Tropical Jacobian Conjecture

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CNRS

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Classical Jacobian Conjecture

Polynomial map \( f := (f_1, \ldots, f_n) : F^n \to F^n \) where the field \( F \) has characteristic 0. Its Jacobian \( J(f) := \det(\partial f_i/\partial x_j)_{1 \leq i, j \leq n} \).

Jacobian conjecture, Keller, 1939: if \( J(f) = 1 \) then \( f \) is an isomorphism and its inverse is also a polynomial map.

Theorem

(Ax, 1968; Grothendieck, 1966). For an algebraically closed field \( F \) if \( f \) is injective then \( f \) is bijective.

Model-theoretic proof: reduction to finite fields using Nullstellensatz.

Jacobian conjecture: a local isomorphism (due to the Implicit Function Theorem) implies a global isomorphism.

Example

(Pinchuk, 1994). When \( F = \mathbb{R} \) the conclusion of \( f \) being an isomorphism is wrong under the assumption \( J(f) > 0 \).
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**Tropical semi-ring**

*Tropical semi-ring* $T$ is endowed with operations $\oplus$, $\otimes$.

If $T$ is an ordered semi-group then $T$ is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If $T$ is an ordered (resp. abelian) group then $T$ is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\ominus := –$.

**Examples**

- $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_{\infty} := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. $\infty$ plays a role of 0, in its turn 0 plays a role of 1;
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**Tropical polynomials**

*Tropical monomial* $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{i_1} \otimes \cdots \otimes x_n^{i_n}$, its *tropical degree* $\text{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

*Tropical polynomial* $f = \bigoplus_j (a_j \otimes x_1^{\bar{j}1} \otimes \cdots \otimes x_n^{\bar{j}n}) = \min_j \{Q_j\}$; $x = (x_1, \ldots, x_n)$ is a *tropical zero* of $f$ if minimum $\min_j \{Q_j\}$ is attained for at least two different values of $j$. 

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Tropical Algebraic Rational Functions

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How to replace the Jacobian for non-smooth tropical algebraic rational maps (=tropical maps) \( f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n \) where \( f_1, \ldots, f_n \) are tropical algebraic rational functions? If \( f \) is an isomorphism then its inverse \( f^{-1} \) is also a tropical map.
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Weak version of a Tropical Jacobian Conjecture

For a tropical map \( f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n \) and a point \( p \in \mathbb{R}^n \) consider all \( n \)-dimensional polyhedra containing \( p \) on which \( f \) is linear, the \( n \times n \) matrices (\( = \)Jacobian matrices) of these linear maps denote by \( A_1, \ldots, A_k \), then \( J_i = \det(A_i) \), \( 1 \leq i \leq k \) are their Jacobians. The convex hull of \( A_1, \ldots, A_k \) denote by \( \partial_p(f) \).

Proposition

If \( \partial_p(f) \) does not contain a singular matrix for any \( p \in \mathbb{R}^n \) then \( f \) is an isomorphism.

The proof relies on Clarke's theorem (1974) that \( f \) (being Lipschitz) is a local homeomorphism. Then being proper (= the preimage of every compact is again compact) \( f \) is a (global) homeomorphism.
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Non-necessity of the Weak Conjecture

A tropical polynomial isomorphism \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a composition of a lower-triangular and an upper-triangular isomorphisms

\[
(x, y) \mapsto (x, y + \min\{\alpha x, \beta x\}), \ \alpha < \beta,
\]
\[
(x, y) \mapsto (x + \min\{ay, by\}, y), \ \ a < b.
\]

Then \( f(x, y) \) is linear on 4 pieces:

\[
f = (x + a(y + \alpha x), y + \alpha x) \quad \text{if} \quad x > 0, \ y + \alpha x > 0;
\]
\[
f = (x + b(y + \alpha x), y + \alpha x) \quad \text{if} \quad x > 0, \ y + \alpha x < 0;
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\( \partial_{(0,0)}(f) \) is the convex hull of the corresponding Jacobian matrices

\[
\begin{pmatrix}
1 + a\alpha & a \\
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\end{pmatrix}, \quad
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\end{pmatrix}, \quad
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\end{pmatrix}.
\]

The sum of the second and the third matrices is singular when

\[(\beta - \alpha)(b - a) = 4\] (in particular, one can put \( \beta = b = 2, \ \alpha = a = 0 \)).
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\begin{align*}
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\begin{align*}
    \left( \begin{array}{cc} 1 + a\alpha & a \\ \alpha & 1 \end{array} \right), \quad \left( \begin{array}{cc} 1 + b\alpha & b \\ \alpha & 1 \end{array} \right), \quad \left( \begin{array}{cc} 1 + a\beta & a \\ \beta & 1 \end{array} \right), \quad \left( \begin{array}{cc} 1 + b\beta & b \\ \beta & 1 \end{array} \right).
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Strong version of the tropical Jacobian conjecture

If a tropical map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism then all the Jacobians $J_i$ have the same sign, say $J_i > 0$ for all $i$.

**Theorem**

If $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a tropical polynomial map and all $J_i > 0$ then $f$ is an isomorphism.

**Example**

A tropical rational map $g : (x, y) \to (|x| - |y|, |x + y| - |x - y|)$ has all the positive Jacobians $J_i > 0$, but $g(x, y) = g(-x, -y)$ is not an isomorphism. Modifying $g$ one can construct a tropical polynomial map $\mathbb{R}^3 \to \mathbb{R}^3$ with all positive $J_i > 0$ being not an isomorphism.
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An algorithm to verify whether a tropical map is an isomorphism

A point $p \in \mathbb{R}^n$ is regular for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ if for any $x \in f^{-1}(p)$ its Jacobian $J_f(x) \neq 0$. By Sard’s lemma the set of regular values is dense.

**Theorem**

A necessary and sufficient condition for a tropical map $f : \mathbb{R}^n \to \mathbb{R}^n$ to be an isomorphism is that all the Jacobians $J_i$ have the same sign and $|f^{-1}(p)| = 1$ for at least one regular value $p \in \mathbb{R}^n$.

An algorithm yields a partition of $\mathbb{R}^n = \bigcup_i P_i$ into polyhedra $P_i$ such that $f$ is linear on each $P_i$. Then any point $p \in \mathbb{R}^n \setminus \bigcup_i f(\partial P_i)$ is regular. The algorithm tests whether $|f^{-1}(p)| = 1$. All this can be performed invoking linear programming.
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Tameness of tropical rational plane automorphisms

A *triangle* tropical rational plane automorphism has a form 
\[(x, y) \rightarrow (x, y + \min\{ax, bx\}), \ a, b \in \mathbb{Z}.
\] A linear tropical rational automorphism has a form
\[(x, y) \rightarrow (ax + by, cx + dy), \ a, b, c, d \in \mathbb{Z}, \ ad - bc = \pm 1.
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The group of tropical rational homogeneous automorphisms is generated by triangular and linear automorphisms.
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