The Vlasov-Poisson system with a uniform magnetic field: propagation of moments and regularity

A. Rege
(Laboratoire Jacques-Louis Lions - Sorbonne Université)

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Outline

1. The magnetized Vlasov-Poisson system
2. Propagation of moments (sharp a priori estimates)

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2. Propagation of moments (sharp a priori estimates)

The magnetized system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q}{m}(E + v \wedge B) \cdot \nabla_v f = 0, \\ \mathrm{div}_x\, E = \frac{q}{\epsilon_0} \int f dv. \end{cases} \tag{VPwB}$$

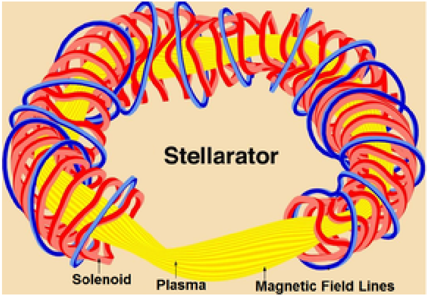
with $f \equiv f(t,x,v) \geq 0$ the distribution function of particles, $E \equiv E(t,x)$ the electric field and $B \equiv B(x)$ the magnetic field with $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$.

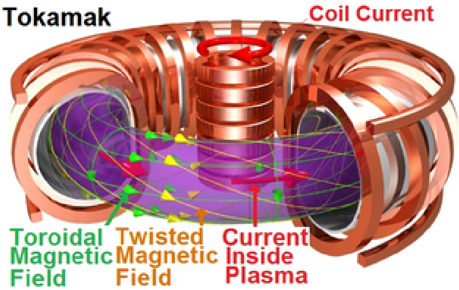
- The electric field E is self-consistent (depends on f) and given by

$$E(t,x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \underbrace{\frac{x-y}{|x-y|}}_{=K_3(x-y)} \rho(t,y) dy \tag{1}$$

  with $\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv$ the macroscopic density.
- We consider an external magnetic field B.

Magnetic confinement





Existing literature on Vlasov-Poisson

- Existence of weak solutions [Arsenev, 75']
- Small initial data [Bardos, Degond, 85']
- Existence of smooth solutions [Pfaffelmoser, 1992']
- Propagation of velocity moments [Lions, Perthame, 1991]
- Propagation of space moments [Castella, 99']

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In our study, we will consider a constant B

$$B = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}. \tag{2}$$

Outline

Theorem : Propagation of moments for VPwB

- Velocity moment $M_k(f) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f dv dx$.
- Energy of the system

$$\mathcal{E}(t) := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)\,dxdv + \frac{1}{2}\int_{\mathbb{R}^3} |E(t,x)|^2 dx. \tag{3}$$

Propagation of moments for VPwB

Let $k_0 > 3$, $T > 0$, $f^{in} = f^{in}(x,v) \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and assume that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f^{in}\,dxdv < \infty. \tag{4}$$

Then for all k such that $0 \leq k \leq k_0$, there exists $C > 0$ and a weak solution f to VPwB such that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)\,dxdv \leq C < +\infty, \quad 0 \leq t \leq T. \tag{5}$$

Differential inequality on M_k

$$
\begin{aligned}
|\frac{d}{dt}M_k(t)| &= |\iint |v|^k(-v\cdot\nabla_x f - (E+v\wedge B)\cdot\nabla_v f)dvdx| \\
&= |\iint |v|^k \,\mathrm{div_v}\left((E+v\wedge B)f\right) dvdx| \\
&= |\iint k|v|^{k-2}v\cdot(E+v\wedge B)fdvdx| \\
&\leq C\left\|E(t)\right\|_{k+3} M_k(t)^{\frac{k+2}{k+3}}
\end{aligned}
$$

Next step : we need to control of $\|E(t)\|_{k+3}$ with $M_k(t)^\alpha$ with $\alpha \leq \frac{1}{k+3}$.

A representation formula for ρ

$$\begin{cases} \frac{d}{ds}(X(s), V(s)) = (V(s), \omega V_2(s), -\omega V_1(s), 0) \\ (X(t), V(t)) = (x, v), \end{cases}$$

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$$\begin{cases} V(s) = \begin{pmatrix} v_1 \cos(\omega(s-t)) + v_2 \sin(\omega(s-t)) \\ -v_1 \sin(\omega(s-t)) + v_2 \cos(\omega(s-t)) \\ v_3 \end{pmatrix} \\ X(s) = \begin{pmatrix} x_1 + \frac{v_1}{\omega} \sin(\omega(s-t)) + \frac{v_2}{\omega}(1 - \cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t)) - 1) + \frac{v_2}{\omega} \sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix} \end{cases} \tag{6}$$

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$$\rho(t,x) = \underbrace{\int_v f^{in}(X(0), V(0))dv}_{=:\rho_0(t,x)} + \mathrm{div}_x \int_0^t \underbrace{\int_v (fH_t)(s, X(s), V(s))\, dv ds}_{=:\sigma(s,t,x)}$$

Singularities at multiples of the cyclotron period

$$E(t,x) = -\left(\nabla K_3 \star \rho\right)(t,x) = E^0(t,x) + \tilde{E}(t,x) \tag{7}$$

$$\|E(t)\|_{k+3} \leq \left\|E^0(t)\right\|_{k+3} + \int_0^t \|\sigma(s,t,x)\|_{k+3}\, ds \tag{8}$$

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C\frac{\sqrt{2}}{s}\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}} \tag{9}$$

Propagation of moments on a finite interval

For $T = \frac{\pi}{\omega} =: T_\omega$ we have

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dxdv \leq C < +\infty, \quad 0 \leq t \leq T_\omega, \tag{10}$$

with $C = C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

Propagation of moments for all time

We have that

- $\|f^{in}\|_1 = \|f(T_\omega)\|_1$ and $\|f^{in}\|_\infty = \|f(T_\omega)\|_\infty$.
- $\mathcal{E}(T_\omega) \leq \mathcal{E}_{in}$
- $M_k(f(T_\omega)) \leq C(k, \omega, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$

This means $f(T_\omega)$ verifies the assumptions of the theorem $\Rightarrow$ we can show propagation of moments for all time by induction.

1. G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density, 2006.

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Additional results

- Propagation of regularity if f^{in} is C^1.
- Uniqueness if $\rho \in L^\infty([0\,;\,T] \times \mathbb{R}^3)$ [1].

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Perspectives

- Propagation of moments larger than 2[2].
- Space moments to consider solutions with infinite kinetic energy[3].
- Non-uniform magnetic field B.
- Coupling with an magneto-hydrodynamics equation on the magnetic field B[4].

2. I. Gasser, P.-E. Jabin, B. Perthame, Regularity and propagation of moments in some nonlinear Vlasov systems, 2000.
3. F. Castella, Propagation of space moments in the Vlasov-Poisson Equation and further results, 1999.
4. F. Charles, B. Després, B. Perthame, R. Sentis, Nonlinear stability of a Vlasov equation for magnetic plasmas, 2013.