

# A quotient of the ring of symmetric functions generalizing quantum cohomology

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**slides:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/cap2020.pdf)

[//www.cip.ifi.lmu.de/~grinberg/algebra/cap2020.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/cap2020.pdf)

**paper:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf)

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**overview:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/fpsac19.pdf)

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## What is this about?

- From a modern point of view, **Schubert calculus** (a.k.a. classical enumerative geometry, or Hilbert's 15th problem) is about two cohomology rings:

$$H^* \left( \underbrace{\text{Gr}(k, n)}_{\text{Grassmannian}} \right) \text{ and } H^* \left( \underbrace{\text{Fl}(n)}_{\text{flag variety}} \right)$$

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- In this talk, we are concerned with the first.
- Classical result: as rings,

$$\begin{aligned} H^*(\text{Gr}(k, n)) \\ \cong (\text{symmetric polynomials in } x_1, x_2, \dots, x_k \text{ over } \mathbb{Z}) \\ / (h_{n-k+1}, h_{n-k+2}, \dots, h_n)_{\text{ideal}}, \end{aligned}$$

where the  $h_i$  are complete homogeneous symmetric polynomials (to be defined soon).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

$$QH^*(Gr(k, n))$$

$$\cong (\text{symmetric polynomials in } x_1, x_2, \dots, x_k \text{ over } \mathbb{Z}[q])$$

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$$\begin{aligned} & \text{QH}^*(\text{Gr}(k, n)) \\ & \cong (\text{symmetric polynomials in } x_1, x_2, \dots, x_k \text{ over } \mathbb{Z}[q]) \\ & \quad / \left( h_{n-k+1}, h_{n-k+2}, \dots, h_{n-1}, h_n + (-1)^k q \right)_{\text{ideal}}. \end{aligned}$$

- Many properties of classical cohomology still hold here. In particular:  $\text{QH}^*(\text{Gr}(k, n))$  has a  $\mathbb{Z}[q]$ -module basis  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$  of (projected) Schur polynomials (to be defined soon), with  $\lambda$  ranging over all partitions with  $\leq k$  parts and each part  $\leq n - k$ . The structure constants are the **Gromov–Witten invariants**. References:

- Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
- Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

- **Goal:** Deform  $H^*(\text{Gr}(k, n))$  using  $k$  parameters instead of one, generalizing  $QH^*(\text{Gr}(k, n))$ .

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- The new ring has no geometric interpretation known so far, but various properties suggesting such an interpretation likely exists.
- I will now start from scratch and define standard notations around symmetric polynomials, then introduce the deformed cohomology ring algebraically.
- There is a number of open questions and things to explore.

## A more general setting: $\mathcal{P}$ and $\mathcal{S}$

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Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Let  $k \in \mathbb{N}$ .

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- Let  $\mathcal{S}$  denote the ring of *symmetric* polynomials in  $\mathcal{P}$ .  
These are the polynomials  $f \in \mathcal{P}$  satisfying

$$f(x_1, x_2, \dots, x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})$$

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- **Theorem (Artin  $\leq 1944$ ):** The  $\mathcal{S}$ -module  $\mathcal{P}$  is free with basis

$$(x^\alpha)_{\alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i} \quad (\text{or, informally: } "(x_1^{<1} x_2^{<2} \cdots x_k^{<k})").$$



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**Example:** For  $k = 3$ , this basis is  $(1, x_3, x_3^2, x_2, x_2x_3, x_2x_3^2)$ .

- The ring  $\mathcal{S}$  of symmetric polynomials in  $\mathcal{P} = \mathbf{k}[x_1, x_2, \dots, x_k]$  has several bases, usually indexed by certain sets of (integer) partitions.

First, let us recall what partitions are:

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**Examples:**  $(4, 2, 2, 0, 0, 0, \dots)$  and  $(3, 2, 0, 0, 0, 0, \dots)$  and  $(5, 0, 0, 0, 0, 0, \dots)$  are three partitions.  
 $(2, 3, 2, 0, 0, 0, \dots)$  and  $(2, 1, 1, 1, \dots)$  are not.

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- Thus there is a bijection

$$\{k\text{-partitions}\} \rightarrow \{\text{partitions with at most } k \text{ nonzero entries}\},$$
$$\lambda \mapsto (\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, 0, \dots).$$

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 $(2, 3, 2)$  is not.
- If  $\lambda \in \mathbb{N}^k$  is a  $k$ -partition, then its **Young diagram**  $Y(\lambda)$  is defined as a table made out of  $k$  left-aligned rows, where the  $i$ -th row has  $\lambda_i$  boxes.

**Example:** If  $k = 6$  and  $\lambda = (5, 5, 3, 2, 0, 0)$ , then

$$Y(\lambda) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & \\ \hline \square & \square & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} .$$

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- The same convention applies to partitions.

- For each  $m \in \mathbb{Z}$ , we let  $e_m$  denote the  $m$ -th *elementary symmetric polynomial*:

$$e_m = \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha|=m}} x^\alpha \in \mathcal{S}.$$

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- Note that  $e_m = 0$  when  $m > k$ .

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- **Theorem:**  $(h_\lambda)_\lambda$  is a  $k$ -partition is a basis of the  $\mathbf{k}$ -module  $\mathcal{S}$ . (Another basis!)

## Symmetric polynomials: the $s$ -basis (Schur polynomials)

- For each  $k$ -partition  $\lambda$ , we let  $s_\lambda$  be the  $\lambda$ -th Schur polynomial:

$$s_\lambda = \frac{\det \left( \left( x_i^{\lambda_j + k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)}{\det \left( \left( x_i^{k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)} \quad (\text{alternant formula})$$

$$= \det \left( (h_{\lambda_i - i + j})_{1 \leq i \leq k, 1 \leq j \leq k} \right) \quad (\text{Jacobi-Trudi}).$$

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$$s_\lambda = \frac{\det \left( \left( x_i^{\lambda_j + k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)}{\det \left( \left( x_i^{k - j} \right)_{1 \leq i \leq k, 1 \leq j \leq k} \right)} \quad (\text{alternant formula})$$

$$= \det \left( (h_{\lambda_i - i + j})_{1 \leq i \leq k, 1 \leq j \leq k} \right) \quad (\text{Jacobi-Trudi}).$$

- Theorem:** The equality above holds, and  $s_\lambda$  is a symmetric polynomial with nonnegative coefficients. Explicitly,

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard } \lambda\text{-tableau} \\ \text{with entries } 1, 2, \dots, k}} \prod_{i=1}^k x_i^{(\text{number of } i\text{'s in } T)},$$

where a *semistandard  $\lambda$ -tableau with entries  $1, 2, \dots, k$*  is a way of putting an integer  $i \in \{1, 2, \dots, k\}$  into each box of  $Y(\lambda)$  such that the entries **weakly** increase along rows and **strictly** increase along columns.

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- Theorem:** The equality above holds, and  $s_\lambda$  is a symmetric polynomial with nonnegative coefficients.
- Theorem:**  $(s_\lambda)_\lambda$  is a  $k$ -partition is a basis of the  $\mathbf{k}$ -module  $\mathcal{S}$ .

- If  $\lambda$  and  $\mu$  are two  $k$ -partitions, then the product  $s_\lambda s_\mu$  can be again written as a  $\mathbf{k}$ -linear combination of Schur polynomials (since these form a basis):

$$s_\lambda s_\mu = \sum_{\nu \text{ is a } k\text{-partition}} c_{\lambda, \mu}^\nu s_\nu,$$

where the  $c_{\lambda, \mu}^\nu$  lie in  $\mathbf{k}$ . These  $c_{\lambda, \mu}^\nu$  are called the *Littlewood-Richardson coefficients*.

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- **Theorem:** These Littlewood-Richardson coefficients  $c_{\lambda, \mu}^\nu$  are nonnegative integers (and count something).



- We have defined

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- **Proposition:** If  $\alpha \in \mathbb{Z}^k$ , then  $s_\alpha$  is either 0 or equals  $\pm s_\lambda$  for some  $k$ -partition  $\lambda$ .  
(So we get nothing really new.)

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$$\beta = (\alpha_1 + (k - 1), \alpha_2 + (k - 2), \dots, \alpha_k + (k - k)).$$

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- If  $\beta$  has a negative entry, then  $s_\alpha = 0$ .
- If  $\beta$  has two equal entries, then  $s_\alpha = 0$ .
- Otherwise, let  $\gamma$  be the  $k$ -tuple obtained by sorting  $\beta$  in decreasing order, and let  $\sigma$  be the permutation of the indices that causes this sorting. Let  $\lambda$  be the  $k$ -partition  $(\gamma_1 - (k-1), \gamma_2 - (k-2), \dots, \gamma_k - (k-k))$ . Then,  $s_\alpha = (-1)^\sigma s_\lambda$ .

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- Also, the alternant formula still holds if all  $\lambda_i + (k-i)$  are  $\geq 0$ .

## A more general setting: $a_1, a_2, \dots, a_k$ and $J$

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- Let  $J$  be the ideal of  $\mathcal{P}$  generated by the  $k$  differences

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- **Theorem (G.):** The  $\mathbf{k}$ -module  $\mathcal{P}/J$  is free with basis

$$\begin{aligned} & (\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i} \\ & \left( \text{informally: } \overline{\left( x_1^{<n-k+1} x_2^{<n-k+2} \dots x_n^{<n} \right)} \right) \end{aligned}$$

where the overline  $\overline{\quad}$  means “projection” onto whatever quotient we need (here: from  $\mathcal{P}$  onto  $\mathcal{P}/J$ ).

(This basis has  $n(n-1)\cdots(n-k+1)$  elements.)

- FROM NOW ON, assume that  $a_1, a_2, \dots, a_k \in \mathcal{S}$ .

## A slightly less general setting: symmetric $a_1, a_2, \dots, a_k$ and $J$

- **FROM NOW ON, assume that**  $a_1, a_2, \dots, a_k \in \mathcal{S}$ .
- Let  $I$  be the ideal of  $\mathcal{S}$  generated by the  $k$  differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \dots, \quad h_n - a_k.$$

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- Let  $\omega = \underbrace{(n-k, n-k, \dots, n-k)}_{k \text{ entries}}$  and

$$\begin{aligned} P_{k,n} &= \{ \lambda \text{ is a } k\text{-partition} \mid \lambda_1 \leq n-k \} \\ &= \{ k\text{-partitions } \lambda \subseteq \omega \}. \end{aligned}$$

- Here, for two  $k$ -partitions  $\alpha$  and  $\beta$ , we say that  $\alpha \subseteq \beta$  if and only if  $\alpha_i \leq \beta_i$  for all  $i$ .
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- This setting still is general enough to encompass ...
  - **classical cohomology:** If  $\mathbf{k} = \mathbb{Z}$  and  $a_1 = a_2 = \dots = a_k = 0$ , then  $\mathcal{S}/I$  becomes the cohomology ring  $H^*(\text{Gr}(k, n))$ ; the basis  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$  corresponds to the Schubert classes.
  - **quantum cohomology:** If  $\mathbf{k} = \mathbb{Z}[q]$  and  $a_1 = a_2 = \dots = a_{k-1} = 0$  and  $a_k = -(-1)^k q$ , then  $\mathcal{S}/I$  becomes the quantum cohomology ring  $\text{QH}^*(\text{Gr}(k, n))$ .

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- The above theorem lets us work in these rings (and more generally) without relying on geometry.

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- For every  $k$ -partition  $\nu = (\nu_1, \nu_2, \dots, \nu_k) \in P_{k,n}$ , we define

$$\nu^\vee := (n - k - \nu_k, n - k - \nu_{k-1}, \dots, n - k - \nu_1) \in P_{k,n}.$$

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- For any three  $k$ -partitions  $\alpha, \beta, \gamma \in P_{k,n}$ , let

$$g_{\alpha, \beta, \gamma} := \text{coeff}_{\gamma^\vee} (\overline{s_\alpha s_\beta}) \in \mathbf{k}.$$

These generalize the Littlewood–Richardson coefficients and (3-point) Gromov–Witten invariants.

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- Equivalent restatement:** Each  $\nu \in P_{k,n}$  and  $f \in \mathcal{S}/I$  satisfy  $\text{coeff}_\omega (\overline{s_\nu} f) = \text{coeff}_{\nu^\vee} (f)$ .

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- **Proposition (G.):** Let  $m$  be a positive integer. Then,

$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \overline{s_{(m,1^j)}},$$

where  $(m, 1^j) := (m, \underbrace{1, 1, \dots, 1}_{j \text{ ones}}, 0, 0, 0, \dots)$  (a hook-shaped  $k$ -partition).



## The Pieri rule for symmetric polynomials

- If  $\alpha$  and  $\beta$  are two  $k$ -partitions, then we say that  $\alpha/\beta$  is a *horizontal strip* if and only if the Young diagram  $Y(\alpha)$  is obtained from  $Y(\beta)$  by adding some (possibly none) extra boxes with no two of these new boxes lying in the same column.

**Example:** If  $k = 4$  and  $\alpha = (5, 3, 2, 1)$  and  $\beta = (3, 2, 2, 0)$ , then  $\alpha/\beta$  is a horizontal strip, since

$$Y(\beta) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & \square & \\ \hline \end{array} \subseteq \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & X & X \\ \hline \square & \square & X & & \\ \hline \square & \square & & & \\ \hline X & & & & \\ \hline \end{array} = Y(\alpha)$$

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- **Theorem (Pieri).** Let  $\lambda$  be a  $k$ -partition. Let  $j \in \mathbb{N}$ . Then,

$$s_\lambda h_j = \sum_{\substack{\mu \text{ is a } k\text{-partition;} \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} s_\mu.$$

- **Theorem (G.):** Let  $\lambda \in P_{k,n}$ . Let  $j \in \{0, 1, \dots, n - k\}$ . Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^k (-1)^i a_i \sum_{\nu \subseteq \lambda} c_{(n-k-j+1, 1^{i-1}), \nu}^\lambda \overline{s_\nu}.$$

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- This generalizes the h-Pieri rule from Bertram, Ciocan-Fontanine and Fulton, but note that  $c_{(n-k-j+1, 1^{i-1}), \nu}^\lambda$  may be  $> 1$ .

- **Example:** For  $n = 7$  and  $k = 3$ , we have

$$\begin{aligned} \overline{s_{(4,3,2)} h_2} &= \overline{s_{(4,4,3)}} + a_1 (\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}) \\ &\quad - a_2 (\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}) \\ &\quad + a_3 (\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}). \end{aligned}$$

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- Multiplying by  $e_j$  appears harder: For  $n = 5$  and  $k = 3$ , we have

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 (\overline{s_{(2)}} + \overline{s_{(1,1)}}) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$



- **Example:** For  $n = 7$  and  $k = 3$ , we have

$$\begin{aligned} \overline{s_{(4,3,2)}} h_2 &= \overline{s_{(4,4,3)}} + a_1 (\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}) \\ &\quad - a_2 (\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}) \\ &\quad + a_3 (\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}). \end{aligned}$$

- Multiplying by  $e_j$  appears harder: For  $n = 5$  and  $k = 3$ , we have

$$\overline{s_{(2,2,1)}} e_2 = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 (\overline{s_{(2)}} + \overline{s_{(1,1)}}) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

- Even multiplying by  $e_k$  can give a mess...

## A “rim hook algorithm”

- For  $QH^*(Gr(k, n))$ , Bertram, Ciocan-Fontanine and Fulton give a “rim hook algorithm” that rewrites an arbitrary  $\overline{s}_\mu$  as  $(-1)^{\text{something}} q^{\text{something}} \overline{s}_\lambda$  with  $\lambda \in P_{k,n}$ .  
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Looks hopeless...

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- **Theorem (G.):** Let  $\mu$  be a  $k$ -partition with  $\mu_1 > n - k$ . Let

$$W = \left\{ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 = \mu_1 - n \right. \\ \left. \text{and } \lambda_i - \mu_i \in \{0, 1\} \text{ for all } i \in \{2, 3, \dots, k\} \right\}.$$

(Not all elements of  $W$  are  $k$ -partitions, but all belong to  $\mathbb{Z}^k$ , so we know how to define  $s_\lambda$  for them.)

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Then,

$$\overline{s}_\mu = \sum_{j=1}^k (-1)^{k-j} a_j \sum_{\substack{\lambda \in W; \\ |\lambda| = |\mu| - (n-k+j)}} \overline{s}_\lambda.$$

- **Conjecture:** Let  $b_i = (-1)^{n-k-1} a_i$  for each  $i \in \{1, 2, \dots, k\}$ . Let  $\lambda, \mu, \nu \in P_{k,n}$ . Then,  $(-1)^{|\lambda|+|\mu|-|\nu|} \text{coeff}_\nu(\overline{s_\lambda s_\mu})$  is a polynomial in  $b_1, b_2, \dots, b_k$  with coefficients in  $\mathbb{N}$ .
- Verified for all  $n \leq 8$  using SageMath.
- This would generalize positivity of Gromov–Witten invariants.

- **Theorem (G.):** The  $\mathbf{k}$ -module  $\mathcal{S}/I$  is free with basis

$$(\overline{m_\lambda})_{\lambda \in P_{k,n}},$$

where

$$m_\lambda = \left( \begin{array}{l} \text{the sum of all distinct permutations of} \\ \text{the monomial } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \end{array} \right)$$

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- What about other bases? Forgotten symmetric functions?

- **Question:** Does  $\mathcal{S}/I$  have a geometric meaning? If not, why does it behave so nicely?

## More questions

- **Question:** Does  $\mathcal{S}/I$  have a geometric meaning? If not, why does it behave so nicely?
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- **Question:** Is there an equivariant analogue?
- **Question:** What about quotients of the quasisymmetric polynomials?

- The symmetric group  $S_k$  acts on  $\mathcal{P}$ , with invariant ring  $\mathcal{S}$ .
- What is the  $S_k$ -module structure on  $\mathcal{P}/J$ ?

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- What is the  $S_k$ -module structure on  $\mathcal{P}/J$ ?
- **Almost-theorem (G., needs to be checked):** Assume that  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra. Then, as  $S_k$ -modules,

$$\mathcal{P}/J \cong (\mathcal{P}/\mathcal{PS}^+)^{\times \binom{n}{k}} \cong \left( \underbrace{\mathbf{k}S_k}_{\text{regular rep}} \right)^{\times \binom{n}{k}},$$

where  $\mathcal{PS}^+$  is the ideal of  $\mathcal{P}$  generated by symmetric polynomials with constant term 0.

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

$$\mathcal{S} := \{\text{symmetric polynomials in } x_1, x_2, \dots, x_k\};$$

$$\mathcal{A} := \{\text{symmetric functions in } x_1, x_2, x_3, \dots\}.$$

## Deforming symmetric functions, 1

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**e** = elementary symmetric,

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$$\begin{aligned} \mathcal{S} &\cong \Lambda / (\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \dots)_{\text{ideal}}, & \text{thus} \\ \mathcal{S}/I &\cong \Lambda / (\mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \dots, \mathbf{h}_n - a_k, \\ &\quad \mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \dots)_{\text{ideal}}. \end{aligned}$$

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- So why not replace the  $\mathbf{e}_j$  by  $\mathbf{e}_j - b_j$  too?



- **Theorem (G.):** Assume that  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$  as well as  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots$  are elements of  $\Lambda$  such that

$$\deg \mathbf{a}_i < n - k + i \quad \text{and} \quad \deg \mathbf{b}_i < k + i.$$

Then,

$$\Lambda / (\mathbf{h}_{n-k+1} - \mathbf{a}_1, \mathbf{h}_{n-k+2} - \mathbf{a}_2, \dots, \mathbf{h}_n - \mathbf{a}_k, \\ \mathbf{e}_{k+1} - \mathbf{b}_1, \mathbf{e}_{k+2} - \mathbf{b}_2, \mathbf{e}_{k+3} - \mathbf{b}_3, \dots)_{\text{ideal}}$$

is a free  $\mathbf{k}$ -module with basis  $(\overline{\mathbf{s}}_\lambda)_{\lambda \in P_{k,n}}$ .

- Proofs of all the above (except for the  $S_k$ -action) can be found in
  - Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, <http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf> ,  
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- **Main ideas:**
  - Use Gröbner bases to show that  $\mathcal{P}/J$  is free with basis  $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$ .  
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  - Using that + Jacobi–Trudi, show that  $\mathcal{S}/I$  is free with basis  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ .
  - As for the rest, compute in  $\Lambda \dots$  a lot.

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- Gröbner bases are “particularly uncomplicated” generating sets for ideals in polynomial rings.  
(But take the word “basis” with a grain of salt – they can have redundant elements, for example.)

- A *monomial order* is a total order on the monomials in  $\mathcal{P}$  with the properties that
  - $1 \leq m$  for each monomial  $m$ ;
  - $a \leq b$  implies  $am \leq bm$  for any monomials  $a, b, m$ ;
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- The *degree-lexicographic order* is the monomial order defined as follows: Two monomials  $a = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$  and  $b = x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$  satisfy  $a > b$  if and only if
  - either  $\deg a > \deg b$
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  - each nonzero polynomial  $f \in \mathcal{P}$  has a well-defined *leading monomial* (= the highest monomial appearing in  $f$ ).
  - a polynomial  $f$  is called *quasi-monic* if the coefficient of its leading term in  $f$  is invertible.

- If  $\mathcal{I}$  is an ideal of  $\mathcal{P}$ , then a *Gröbner basis* of  $\mathcal{I}$  (for a fixed monomial order) means a family  $(f_i)_{i \in G}$  of quasi-monic polynomials that
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- **Example:** Let  $k = 3$ , and rename  $x_1, x_2, x_3$  as  $x, y, z$ . Use the degree-lexicographic order. Let  $\mathcal{I}$  be the ideal generated by  $x^2 - yz, y^2 - zx, z^2 - xy$ . Then:

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  - The triple  $(x^2 - yz, y^2 - zx, z^2 - xy)$  is not a Gröbner basis of  $\mathcal{I}$ , since its leading monomials are  $x^2, yz, xy$ , but the leading term  $y^3$  of the polynomial  $y^3 - z^3 \in \mathcal{I}$  is not divisible by any of them.

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- **Example:** Let  $k = 3$ , and rename  $x_1, x_2, x_3$  as  $x, y, z$ . Use the degree-lexicographic order. Let  $\mathcal{I}$  be the ideal generated by  $x^2 - yz, y^2 - zx, z^2 - xy$ . Then:
  - The triple  $(x^2 - yz, y^2 - zx, z^2 - xy)$  is not a Gröbner basis of  $\mathcal{I}$ , since its leading monomials are  $x^2, yz, xy$ , but the leading term  $y^3$  of the polynomial  $y^3 - z^3 \in \mathcal{I}$  is not divisible by any of them.
  - The quadruple  $(y^3 - z^3, x^2 - yz, xy - z^2, xz - y^2)$  is a Gröbner basis of  $\mathcal{I}$ . (Thanks SageMath, and whatever packages it uses for this.)

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- **Theorem (Buchberger's first criterion).** Let  $\mathcal{I}$  be an ideal of  $\mathcal{P}$ .

Let  $(f_i)_{i \in G}$  be a family of quasi-monic polynomials that generates  $\mathcal{I}$ .

Assume that the leading monomials of all the  $f_i$  are mutually coprime (i.e., each indeterminate appears in the leading monomial of  $f_i$  for at most one  $i \in G$ ).

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- **Theorem (Macaulay's basis theorem).** Let  $\mathcal{I}$  be an ideal of  $\mathcal{P}$  that has a Gröbner basis  $(f_i)_{i \in G}$ . A monomial  $\mathfrak{m}$  will be called *reduced* if it is not divisible by the leading term of any  $f_i$ . Then, the projections of the reduced monomials form a basis of the  $\mathbf{k}$ -module  $\mathcal{P}/\mathcal{I}$ .

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- It is easy to prove the identity

$$h_p(x_{i..k}) = \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) h_{p-t}(x_{1..k})$$

for all  $i \in \{1, 2, \dots, k+1\}$  and  $p \in \mathbb{N}$ .

Here,  $x_{a..b}$  means  $x_a, x_{a+1}, \dots, x_b$ .

- Use this to show that

$$\left( h_{n-k+i}(x_{i..k}) - \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) a_{i-t} \right)_{i \in \{1, 2, \dots, k\}}$$

is a Gröbner basis of the ideal  $J$  wrt the degree-lexicographic order.

- Thus, Macaulay's basis theorem shows that

$(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$  is a basis of the  $\mathbf{k}$ -module  $\mathcal{P}/J$ .

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- Thus,  $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$  is a basis of  $\mathcal{P}/J$ .

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- Maybe the most important one:

**Bernstein's identity:** Let  $\lambda$  be a partition. Let  $m \in \mathbb{Z}$  be such that  $m \geq \lambda_1$ . Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i}(\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m, \lambda_1, \lambda_2, \lambda_3, \dots)}.$$

Here,  $\mathbf{f}^\perp \mathbf{g}$  means “ $\mathbf{g}$  skewed by  $\mathbf{f}$ ” (so that  $(\mathbf{s}_\mu)^\perp \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu}$ ).

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