Unramified cohomology and $\mathbb{P}^1$-invariance

joint work with Wataru Kai and Shusuke Otabe.

Motives, quadratic forms and arithmetic.

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§1 Introduction

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§1 Introduction

Artin-Mumford's counter example to Lüroth problem

$X \in \mathrm{SmProp}/\mathbb{C}$, not birat. to $\mathbb{P}^4$

but $\exists\ \mathbb{P}^4 \dashrightarrow X$ dominant rat. map

is based on

$\bigstar$ if X : birat. to $\mathbb{P}^4 \Rightarrow \mathrm{Br}(X) = 0$

[(ii) construct X with $\mathrm{Br}(X)[2] \neq 0$] $\square$

Generalization of $\bigstar$ by Merkurjev, Kahn :

for $X \in \mathrm{SmProp}/k$: field , TFAE

(a) $\deg : CH_0(X_K) \xrightarrow{\cong} \mathbb{Z}$ $\forall K/k$: field ext.

universally CH_0-trivial, UCT

(b) $F(k) \to F(X)$ induced by structure morphism

for $\forall F \in HI = \{ F \in NST : F(U) \xrightarrow{\cong} F(U \times \mathbb{A}^1)\ \forall U \in Sm \}$

$\cap$ $\quad$ $\mathbb{A}^1$-invariant

$NST = \{$Nisnevich sheaf w. transfers on $Sm/k\}$

<u>NB.</u>
- $X \sim \mathbb{P}^4$ birat. $\Rightarrow$ UCT
- $F = \mathrm{Br}[n] \in HI$. if $n \in k^\times$
- $\mathrm{Br}(\mathbb{C}) = 0$

if $\mathrm{ch}\, k = p > 0$, then $\mathrm{Br}[p]$ is NOT in HI. Nevertheless:

Auel, Bigazzi, Böhning, Graf von Bothmer (2021) showed

$X \in \mathrm{SmProp}/k$ $\qquad\qquad$ $\mathrm{ch}(k) = p > 0$

if X : VCT $\Rightarrow$ $\mathrm{Br}(k)[p^r] \cong \mathrm{Br}(X)[p^r]$

and raised a question:

Q Can $\mathrm{Br}[p^r]$ be replaced by $H^{ij}_r = H^i_{ur}(-, W_r\Omega^j_{\log})$?

$\left(\mathrm{NB}: \ H^{11}_r = \mathrm{Br}[p^r]\right)$ $\qquad$ unramified log de Rham coh

Affirmative answers by Binda-Rülling-Saito and Otabe independently

BRS $\quad F \in \mathrm{RSC} = $ (reciprocity sheaves)

$\Rightarrow \boxed{F(k) \xrightarrow{\cong} F(X) \quad \forall X \in \mathrm{SmProp.}\ \mathrm{VCT}}$ (※) □

N.B. $\quad \mathrm{HI} \subset \mathrm{RSC} \subsetneq \mathrm{PI} \subset \mathrm{NST}$

$\not\ni \quad \rotatebox{90}{$\in$}$ H^{ij}_r

$\mathrm{PI} := \{F \in \mathrm{NST} \mid F(U) \xrightarrow{\cong} F(U \times \mathbb{P}^1)\ \forall U \in \mathrm{Sm}\}$ $\quad$ $\mathbb{P}^1$-invariant

$(\bigstar)$ $F(k) \xrightarrow{\cong} F(X)$ $\forall X \in \mathrm{SmProp.}$ VCT

Main Thm. $F \in PI \Rightarrow (\bigstar)$ holds $\square$

NB. • A third (short) proof to above Question

• use only elementary theory of NST, HI
no use of RSC. (no perfectness assumption on k)

• Possible to show $H^{ij}_r \in PI$ directly
(use Izhboldin's Thm $H^{ij}_r(k) \xrightarrow{\cong} H^{ij}_r(\mathbb{P}^1)$ from 1990s
and Gros-Suwa's purity)

• $\exists F \in PI \setminus RSC$: $F = Z_{tr}(\mathbb{A}^1)$ $\in PI$ $\forall$ q-affine
but $F(k) \xrightarrow{\cong} F(X)$ holds for
$\forall X \in \mathrm{SmProp.}$ connected (not nec. VCT)

§2 Main Thm.

Cor. additive category.

Obj = Sm/k = smooth f.t. sep / k.

$Cor(U,V) = \mathbb{Z}\left[\Gamma \subset U \times V \;\middle|\; \begin{array}{l}\text{closed integral,}\\ \text{finite surj. / a component of } U\end{array}\right]$

$NST := \left\{ F : Cor^{op} \to Ab : \begin{array}{l}\text{additive functors s.t.}\\ Sm \hookrightarrow Cor \to Ab \;:\; \text{Nis. sheaf}\end{array} \right\}$

$\mathbb{Z}_{tr}(U) = Cor(-, U) \in NST$

$HI := \{ F \in NST : F(U) \xrightarrow[pr^*]{\cong} F(U \times \mathbb{A}^1) \;\; \forall U \in Sm \}$

$HI \hookrightarrow NST$ admits

a left adjoint : $h_0(F)$ = a maximal HI quotient of $F \in NST$

$h_0(F) = [\, U \mapsto \mathrm{Coker}(F(U \times \mathbb{A}^1) \xrightarrow[i_0^* - i_1^*]{} F(U)) \,]_{Nis}$

$i_\varepsilon : \mathrm{Spec}\, k \hookrightarrow \mathbb{A}^1 \quad (\varepsilon = 0, 1)$

and

a right adjoint : $h^0(F)$ = a maximal HI subobject of F

$h^0(F)(U) = \mathrm{Hom}_{NST}(h_0(U), F)$

$h_0(U) = h_0(\mathbb{Z}_{tr}(U))$.

already a sheaf.

Suslin homology of $U \in Sm$ is by definition

$$H_0^S(U) := h_0(U)(k) = \mathrm{Coker}\left(Cor(\mathbb{A}^1, U) \xrightarrow{i_0^* - i_1^*} Cor(\mathrm{Spec}\, k, U) \right).$$

<u>Fact</u> if X is proper $/k$, then $H_0^S(X) \cong CH_0(X)$.

<u>Thm</u> (Merkurjev, Kahn) $f \in Cor(U, V)$ $U, V \in Sm$ TFAE.

(a) $f_* : H_0^S(U_K) \xrightarrow{\cong} H_0^S(V_K)$ $\forall K/k$

(a)' $f_* : h_0(U) \xrightarrow{\cong} h_0(V)$ in HI

(b) $f^* : F(V) \xrightarrow{\cong} F(U)$ $\forall F \in HI$. $\square$

<u>NB.</u> • Take $V = \mathrm{Spec}\, k$ to recover Introduction

• (a) $\Leftrightarrow$ (a)' Voevodsky. (a)' $\Leftrightarrow$ (b) follows from Yoneda.

• U, V may not be proper $/k$ (Kahn)

<u>Thm A.</u> if U, V proper $/k$, then (a) – (b) are equiv. to

(c) $f^* : G(V) \xrightarrow{\cong} G(U)$ $\forall G \in PI$

$$PI := \left\{ F \in NST : F(U) \xrightarrow[pr^*]{\cong} F(U \times \mathbb{P}^1) \ \forall U \in Sm \right\}$$ $\square$

<u>NB.</u> • Take $Y = \mathrm{Spec}\, k$ to recover Introduction

• $HI \subset RSC \subset PI \subset NST$.

Define for $F \in NST$ $\qquad \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ $\qquad F(U \times \mathbb{A}^1)$

$$\bar{h}_0(F) = \left[U \mapsto \mathrm{Coker}\left(F(U \times \mathbb{P}^1) \xrightarrow{i_0^* - i_1^*} F(U) \right) \right]_{Nis}$$

$$F \twoheadrightarrow \bar{h}_0(F) \twoheadrightarrow h_0(F) \qquad \bar{h}_0(X) = \bar{h}_0(\mathbb{Z}_{tr}(X))$$

<u>Thm. B.</u> $X \in SmProp \Rightarrow \bar{h}_0(X) \xrightarrow{\cong} h_0(X)$ □

pf. of Thm B ⇒ Thm A is formal :

(c) ⇒ (b) holds since $HI \subset PI$

Converse : Take $G \in PI$. for $X \in SmProp$:

$$\begin{aligned}
G(X) &= \mathrm{Hom}_{NST}(\mathbb{Z}_{tr}(X), G) && \text{Yoneda} \\
&= \mathrm{Hom}(\bar{h}_0(X), G) && G \in PI \\
&= \mathrm{Hom}(h_0(X), G) && \text{Thm. B} \\
&\left[= \mathrm{Hom}_{HI}(h_0(X), h^0(G)) \right. && \left. \text{adjoint} \right] \\
&= \mathrm{Hom}(\mathbb{Z}_{tr}(X), h^0(G)) && \text{double adjoint} \\
&= h^0(G)(X) && \text{Yoneda}
\end{aligned}$$

Apply (b) to $h^0(G) \in HI$. □

§3 Moving lemma

Fix $X \in \mathrm{SmProp}$. Define $C_\bullet$, $\overline{C}_\bullet$, $Q_\bullet$ cpx of NST

by $C_n = \mathrm{Cor}(- \times \Delta^n, X)$, $\overline{C}_n = \mathrm{Cor}(- \times \overline{\Delta}^n, X)$ $\qquad Q_n := C_n / \overline{C}_n$

$\Delta^n \cong \mathbb{A}^n$ (standard simplex), $\overline{\Delta}^n \underset{\neq}{\subseteq} \mathbb{P}^n$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
\cdots \arrow[r] & & \overline{C}_2 \arrow[r] \arrow[d] & \overline{C}_1 \arrow[r] \arrow[d] & \overline{C}_0 \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & & C_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] & 0 \\
\cdots \arrow[r] & & Q_2 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}
\qquad \Delta^0 = \overline{\Delta}^0
$$

$$0 \to \overline{C}_\bullet \to C_\bullet \to Q_\bullet \to 0 \quad \text{in NST}$$

$h_0(X) = H_0(C_\bullet)$, $\quad \overline{h}_0(X) = H_0(\overline{C}_\bullet)$

Thm. B is reduced to:

Zariski is sufficient.

Thm. C. $H_1(Q_\bullet) = 0$ in NST if $X \in \mathrm{SmProp}$. $\square$

NB in general $H_n(Q_\bullet) \neq 0$ for

- $n \geq 2$ (e.g. $H_2 \neq 0$ for $X = \mathbb{P}^1$) or
- X not proper$/k$ (e.g. $H_1 \neq 0$ for $X = \mathbb{A}^1$)

$$
\begin{tikzcd}
 & \mathrm{Cor}(U\times\overline{\Delta}^1, X) = \overline{C}_1(U) & U \in Sm \\
C_2(U) = \mathrm{Cor}(U\times\Delta^2, X) \arrow[r, "\partial"] & \mathrm{Cor}(U\times\Delta^1, X) = C_1(U) \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}
$$

$\Sigma \in \mathrm{Cor}(U\times\overline{\Delta}^1, X)$, $\quad \Lambda \in \mathrm{Cor}(U\times\Delta^2, X)$, $\quad \Gamma \in \mathrm{Cor}(U\times\Delta^1, X)$

irred.

$$
\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[dr, "\text{finite surj.}"'] & U\times\Delta^1\times X \arrow[d] \\
 & U\times\Delta^1
\end{tikzcd}
\qquad
\text{closure} =:
\begin{tikzcd}
\overline{\Gamma} \arrow[r, hook] \arrow[dr, "\text{may not be finite}"'] & U\times\overline{\Delta}^1\times X \arrow[d] \\
 & U\times\overline{\Delta}^1
\end{tikzcd}
$$

Want to find Λ, Σ s.t. $\Gamma = \partial\Lambda + \Sigma$

after replacing U by $U_p = \mathrm{Spec}\, \mathcal{O}_{U,p}$ $(p \in U)$

<u>Easy Case</u> $\quad U = \mathrm{Spec}\, k$

$$
\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d, "\text{fin. surj}"'] & \overline{\Gamma} \arrow[d] \\
\Delta^1 = \mathbb{A}^1 \arrow[r, hook] & \overline{\Delta}^1 = \mathbb{P}^1
\end{tikzcd}
$$

generically finite, surj.

$\Downarrow$

finite

no need to use C_2.

General Case.

Step 1. $\quad U = \mathbb{A}^n$

Step 2. $\quad$ Reduce general case to $\mathbb{A}^n$ using Noether normalization $\quad$ (omit today)

Assume k : infinite and $U = \mathbb{A}^n$

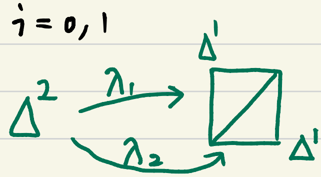
Take $\Gamma \subset \mathbb{A}^n \times \Delta^1 \times X$ as above

<u>Lem.</u> $v \in \mathbb{A}^n(k)$

$$\Gamma = \partial\Big(\underbrace{\Lambda_v^{1*}\Gamma - \Lambda_v^{2*}\Gamma}_{\in\, C_2(X)(U)}\Big) + \Big(\underbrace{\tau_v^{*}\Gamma - \Sigma_v^{0*}\Gamma + \Sigma_v^{1*}\Gamma}_{\text{Lem. 2}}\Big) \text{ in } C_1(X)(U)$$

$$\begin{tikzcd}
\mathbb{A}^n \times X \arrow[r, shift right, "\sigma_i"'] & \mathbb{A}^n \times \Delta^1 \times X \arrow[l, shift right, "\varphi_v"'] \arrow[loop above, "\tau_v"] \arrow[loop below, "\Sigma_v^i"] & & \mathbb{A}^n \times \Delta^2 \times X \arrow[ll, "\Lambda_v^i"'] \arrow[dl, "\lambda_i"] \\
& & \mathbb{A}^n \times \Delta^1 \times \Delta^1 \times X \arrow[ul, "\varphi_v"] &
\end{tikzcd}$$

- $\tau_v : \mathbb{A}^n \longrightarrow \mathbb{A}^n \qquad x \longmapsto x + v$
- $\varphi_v : \mathbb{A}^n \times \Delta^1 \longrightarrow \mathbb{A}^n \qquad (x,t) \longmapsto x + tv \qquad t \in \Delta^1 = \mathbb{A}^1$
- $\sigma_i : \operatorname{Spec} k \overset{0}{\underset{1}{\rightrightarrows}} \Delta^1 \qquad i = 0, 1$
- $\lambda_i : \Delta^2 \xrightarrow{\cong} \Delta^1 \times \Delta^1$



<u>pf.</u> Straight forward computation. $\square$

Thus, the claim for $U = \mathbb{A}^n$ follows from :

<u>Key Prop.</u> $p \in U = \mathbb{A}^n \quad U_p := \operatorname{Spec} \mathcal{O}_{\mathbb{A}^n, p}$

$\emptyset$

$\neq \quad |k| = \infty$

$\Rightarrow \exists B = B(\Gamma, p) \subsetneq \mathbb{A}^n$ closed s.t. $\forall v \in (\mathbb{A}^n \setminus B)(\bar{k})$

$$
\begin{tikzcd}
T_v^* \Gamma, \ \Sigma_v^{i*} \Gamma \arrow[d, mapsto] \arrow[d, mapsto, shift left=6] & \in & C_1(U) \arrow[d] \\
\overline{C}_1(U_p) & \subset & C_1(U_p)
\end{tikzcd}
$$

lands in $\overline{C}_1(U_p)$

$\square$

A step in the proof of key Prop.

<u>Lem. 2</u> $\widetilde{B} := \{ x \in \mathbb{A}^n \times \overline{\Delta^1} : \overline{\Gamma} \text{ is not finite over } p \}$

$\Rightarrow \quad \widetilde{B} \subsetneq \mathbb{A}^n \times (\overline{\Delta^1} \setminus \Delta^1) \cong \mathbb{A}^n$ closed.

pf closed since X is proper.

if $=$ holds, $\overline{\Gamma} \setminus \Gamma \twoheadrightarrow U \times (\overline{\Delta}^1 \setminus \Delta^1) \cong U$.

but $\dim(\overline{\Gamma} \setminus \Gamma) < \dim \Gamma = \dim U$ contradiction

$\square$

$\widetilde{B}$ works for $T_v^* \Gamma$.