Representing Hodge realization

I Hodge realization. $k \hookrightarrow \mathbb{C}$

functor $DM^{gm}(k) \longrightarrow MHS(k)$ (vanishing gen. motivic)

$M \longmapsto (H_B(M), W\uparrow, H_{DR}(M), F\downarrow)$

$H_{\mathbb{C}} \simeq H_B \otimes \mathbb{C} \simeq H_{DR} \otimes \mathbb{C}$.

$G_{\mathbb{R}} H_{\mathbb{C}}$ pure Hodge structure of weight n $\bigoplus_{p+q=n} H^{p,q}$ $\overline{H^{p,q}} = H^{q,p}$

Constructed by A. Huber (00) $DM^{gm} \to MR(k)$

$M(X) \longmapsto Hodge(X)$ (Deligne)

Finite representable (eff) $DM^{(eff)}$ $\mathbb{A}^1$-localization of $D(NST)$ (Voevodsky, Kimura, Voevodsky, Suslin, Friedlander, Mazza, Cisinski, Deglise)

H_B ✓ $(D(Ab))$ H_{DR} ✓ W (Bondarko) ✓

F X smooth projective Hodge filtration is induced by the stupid filtration $\sigma^{\geq p}$ X smooth quasi proj

$0 \to \cdots \to \Omega^{\geq p} \to \cdots \to \Omega^{\bullet}$ ∞ no a metric

$X \hookrightarrow \overline{X}$ $\overline{X} \setminus X = D$ n.c.d.

$\Omega^{\bullet}_{\overline{X}}(\log D) \overset{qis}{\sim} \Omega^{\bullet}_X$

A. Beilinson (2009) shows $X \longmapsto R\Gamma(\Omega^{\bullet}_X(\log D))$

II Hodge filtration induces a functor $DM \to D(k\text{-vect})$.

Wanted: D $\exists$ $DM \to D$

$F^p H_{dR}(M) = Im\left(Hom_{D}(\mathbb{1}(M), \mathbb{1}(\Omega^{\geq p})[i]) \to Hom_D(\mathbb{1}(M), \mathbb{1}(\Omega^{\bullet})[i]) = H_{dR}(M) \right)$

$M = M(X)$ Hodge filtration.

① X smooth projective. Bondarko 2009-2010 $M \sim P^0 \dashrightarrow P^m$.

② $0 \to \Omega^{\geq p+1} \to \Omega^{\geq p} \to \Omega^p \to 0$ P^i motivic weight i.

$X \mapsto H^i(X, \Omega^p)$.

Two known candidates: (T. Binda, Doosung Park, P.A. Østvær)

① $\exists$ category of log motives (replace schemes by log schemes, diff by log differentials)

$DM \to \log DM \leftarrow \exists$ representable Hodge realization.

② modulus

II motives with modulus (B. Kahn, H. Miyazaki, Shuji Saito, T. Yamazaki)

$\mathcal{X} = (\overline{X}, D)$ modulus pair $X_i = \overline{X}_i \setminus |D_i|$.

$MCor(\mathcal{X}_1, \mathcal{X}_2) \subset Cor(X_1, X_2)$ generated by elementary correspondences $\overline{Z}$

$Z \overset{N\text{-finite}}{\longrightarrow} \overline{Z} \hookrightarrow \overline{X}_1 \times \overline{X}_2$

$D_1|_{\overline{Z}^N} \geqslant D_2|_{\overline{Z}^N}$

$w: MCor \to Cor$ $(\overline{X}, D) \longmapsto X$

Voevodsky $D(w^{\geq 0})$ Kahn et al. $\square = (\mathbb{P}^1, \infty)$.

$D(NST)$ $\rightleftarrows$ $D(MNST)$ ($D(w^{\geq 0})$, $D(w^{=0})$)

$L^{\mathbb{A}^1} \downarrow \uparrow j^{\mathbb{A}^1}$ $L^{\square} \downarrow \uparrow j^{\square}$

DM $\rightleftarrows$ MDM $\simeq_{\mathbb{Q}}$ MDA (w^{eff}, $w^{=0}$)

Can represent $H^i(-, \Omega^q)$ in MDA (Miyazaki-Kelly)

$H_{DNST}(C_*(\overline{X}), \Omega^q_{\overline{X}, D})$

Prop: $Hom_{D(NST)}(C_*(\overline{X}), \Omega^q) = H^0(\overline{X}, \Omega^q)$ (projective)

$C_1(\overline{X}) \to C_0(\overline{X})$ $\quad$ $0 \to I^0(q) \to I^1(q)$ injective resolution of Ω^q

$C_0(\overline{X}) \to I^0(q) \hookrightarrow Coker\left(C_1(\overline{X}) \xrightarrow{i_0 - i_1} C_0(\overline{X})\right) \to \Omega^q$

$C_1(\overline{X})(-\infty)$ $\quad$ $\mathbb{Z}_{tr}(X)$

$\exists$ $h^0_{\mathbb{A}^1}(\Omega^q)(\overline{X}) = Hom_{NST}(h^{\mathbb{A}^1}_0(X), \Omega^q)$ $\quad$ $h^{\mathbb{A}^1}_0(X)$

unramified H^q subpresheaf of Ω^q

Thm (K. Rülling & S. Saito). X smooth

$$h^0_{\mathbb{A}^1}(\Omega^q)(X) = H^0\left(\overline{X}, \Omega^q_{\overline{X}}(\log D)\right)$$

where $\overline{X}$ sm. $\overline{X} \setminus X = D$ n.c.d

Ingredients conductor $c = \left\{ c_L : F(L) \to \mathbb{N} \cup \{\infty\} \right\}$

L Henselian d.v.f. $Frac(\mathcal{O}^h_{X,x})$.

$a \in F(L)$ $c_L(a) = 0$ if a comes from $F(\mathcal{O}_L)$ $\quad$ $v_L(x_\infty) = 1$

HI sheaf $\hookrightarrow$ $c \leq 1$.

$h^0_{\mathbb{A}^1}(F) = F^{c \leq 1}$