)}.$$

Proposition 1

Let σ be the letter morphism s.t. $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$. Then

$$\mathrm{L}(1 - z) = \sigma(\mathrm{L}(z)) Z_{\sqcup\!\sqcup} .$$

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[15] It can be obtained by a convergent Picard iteration, for a discrete topology, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*} \rangle = 1_\Omega 1_{\mathcal{X}^*}$.

Abel like results and bridge equations

Since[16] $\mathrm{L}(z) = \sigma(\mathrm{L}(1-z))Z_{ш} = e^{x_0 \log(z)}\sigma(\mathrm{L}_{\mathrm{reg}}(1-z))e^{-x_1\log(1-z)}Z_{ш}$ then $\mathrm{L}(z) \sim_1 e^{-x_1\log(1-z)}Z_{ш}$ and then $\mathrm{H}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y Z_{ш}$.

Theorem 1 (first Abel like theorem)

$$\lim_{z\to 1} e^{y_1\log(1-z)}\pi_Y \mathrm{L}(z) = \pi_Y Z_{ш} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\mathrm{H}(n).$$

Corollary 2 (bridge equations)

$$Z_\gamma = B(y_1)\pi_Y Z_{ш} \iff Z_{⧢} = B'(y_1)\pi_Y Z_{ш}.$$

Remark 1

On the one hand, by identification coefficients, for $w \in X^*x_1$,
$\zeta_{ш}(w) = \langle Z_{ш} | w\rangle = \mathrm{f.p.}_{z\to 1}\mathrm{Li}_w(z), \quad \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}$.
On the other hand, by an ⧢-modified Radford theorem, for $w \in Y^*$,
$\zeta_{⧢}(w) = \langle Z_{⧢} | w\rangle = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n), \quad \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}$.
In particular[17], $\zeta_{ш}(x_1) = \zeta_{⧢}(y_1) = 0$.

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[16] By Hoffman's duality, *i.e.* $\zeta(\rho(\tilde{w})) = \zeta(w)$ (where ρ is the morphism defined by $\rho(x_0) = x_1, \rho(x_1) = x_0$ and $\tilde{w}$ is mirror of w), we get $\sigma(Z_{ш}^{-1}) = Z_{ш}$.

[17] These coefficients of singular and asymptotic expansions can be changed if we use other comparison scales.

Cloned Abel like results and cloned bridge equations

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle}$ and $\overline{\mathrm{L}} := \mathrm{L}e^C, \overline{Z}_{ш} := Z_{ш}\, e^C$.
Hence, $\overline{\mathrm{L}}(z) \sim_1 e^{-x_1 \log(1-z)}\overline{Z}_{ш}$ and then $\overline{\mathrm{H}}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y \overline{Z}_{ш}$.

Theorem 3 (cloned first Abel like theorem)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \overline{\mathrm{L}}(z) = \pi_Y \overline{Z}_{ш} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\overline{\mathrm{H}}(n).$$

If[18] $\overline{Z}_{ш} \in dm(A) := \{Z_{ш}\, e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0\}$
then[19] $\overline{Z}_\gamma = e^{\gamma y_1}\overline{Z}_{\text{⧢}}$ and it follows that

Corollary 4 (cloned bridge equations)

If $\overline{Z}_{ш} \in dm(A)$ *then* $(\overline{Z}_\gamma = B(y_1)\pi_Y \overline{Z}_{ш} \iff \overline{Z}_{\text{⧢}} = B'(y_1)\pi_Y \overline{Z}_{ш})$.

Remark 2

The local coordinates of $\overline{Z}_{\text{⧢}}$ *and* $\overline{Z}_{ш}$ *are homogenous polynomial on convergent polyzetas, with coefficients in* A. *Hence, if* $\gamma \notin A$ *then* γ *is transcendent over the* A*-algebra generated by convergent polyzetas.*

[18] $dm(A)$ contains $DM(A)$ introduced by P. Cartier and G. Racinet and it is a strict normal subgroup of $\mathrm{Gal}_A(DE)$ (recall that $\mathbb{Q} \subset A \subset \mathbb{C}$).

[19] For $w \in Y^*$, one has $\langle \overline{Z}_{\text{⧢}}|w\rangle = \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_w(n)$, $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}$ and $\langle \overline{Z}_\gamma|w\rangle = \mathrm{f.p.}_{n\to+\infty}\overline{\mathrm{H}}_w(n)$, $\{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}$.

COMPUTATIONAL EXAMPLES[20]

[20] Examples, in the sequel, use maple packages developed in the PhD theses of C. Bùi (2016), C. Costermans (2008) and H. Ngô (2016).

Generalized Euler's gamma constant

Identifying the coefficients of $y_1^k w, w \in X^*, k \in \mathbb{N}$ in $Z_\gamma = B(y_1)\pi_Y Z_{ш}$, one has

1. $$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Example 5

$$\gamma_{1,1} = \frac{1}{2}(\gamma^2 - \zeta(2)), \qquad \gamma_{1,1,1} = \frac{1}{6}(\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)).$$

2. $$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \mathbin{ш} \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

where $k \in \mathbb{N}_+, w \in Y^+$ and $b_{n,k}(t_1, \ldots, t_k)$ are Bell polynomials.

Example 6

$$\begin{aligned}
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4, \\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2 + (\zeta(2)\zeta(5) + \frac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7))\gamma \\
&\quad + \zeta(6,2) + \frac{19}{35}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5).
\end{aligned}$$

Homogenous polynomials relations[21] on local coordinates

Identifying the local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{ш}$, one has

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

[21] These polynomials relations are independent from γ and similarly for the case where the ring of their coefficients is the ring A.

Homogenous polynomials generating inside $\ker\zeta$

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2})=0$ $\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\uplus\Sigma_{y_2}-5\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\uplus\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$ $\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$ $\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1})=0$ $\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$ $\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$ $\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\uplus 2}-\frac{4}{21}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\uplus 3}-\frac{1}{2}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\uplus 3}+\frac{9}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\uplus 2}-\frac{9}{10}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2}-\frac{3}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\uplus 2}-\frac{1}{4}\Sigma_{y_3}^{\uplus 2})=0$ $\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\uplus 3})=0$ $\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\uplus 3}+\frac{3}{16}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$ $\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3}+\frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$ $\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$

One has $\mathcal{R}_{\mathcal{X}}\subseteq\ker\zeta$, where $\begin{cases}\mathcal{R}_Y:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}},\uplus,1_{Y^*})\\ \mathcal{R}_X:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX\setminus X},\sqcup\!\sqcup,1_{X^*})\end{cases}$.

Noetherian rewriting system & irreducible coordinates[22]

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

$$\mathcal{Z}_{irr}^{\le 2}(\mathcal{X}) \subset \cdots \mathcal{Z}_{irr}^{\le p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\ge 2}\mathcal{Z}_{irr}^{\le p}(\mathcal{X}).$$

[22] The set of irreducible local coordinates forms algebraic generator system for $\mathcal{Z}$.

Noetherian rewriting system & totally ordered $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$

	Rewriting among $\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{S_l\}_{\mathcal{L}ynX-X}$
3	$\Sigma_{y_2y_1} \to \frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2} \to S_{x_0^2x_1}$
4	$\Sigma_{y_4} \to \frac{2}{5}\Sigma_{y_2}^2$	$S_{x_0^3x_1} \to \frac{2}{5}S_{x_0x_1}^2$
	$\Sigma_{y_3y_1} \to \frac{3}{10}\Sigma_{y_2}^2$	$S_{x_0^2x_1^2} \to \frac{1}{10}S_{x_0x_1}^2$
	$\Sigma_{y_2y_1^2} \to \frac{2}{3}\Sigma_{y_2}^2$	$S_{x_0x_1^3} \to \frac{2}{5}S_{x_0x_1}^2$
5	$\Sigma_{y_3y_2} \to 3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1} \to -\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1} \to -\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1} \to \frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2} \to \frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2} \to \frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3} \to \frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4} \to S_{x_0^4x_1}$
6	$\Sigma_{y_6} \to \frac{8}{35}\Sigma_{y_2}^3$	$S_{x_0^5x_1} \to \frac{8}{35}S_{x_0x_1}^3$
	$\Sigma_{y_4y_2} \to \Sigma_{y_3}^2 - \frac{4}{21}\Sigma_{y_2}^3$	$S_{x_0^4x_1^2} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_5y_1} \to \frac{2}{7}\Sigma_{y_2}^3 - \frac{1}{2}\Sigma_{y_3}^2$	$S_{x_0^3x_1x_0x_1} \to \frac{4}{105}S_{x_0x_1}^3$
	$\Sigma_{y_3y_1y_2} \to -\frac{17}{30}\Sigma_{y_2}^3 + \frac{9}{4}\Sigma_{y_3}^2$	$S_{x_0^3x_1^3} \to \frac{23}{70}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_2y_1} \to 3\Sigma_{y_3}^2 - \frac{9}{10}\Sigma_{y_2}^3$	$S_{x_0^2x_1x_0x_1^2} \to \frac{2}{105}S_{x_0x_1}^3$
	$\Sigma_{y_4y_1^2} \to \frac{3}{10}\Sigma_{y_2}^3 - \frac{3}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^2x_0x_1} \to -\frac{89}{210}S_{x_0x_1}^3 + \frac{3}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_2^2y_1^2} \to \frac{11}{63}\Sigma_{y_2}^3 - \frac{1}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^4} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_1^3} \to \frac{1}{21}\Sigma_{y_2}^3$	$S_{x_0x_1x_0x_1^3} \to \frac{8}{21}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_2y_1^4} \to \frac{17}{50}\Sigma_{y_2}^3 + \frac{3}{16}\Sigma_{y_3}^2$	$S_{x_0x_1^5} \to \frac{8}{35}S_{x_0x_1}^3$

$$\mathcal{L}_{irr}^{\le 2}(\mathcal{X}) \subset \cdots \mathcal{L}_{irr}^{\le p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\ge 2}\mathcal{L}_{irr}^{\le p}(\mathcal{X}).$$

$$\forall l \in \left\{\begin{matrix}\mathcal{L}ynY \setminus \{y_1\} \\ \mathcal{L}ynX \setminus X\end{matrix}\right\}, \quad \left\{\begin{matrix}\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y) \\ S_l \in \mathcal{L}_{irr}^{\infty}(X)\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\Sigma_l \to \Sigma_l \\ S_l \to S_l\end{matrix}\right\} \Leftrightarrow Q_l = 0.$$

STRUCTURE OF POLYZETAS

Identification of local coordinates $\{\zeta(S_l)\}_{l\in\in\mathcal{L}ynX-X}$ $\{\zeta(\Sigma_l)\}_{l\in\in\mathcal{L}ynY-\{y_1\}}$

In summary, the identification of local coordinates leads to

1. A family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(\mathcal{X})$ of $\mathcal{Z}$ constructed as follows
$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{Z}_{irr}^{\leq p}(\mathcal{X})$$
and their inverse image, by a section of ζ,
$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{L}_{irr}^{\leq p}(\mathcal{X})$$
such that the following restriction is bijective
$$\zeta : \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})] \to \mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}].$$
2. A family $\{Q_l\}_{\substack{l\in\mathcal{L}yn\mathcal{X}\\ l\neq y_1,x_0,x_1}}$ generates $\mathcal{R}_{\mathcal{X}} = (\mathbb{Q}\{Q_l\}_{\substack{l\in\mathcal{L}yn\mathcal{X}\\ l\neq y_1,x_0,x_1}}, ⧢, 1_{\mathcal{X}^*})$.
For any $l \in \mathcal{L}yn\mathcal{X}, l \neq y_1, x_0, x_1$, Q_l is homogenous of weight (l).
The following assertions are equivalent
   1. $Q_l = 0$,
   2. $\Sigma_l \to \Sigma_l$ (resp. $S_l \to S_l$),
   3. $\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y)$ (resp. $S_l \in \mathcal{L}_{irr}^{\infty}(X)$).

   If $Q_l \neq 0$ then its leading term is Σ_l (resp. S_l), transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, and there is $\Upsilon_l \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)]$ (resp. $U_l \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)]$), homogenous of weight (l), s.t. $Q_l = \Sigma_l - \Upsilon_l$ (resp. $S_l - U_l$), *i.e.* $\Sigma_l = Q_l + \Upsilon_l$ (resp. $S_l = Q_l + U_l$) and then $\Sigma_l \to \Upsilon_l$ (resp. $S_l \to U_l$).

Im and ker of ζ : $\begin{matrix}(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX-x}],⧢,1_{X^*})\\(\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}],\uplus,1_{Y^*})\end{matrix}\twoheadrightarrow(\mathcal{Z},.,1)$

Proposition 2

$$\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX-x}]=\mathcal{R}_X\oplus\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)]$$
$$\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}]=\mathcal{R}_Y\oplus\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)]$$

(as vector spaces associated to respective ⧢ or $\uplus$-subalgebras)[23].

We have seen that $\mathcal{R}_{\mathcal{X}}\subseteq\ker\zeta$. Now, let $Q\in\ker\zeta$, $\langle Q|1_{\mathcal{X}^*}\rangle=0$. Then $Q=Q_1+Q_2$ with $Q_1\in\mathcal{R}_{\mathcal{X}}$ and $Q_2\in\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Thus, $Q\equiv_{\mathcal{R}_{\mathcal{X}}}Q_1\in\mathcal{R}_{\mathcal{X}}$.

Corollary 7

$\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}]=\mathcal{Z}=\mathrm{Im}\,\zeta$ *and* $\mathcal{R}_{\mathcal{X}}=\ker\zeta$.

It would prove that $\mathcal{Z}$ is also graded because

$$\begin{aligned}\mathrm{Im}\,\zeta&\cong\mathbb{Q}1_{Y^*}\oplus(Y-\{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta\\&\cong\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta.\end{aligned}$$

Now, let $\xi:=\zeta(P)$, where $\mathbb{Q}\langle\mathcal{X}\rangle\ni P\notin\ker\zeta$, homogenous in weight. Each monomial $\xi^n, n\geq1$, is of different weight (because $\mathcal{Z}_p\mathcal{Z}_n\subset\mathcal{Z}_{p+n}$). Thus ξ could not satisfy $\xi^n+a_{n-1}\xi^{n-1}+\ldots=0$, with $a_{n-1},\ldots\in\mathbb{Q}$.

Any $s\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})$ is homogenous in weight then $\zeta(s)$ is transcendent over $\mathbb{Q}$.

[23] These decompositions and the rewriting systems can be viewed as variations of, respectively, the Lazard elimination and the Hall collecting process.

Concluding remarks

For any $l \in \mathcal{L}yn\mathcal{X}$, $l \neq y_1, x_0, x_1$, one has $l \succeq y_n$ (resp. $l \succeq x_0^{n-1}x_1$). In particular, $\Sigma_{y_n} = y_n \in \mathcal{L}ynY$ and $S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}ynX$. Next,

1. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is then irreducible and, by the Euler's identity about the ratio $\zeta(2k)/\pi^{2k}$, one deduces then, for $k > 1$, $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$,
2. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$.

Up to weight 12, the Zagier's dimension conjecture holds meaning that $\mathcal{Z}_{irr}^{\leq 12}(\mathcal{X})$ is algebraically independent over $\mathbb{Q}$:

$$\mathcal{Z}_{irr}^{\leq 12}(X) = \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(X) = \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1}, S_{x_0x_1^2x_0x_1^6}, S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(Y) = \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(Y) = \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.$$

THANK YOU FOR YOUR ATTENTION