

A periodic homogenization problem with defects rare at infinity

Rémi Goudey

Ecole des ponts ParisTech and MATHERIALS Team, INRIA Paris

December 3, 2020



1 Introduction

2 The perturbed problem

3 Existence results for the corrector equation

4 Application to homogenization

Purpose : To address an homogenization problem for a second order elliptic equation in divergence form when the coefficient is a non-local perturbation of a periodic coefficient :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(\cdot/\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \quad (1)$$

Where :

- $\Omega \subset \mathbb{R}^d$ is a bounded domain ($d \geq 1$).
- $f \in L^2(\Omega)$.
- $\varepsilon > 0$ is a small scale parameter.
- $a = a_{per} + \tilde{a}$ is a bounded, elliptic coefficient.
- a_{per} is \mathbb{Z}^d -periodic and \tilde{a} represents a non-local perturbation.

Purpose : To address an homogenization problem for a second order elliptic equation in divergence form when the coefficient is a non-local perturbation of a periodic coefficient :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(\cdot/\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \quad (1)$$

We want :

1. To identify the limit of u^ε when the scale parameter $\varepsilon \rightarrow 0$ and study the convergence for several topologies ($L^2(\Omega)$, $H^1(\Omega)$,...).
2. To make precise the convergence rates.

Introduction

The periodic case

The periodic problem, when $a = a_{per}$, is well known ¹ :

- u^ε converges strongly in $L^2(\Omega)$, weakly in $H^1(\Omega)$ to u^* solution to the homogenized equation :

$$\begin{cases} -\operatorname{div}(a_{per}^* \nabla u^*) = f & \text{on } \Omega \\ u^*(x) = 0 & \text{in } \partial\Omega \end{cases} \quad (2)$$

where (a_{per}^*) is a constant matrix.

- The behavior in $H^1(\Omega)$ is obtained introducing a corrector $w_{per,p}$ defined for all $p \in \mathbb{R}^d$ as the periodic solution (unique up to the addition of a constant) to :

$$-\operatorname{div}(a_{per}(\nabla w_{per,p} + p)) = 0 \quad \text{in } \mathbb{R}^d. \quad (3)$$

¹[Bensoussan, Lions, Papanicolaou '1978]

Introduction

The periodic case

This corrector w_{per} allows to both make explicit the homogenized coefficient :

$$(a_{per}^*)_{i,j} = \int_Q e_i^T a_{per}(y) (e_j + \nabla w_{per,e_j}) dy,$$

and define an approximation

$$u^{\varepsilon,1} = u^*(.) + \varepsilon \sum_{i=1}^d \partial_i u^*(.) w_{per,e_i}(. / \varepsilon),$$

such that $u^{\varepsilon,1} - u^\varepsilon$ **strongly** converges to 0 in $H^1(\Omega)$.

- 1 Introduction
- 2 The perturbed problem**
- 3 Existence results for the corrector equation
- 4 Application to homogenization

The perturbed problem

- Our purpose is to extend these results to the setting of the perturbed problem when

$$a = a_{per} + \tilde{a}.$$

- **Main difficulty** : The corrector equation

$$-\operatorname{div}((a_{per} + \tilde{a})(\nabla w_p + p)) = 0 \text{ in } \mathbb{R}^d,$$

cannot be reduced to an equation posed on a bounded domain as is the case in periodic context, which prevents us from using classical techniques (Poincaré inequality).

The perturbed problem


The case of local perturbations

First extension²: The case of local perturbations (i.e $\tilde{a}(x) \rightarrow 0$ when $|x| \rightarrow \infty$) when $\tilde{a} \in L^r(\mathbb{R}^d)$ for $r \in]1, \infty[$.

- In this case, the homogenized limit is identical to that of the periodic case without defect ($\tilde{a} = 0$) and the existence of a corrector w_p is established. The corrector is of the form :

$$w_p = w_{per,p} + \tilde{w}_p$$

where $w_{per,p}$ is the periodic corrector and $\nabla \tilde{w} \in L^r(\mathbb{R}^d)$,

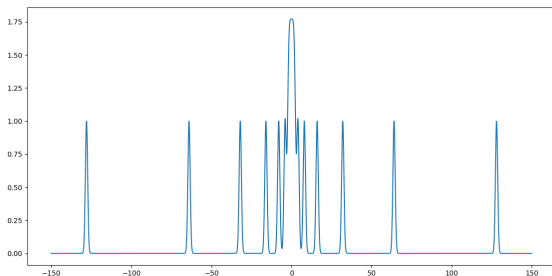
²[Blanc, Le Bris, Lions 2012, 2018] & [Blanc, Josien, Le Bris 2020] 

The perturbed problem

The case of non-local perturbations

- We consider here a perturbation \tilde{a} , that, although it does not vanish, becomes rare at infinity.
- For some fixed $\phi \in \mathcal{D}(\mathbb{R})$, a prototypical one-dimensional example reads as

$$\tilde{a} = \sum_{k \in \mathbb{Z}} \phi(x - \text{sign}(k)2^{|k|}).$$



Mathematical setting

To formalize the mathematical setting of the defects rare at infinity, we introduce :

$$\mathcal{G} = \{x_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{Z}^d,$$

a infinite discrete set of points, and

$$V_k = \left\{ x \in \mathbb{R}^d \mid \forall x_q \in \mathcal{G}, |x - x_k| \leq |x - x_q| \right\},$$

the Voronoi cell containing the point x_k .

- Each point x_k models the presence of a defect in the periodic background.
- To ensure the defects are sufficiently rare at infinity, we need the points are increasingly distant from one another when far from the origin.

Mathematical setting

The set \mathcal{G} is required to satisfy the following three conditions :

$$\forall x_k \in \mathcal{G}, \quad |V_k| < \infty, \quad (\text{H1})$$

$$\exists C_1 > 0, C_2 > 0, \forall x_k \in \mathcal{G}, \quad C_1 \leq \frac{1 + |x_k|}{D(x_k, \mathcal{G} \setminus \{x_k\})} \leq C_2, \quad (\text{H2})$$

$$\exists C_3 > 0, \forall x_k \in \mathcal{G}, \quad \frac{\text{Diam}(V_k)}{D(x_k, \mathcal{G} \setminus \{x_k\})} \leq C_3, \quad (\text{H3})$$

Mathematical setting

The set \mathcal{G} is required to satisfy the following three conditions :

$$\forall x_k \in \mathcal{G}, \quad |V_k| < \infty, \quad (\text{H1})$$

$$\exists C_1 > 0, C_2 > 0, \forall x_k \in \mathcal{G}, \quad C_1 \leq \frac{1 + |x_k|}{D(x_k, \mathcal{G} \setminus \{x_k\})} \leq C_2, \quad (\text{H2})$$

$$\exists C_3 > 0, \forall x_k \in \mathcal{G}, \quad \frac{\text{Diam}(V_k)}{D(x_k, \mathcal{G} \setminus \{x_k\})} \leq C_3, \quad (\text{H3})$$

Remarks

- (H2) is the most significant assumption, it ensures the distance between a point x_k and the others to **scale exponentially** far from the origin.
- (H1) and (H3) are only technical assumptions that limit the size of the Voronoi cells and ensure the worst case scenario, where the set \mathcal{G} contains as many points as possible while satisfying (H2).

Example of admissible set of points

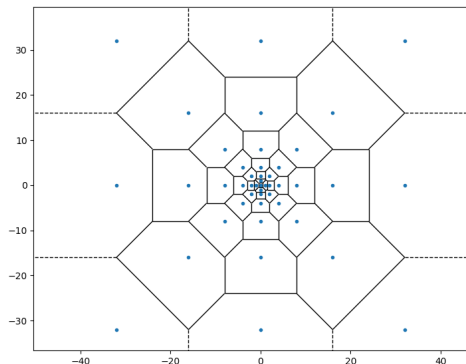


Figure: Example of points that satisfy our assumptions along with their associated Voronoi diagram for $d = 2$. Here the coordinates of the points are powers of 2.

The space of perturbations

In order to describe the perturbations we define :

$$\mathcal{B}^2(\mathbb{R}^d) = \left\{ f \in L^2_{unif}(\mathbb{R}^d) \mid \exists f_\infty \in L^2(\mathbb{R}^d), \lim_{|x_k| \rightarrow \infty} \|f - \tau_{-x_k} f_\infty\|_{L^2(V_k)} = 0 \right\},$$

equipped with the norm

$$\|f\|_{\mathcal{B}^2(\mathbb{R}^d)} = \|f\|_{L^2_{unif}(\mathbb{R}^d)} + \|f_\infty\|_{L^2(\mathbb{R}^d)} + \sup_{x_k \in \mathcal{G}} \|f - \tau_{-x_k} f_\infty\|_{L^2(V_k)}.$$

Where :

- $L^2_{unif}(\mathbb{R}^d) = \left\{ f \in L^2_{loc}(\mathbb{R}^d), \sup_{x \in \mathbb{R}^d} \|f\|_{L^2(B_1(x))} < \infty \right\},$
- $\|f\|_{L^2_{unif}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \|f\|_{L^2(B_1(x))}$
- $\tau_x f = f(\cdot + x)$

The space of perturbations

In order to describe the perturbations we define :

$$\mathcal{B}^2(\mathbb{R}^d) = \left\{ f \in L^2_{unif}(\mathbb{R}^d) \mid \exists f_\infty \in L^2(\mathbb{R}^d), \lim_{|x_k| \rightarrow \infty} \|f - \tau_{-x_k} f_\infty\|_{L^2(V_k)} = 0 \right\},$$

equipped with the norm

$$\|f\|_{\mathcal{B}^2(\mathbb{R}^d)} = \|f\|_{L^2_{unif}(\mathbb{R}^d)} + \|f_\infty\|_{L^2(\mathbb{R}^d)} + \sup_{x_k \in \mathcal{G}} \|f - \tau_{-x_k} f_\infty\|_{L^2(V_k)}.$$

Remarks

- A function in $\mathcal{B}^2(\mathbb{R}^d)$ behaves, locally at the vicinity of each point x_k , as a fixed L^2 -function truncated over the domain V_k .
- $(\mathcal{B}^2(\mathbb{R}^d), \|\cdot\|_{\mathcal{B}^2(\mathbb{R}^d)})$ is a Banach space.

Proposition (Average)

Let $u \in \mathcal{B}^2(\mathbb{R}^d)$, then :

$$\langle |u| \rangle = \lim_{R \rightarrow \infty} \frac{1}{|\mathbb{B}_R|} \int_{\mathbb{B}_R} |u(x)| dx = 0$$

- Consequence of the geometric distribution of the points x_k (Assumption (H2)).
- On average, the perturbations belonging to $\mathcal{B}^2(\mathbb{R}^d)$ do not impact the periodic background.
- If $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)$, the homogenized limit of u^ε is therefore expected to be the same as in the periodic case without defect (i.e. when $\tilde{a} = 0$).

In our work, we consider a matrix-valued coefficient of the form

$$a = a_{per} + \tilde{a},$$

such that

- $a_{per} \in L^2_{per}(\mathbb{R}^d)^{d \times d}$, $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)^{d \times d}$,
- $\exists \lambda > 0$, $\forall x, \xi \in \mathbb{R}^d$, $\lambda |\xi|^2 \leq \langle a(x)\xi, \xi \rangle$, $\lambda |\xi|^2 \leq \langle a_{per}(x)\xi, \xi \rangle$,
- $a_{per}, \tilde{a}, \tilde{a}_\infty \in (L^\infty(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d))^{d \times d}$, $\alpha \in]0, 1[$.

Here, \tilde{a}_∞ is the limit L^2 -function associated with \tilde{a} .

- 1 Introduction
- 2 The perturbed problem
- 3 Existence results for the corrector equation**
- 4 Application to homogenization

Theorem 1 : Existence result for the corrector equation

For every $p \in \mathbb{R}^d$, there exists a unique (up to an additive constant) function $w_p \in H_{loc}^1(\mathbb{R}^d)$ such that $\nabla w_p \in (L_{per}^2(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d))^d$, solution to :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(p + \nabla w_p)) = 0 & \text{in } \mathbb{R}^d, \\ \lim_{|x| \rightarrow \infty} \frac{|w_p(x)|}{1 + |x|} = 0. \end{cases} \quad (4)$$

Theorem 1 : Existence result for the corrector equation

For every $p \in \mathbb{R}^d$, there exists a unique (up to an additive constant) function $w_p \in H_{loc}^1(\mathbb{R}^d)$ such that $\nabla w_p \in (L_{per}^2(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d))^d$, solution to :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(p + \nabla w_p)) = 0 & \text{in } \mathbb{R}^d, \\ \lim_{|x| \rightarrow \infty} \frac{|w_p(x)|}{1 + |x|} = 0. \end{cases} \quad (4)$$

Main Idea : Assume $w_p = w_{p,per} + \tilde{w}_p$ with $\nabla \tilde{w}_p \in \mathcal{B}^2(\mathbb{R}^d)$, then (4) is equivalent to an equation of the form :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

where $u = \tilde{w}_p$ and $f = \tilde{a}(p + \nabla w_{p,per}) \in \mathcal{B}^2(\mathbb{R}^d)$.

Existence of the corrector

Existence result in the periodic problem

Lemma 1 : Existence result in the case $\tilde{a} = 0$

Let $f \in \mathcal{B}^2(\mathbb{R}^d)$, $\exists u \in H_{loc}^1(\mathbb{R}^d)$ such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$, solution to :

$$-\operatorname{div}(a_{per} \nabla u) = \operatorname{div}(f) \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

³[Avellaneda, Lin 1991]

Existence of the corrector

Existence result in the periodic problem

Lemma 1 : Existence result in the case $\tilde{a} = 0$

Let $f \in \mathcal{B}^2(\mathbb{R}^d)$, $\exists u \in H_{loc}^1(\mathbb{R}^d)$ such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$, solution to :

$$-\operatorname{div}(a_{per}\nabla u) = \operatorname{div}(f) \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Main idea (for $d \geq 3$) : Use the Green function G_{per} (i.e. the fundamental solution) associated with $-\operatorname{div}(a_{per}\cdot)$ on \mathbb{R}^d to define a solution u :

$$u(x) = \int_{\mathbb{R}^d} \nabla_y G_{per}(x, y) f(y) dy.$$

In order to show that u is well-defined in H_{loc}^1 and $\nabla u \in \mathcal{B}^2$, we use pointwise estimates satisfied by G_{per} ³ :

$$|\nabla_y G_{per}(x, y)| \leq C \frac{1}{|x - y|^{d-1}}, \quad |\nabla_x \nabla_y G_{per}(x, y)| \leq C \frac{1}{|x - y|^d}.$$

³[Avellaneda, Lin 1991]

Existence of the corrector

A priori estimate

Lemma 2 : A priori estimate

There exists a constant $C > 0$ such that for every $f \in \mathcal{B}^2(\mathbb{R}^d)$ and u solution in $\mathcal{D}'(\mathbb{R}^d)$ to

$$-\operatorname{div}((a_{per} + \tilde{a}) \nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

with $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$, we have the following estimate :

$$\|\nabla u\|_{\mathcal{B}^2(\mathbb{R}^d)} \leq C \|f\|_{\mathcal{B}^2(\mathbb{R}^d)}.$$

- Lemma 2 ensures the continuity of the reciprocal linear operator $f \mapsto \nabla (-\operatorname{div} a \nabla)^{-1} \operatorname{div}(f)$ from $\mathcal{B}^2(\mathbb{R}^d)$ to $\mathcal{B}^2(\mathbb{R}^d)$.

Main ideas for the proof of Theorem 1⁴ :

- For every $s \in [0, 1]$, we consider the assertion $\mathcal{P}(s)$:

" There exists $u \in H_{loc}^1(\mathbb{R}^d)$ solution in $\mathcal{D}'(\mathbb{R}^d)$ to

$$- \operatorname{div}((a_{per} + s\tilde{a})\nabla u) = \operatorname{div}(f),$$

such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$ " .

- We define $\mathcal{I} = \{s \in [0, 1], \mathcal{P}(s) \text{ is true}\}$.
- Lemma 1 $\Rightarrow \mathcal{I}$ is not empty.
- Lemma 2 $\Rightarrow \mathcal{I}$ is both open and closed for the topology of $[0, 1]$.
- Using an argument of connexity, $\mathcal{I} = [0, 1] \Rightarrow s = 1 \in \mathcal{I}$.

⁴Proof adapted from [Blanc, Le Bris, Lions 2018]

Remarks

- The corrector satisfies $\nabla w_p \in L^2_{per} + \mathcal{B}^2(\mathbb{R}^d)$, its gradient shares the same structure "periodic + \mathcal{B}^2 " as the coefficient a .
- The proof of existence is heavily based upon the geometric distribution of the x_k . In particular, Assumption (H2), which ensures the distance between the points x_k to scale exponentially far from the origin, is essential in our approach.

- 1 Introduction
- 2 The perturbed problem
- 3 Existence results for the corrector equation
- 4 Application to homogenization**

Proposition : Homogenization result

Let u^ε the sequence of solutions in $H_0^1(\Omega)$ to

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(\cdot/\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

Then the homogenized (weak- $H^1(\Omega)$ and strong- $L^2(\Omega)$) limit u^* obtained when $\varepsilon \rightarrow 0$ is the solution to

$$\begin{cases} -\operatorname{div}(a_{per}^* \nabla u^*) = f & \text{on } \Omega \\ u^*(x) = 0. & \text{in } \partial\Omega. \end{cases}$$

- As expected, the homogenized coefficient is identical to the periodic homogenized coefficient.

Theorem 2 : Convergence results

Assume $d \geq 3$ and Ω is a $C^{2,1}$ -bounded domain. Let $\Omega_1 \subset\subset \Omega$. We define $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^d \partial_i u^* w_{e_i}(\cdot/\varepsilon)$ where w_{e_i} is solution to corrector equation for $p = e_i$ and $u^* \in H^1(\Omega)$ is the homogenized limit. Then $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ satisfies the following estimates :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \varepsilon \|f\|_{L^2(\Omega)},$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \varepsilon \|f\|_{L^2(\Omega)},$$

where C_1 and C_2 are two positive constants independent of f and ε .

Thank you for your attention !