Some aspects of the analysis of MsFEM methods

Congrès d'Analyse Numérique pour les Jeunes - 2020

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3 December, 2020

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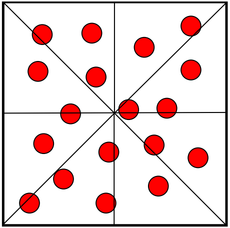
Problem definition

Elliptic boundary value problem for u^ϵ,



(1) $$\begin{cases} -\text{div}(A^\epsilon \nabla u^\epsilon) = f, & \text{in } \Omega, \\ u^\epsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

where
Ω is a bounded convex polygon in $\mathbb{R}^2$, $f \in L^2(\Omega)$.
$A^\epsilon(x) = A_{per}\left(\frac{x}{\epsilon}\right)$, A_{per} bounded and uniformly elliptic.

Homogenized limit $u^* \in H^2(\Omega)$.
Correctors w_1, w_2 give $u^{\epsilon,1} = u^* + \epsilon \sum_{k=1,2} w_k\left(\frac{\cdot}{\epsilon}\right)\partial_k u^*$.

Theorem 1

If $u^* \in W^{1,\infty}(\Omega)$ and $A_{per} \in C^{0,\alpha}$, then $\exists\, C > 0$ such that $\forall\, \epsilon$:

$$\|u^\epsilon - u^{\epsilon,1}\|_{1,\Omega} \leq C\sqrt{\epsilon}\|\nabla u^*\|_{0,\infty,\Omega} + C\epsilon|u^*|_{2,\Omega}.$$

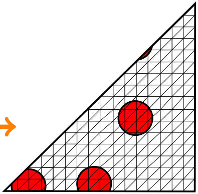
(Adapted from Jikov, Kozlov, and Oleinik 1994)

Method

$(T_H)_{H>0}$ family of regular, quasi-uniform triangulations.

$\mathbb{P}_1$ basis functions $\phi_1, \dots, \phi_N$.

Multiscale basis functions ϕ_j^{ms},



(2)
$$\begin{cases} -\mathrm{div}(A^\epsilon \nabla \phi_j^{ms}) = 0, & \text{in } K, \\ \phi_j^{ms} = \phi_j, & \text{on } \partial K, \quad \forall K \in T_H. \end{cases}$$

Test & trial space $V_{H,\epsilon}^{ms} := \mathrm{span}\{\phi_j^{ms} \mid 1 \le j \le N\} \subset H_0^1(\Omega) \to u_H^\epsilon$.

Homogenization: $\phi_j^{ms} = \phi_j + \epsilon \sum_{k=1,2} w_k\left(\frac{\cdot}{\epsilon}\right) \partial_k \phi_j + \theta_j^\epsilon$.

Multiscale FE $\quad$ $\mathbb{P}_1$ FE $\quad$ small remainder

Convergence

Theorem 2
Assume that $A_{per} \in C^{0,\alpha}$ and that $u^* \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$.
Then $\exists\, C > 0$ independent of ϵ, H and f such that

$$\|u^\epsilon - u_H^\epsilon\|_{1,\Omega} \leq C\sqrt{\epsilon}\|\nabla u^*\|_{0,\infty,\Omega} + C\left(H + \epsilon + \sqrt{\epsilon/H}\right)\|u^*\|_{2,\Omega}.$$

(See e.g. Efendiev and Hou 2009)

Some questions:

(i) Rectangular meshes? $\longrightarrow$ This talk

(ii) Non-homogeneous boundary conditions? $\longrightarrow$ This talk

(iii) Necessity of Hölder continuity? (corrector functions $w_k \in W^{1,\infty}$)

Basic proof

- Céa's lemma: $\|u^\epsilon - u_H^\epsilon\|_{1,\Omega} \le C\|u^\epsilon - v\|_{1,\Omega} \quad \forall\, v \in V_{H,\epsilon}^{ms},$
$$\le C\|u^\epsilon - u^{\epsilon,1}\|_{1,\Omega} + C\|u^{\epsilon,1} - v\|_{1,\Omega}.$$

- $\|u^\epsilon - u^{\epsilon,1}\|_{1,\Omega} \le C\sqrt{\epsilon}\|\nabla u^*\|_{0,\infty,\Omega} + C\epsilon|u^*|_{2,\Omega}$ (homogenization).

- Choose $v =$ interpolant of u^* at the nodes x_j. Then $\nabla(u^{\epsilon,1} - v) =$
$$\left(\nabla u^* - \nabla\sum_{j=1}^N u^*(x_j)\phi_j\right) + \sum_{k=1,2}\nabla w_k\left(\frac{\cdot}{\epsilon}\right)\left[\partial_k u^* - \sum_{j=1}^N u^*(x_j)\partial_k\phi_j\right]$$
$$+\sum_{k=1,2}\epsilon w_k\left(\frac{\cdot}{\epsilon}\right)\nabla\partial_k u^* - \sum_{k=1,2}\epsilon w_k\left(\frac{\cdot}{\epsilon}\right)\nabla\sum_{j=1}^N u^*(x_j)\partial_k\phi_j - \nabla\sum_{j=1}^N u^*(x_j)\theta_j^\epsilon,$$
$$\|\nabla(u^{\epsilon,1} - v)\|_{0,\Omega} \le C(\epsilon + H)|u^*|_{2,\Omega} \quad (\mathbb{P}_1 \text{ interpolation})$$
$$+\left\|\sum_{j=1}^N u^*(x_j)\underbrace{\left(\sum_{k=1,2}\epsilon w_k\left(\frac{\cdot}{\epsilon}\right)\partial_k\nabla\phi_j + \nabla\theta_j^\epsilon\right)}_{=0 \text{ on each } K \text{ for } \mathbb{P}_1}\right\|_{0,\Omega}.$$

Basic proof (ctd.)

- Similarly, $\|u^{\epsilon,1} - v\|_{0,\Omega} \leq C(\epsilon + \epsilon H)|u^*|_{2,\Omega} + \left\|\sum_{j=1}^N u^*(x_j)\theta_j^\epsilon\right\|_{0,\Omega}$.
- On each triangle K, use a local version of Theorem 1:

$$\left\|\sum_{j=1}^N u^*(x_j)\theta_j^\epsilon\right\|_{1,K} \leq C\sqrt{\epsilon H}\left\|\nabla\sum_{j=1}^N u^*(x_j)\phi_j\right\|_{0,\infty,K} + C\epsilon\underbrace{\left|\sum_{j=1}^N u^*(x_j)\phi_j\right|_{2,K}}_{=0 \text{ for } \mathbb{P}_1}$$

$$\leq C\sqrt{\frac{\epsilon}{H}}\left\|\nabla\sum_{j=1}^N u^*(x_j)\phi_j\right\|_{0,K}, \text{ so}$$

$$\left\|\sum_{j=1}^N u^*(x_j)\theta_j^\epsilon\right\|_{1,\Omega,\mathrm{br}} \leq C\sqrt{\frac{\epsilon}{H}}\left\|\nabla\sum_{j=1}^N u^*(x_j)\phi_j\right\|_{0,\Omega} \leq C\left(\sqrt{\epsilon H} + \sqrt{\epsilon/H}\right)\|u^*\|_{2,\Omega}.$$

All estimates together:

$$\|u^\epsilon - u_H^\epsilon\|_{1,\Omega} \leq C\sqrt{\epsilon}\|\nabla u^*\|_{0,\infty,\Omega} + C\left(H + \epsilon + \sqrt{\epsilon/H}\right)\|u^*\|_{2,\Omega}.$$

Proof on rectangles

$\mathbb{Q}_1$ basis functions ϕ_j and ϕ_j^{ms} defined on rectangles.

Difference 1: ϕ_j^{ms} has homogenized limit $\neq \phi_j$.

In the above proof, squeeze in $\sum_{j=1}^{N} u^*(x_j)\tilde{\phi}_j^{ms}$ where

$$(3) \qquad \begin{cases} -\mathrm{div}(A^\epsilon \nabla \tilde{\phi}_j^{ms}) = -\mathrm{div}(A^* \nabla \phi_j), & \text{in } K, \\ \tilde{\phi}_j^{ms} = \phi_j, & \text{on } \partial K. \end{cases}$$

$\rightarrow$ Extra error term $e = \sum_{j=1}^{N} u^*(x_j)\left(\tilde{\phi}_j^{ms} - \phi_j^{ms}\right)$, which satisfies

$$(4) \qquad \begin{cases} -\mathrm{div}(A^\epsilon \nabla e) = -\mathrm{div}\left(A^* \nabla \sum_{j=1}^{N} u^*(x_j)\phi_j\right), & \text{in } K, \\ e = 0, & \text{on } \partial K. \end{cases}$$

$\rightarrow \|e\|_{1,\Omega} \leq CH|u^*|_{2,\Omega}$

Proof on rectangles (ctd.)

Difference 2: $\partial_k \nabla \phi_j \neq 0$.

- $$\left\| \sum_{j=1}^{N} \sum_{k=1,2} \epsilon w_k \left(\frac{\cdot}{\epsilon} \right) \cdot u^*(x_j) \partial_k \nabla \phi_j \right\|_{0,K} \leq C\epsilon |u^*|_{2,K}$$

- $$C\epsilon \left| \sum_{j=1}^{N} u^*(x_j) \phi_j \right|_{2,K} \leq C\epsilon |u^*|_{2,K}$$

Conclusion: same error estimate.

Proof for non-homogeneous BC: straight-forward adaptation

For g the trace of an H^2 function consider

$$
\text{(5)} \qquad \begin{cases} -\mathrm{div}(A^\epsilon \nabla u^\epsilon) = f, & \text{in } \Omega, \\ u^\epsilon = g, & \text{on } \partial\Omega. \end{cases}
$$

- Trial space $V_{H,\epsilon}^{ms} \to u_H^\epsilon$ with $u_H^\epsilon =$ FE interpolant of g on $\partial\Omega$.
- Test space $V_{H,\epsilon,0}^{ms} \subsetneq V_{H,\epsilon}^{ms} \to$ redo Céa's lemma.

- Set $v = \sum_{j=1}^{N} u^*(x_j)\phi_j^{ms}$. Then $u_H^\epsilon - v \in V_{H,\epsilon,0}^{ms}$, so $a^\epsilon(u^\epsilon - u_H^\epsilon, u_H^\epsilon - v) = 0$ (a^ϵ = bilinear form associated to (5)).
- Consequently, $\|u_H^\epsilon - v\|_{1,\Omega} \leq C\|u^\epsilon - v\|_{1,\Omega}$ and

$$
\|u^\epsilon - u_H^\epsilon\|_{1,\Omega} \leq C\|u^\epsilon - v\|_{1,\Omega}.
$$

The rest of the proof is unchanged: *same error estimate.*

Outlook

A_{per} Hölder-continuous for the estimate

$$\|u^\epsilon - u_H^\epsilon\|_{1,\Omega} \leq \mathcal{O}\left(\sqrt{\epsilon} + H + \sqrt{\frac{\epsilon}{H}}\right) \quad ?$$

In order to use only $A_{per} \in L^\infty$, we
(i) observe that the equation is in divergence form,
(ii) will do *all* manipulations in the associated energy norm.

References

Efendiev, Y. and T. Y. Hou (2009). *Multiscale Finite Element Methods*. Berlin: Springer-Verlag.

Jikov, V. V., S. M. Kozlov, and O. A. Oleinik (1994). *Homogenization of Differential Operators and Integral Functionals*. Berlin: Springer-Verlag.