On some realizations of motives with modulus
(jt w/ S. Kelly)

motives	cohomology
pure motive	Weil coh
mixed motive	coh of $\mathbb{A}^1$-hom. inv.
motives w/ mod	coh of $(\mathbb{P}^1,\infty)$-inv.

$\{(\mathbb{P}^1,\infty)\text{-inv. sheaves}\}$

$\downarrow$

$\boxed{\Omega^n}$, $W_{p^n}\Omega^q \in \{\text{reciprocity shvs}\} \supset \{\mathbb{A}^1\text{-hom. inv shvs}\}$

$\cup \qquad\qquad \cup$

$\{\text{comm. alg groups}\} \supset \{\text{semi-ab. gps}\}$

$\mathbb{G}_a$, W_{p^n} $\qquad$ $\mathbb{G}_m$, Ab var

Today : representation of Hodge coh. in the cat of motives w/ modulus.

- §1 Motives with modulus (w/ Kahn-Saito-Yamazaki)
- §2 Modulus Kähler differentials
- §3 Representation of $H^i(-,\Omega^q)$

§1.

A modulus pairs is a pair (X,D) s.t.

(1) X is a normal sch. (sep. of f.t.) / k ← fixed base field.

(2) $D\subset X$: eff. Cartier div.

(3) $U:=X\setminus D$ is smooth / k

A morphism $f:(X,D)\to(Y,E)$ is a morphism of k-schemes $f: X\to Y$ s.t. $D\geq f^*E$. $\boxed{f(X\setminus D)\subset Y\setminus E}$

$\mathrm{MSm} :=$ (cat. of mod. pairs)$\left[\{\text{admissible blow-ups}\}^{-1}\right]$

An adm. blow-up of (X,D) is a morphism $f:(X')^N\to X$: normalized blow-up along a center $Z\subset|D|$.

$f:((X')^N, f^*D)\to(X,D)$

This has "fiber products".

$$\begin{tikzcd} (X,D) \arrow[dr] & & (Y,E) \arrow[dl] \\ & (Z,F) & \end{tikzcd}$$

$\Rightarrow$ ∃ fiber product of this

$(X\setminus D)\times_{Z\setminus F}(Y\setminus E)$ sm

morphism in MSm

$$\begin{tikzcd} {} & \cdot \arrow[l, "\text{adm b-up}"'] \arrow[r, "\text{morphism}"] & {} \end{tikzcd}$$

sheaf theory F : presheaf on MSm. $\mathcal{X}=(X,D)$: mod pair

additive $F(\mathcal{X}\sqcup\mathcal{Y})=F(\mathcal{X})\oplus F(\mathcal{Y})$

$F_{\mathcal{X}}$: a presheaf on $X_{\text{ét}}$ by

$$F_{\mathcal{X}}(U):=F(U,D\times_X U).$$

~~Def~~ F is an ét. sheaf if $\forall\mathcal{X}=(X,D)\in\mathrm{MSm}$, $F_{\mathcal{X}}$ is an ét. sheaf on X.

$\mathrm{MSm}\hookrightarrow \mathrm{Sh}(\mathrm{MSm}):=\{\text{ét. sheaves on MSm}\}\underset{\overset{\exists!}{\perp}}{\subset}\{\text{presheaves}\}$

$\mathcal{X}\mapsto[\mathbb{Z}\mathrm{Hom}_{\mathrm{MSm}}(-,\mathcal{X})]_{\text{ét}}$ Groth. ab. $\qquad$ Groth. abelian

$$\mathrm{MDA}^{\text{eff}}:=\frac{D(\mathrm{Sh}(\mathrm{MSm}))}{\langle\mathcal{X}\otimes(\mathbb{P}^1,\infty)\to\mathcal{X};\ \mathcal{X}\in\mathrm{MSm}\rangle}$$

(X,D)

localization $\dashv$ full

$(X,D)\otimes(Y,E):=(X\times Y,\ D\times Y+X\times E)$.

$$\mathrm{DA}^{\text{eff}}:=\frac{D(\mathrm{Sh}(\mathrm{Sm}))}{\langle X\times\mathbb{A}^1\to X;\ X\in\mathrm{Sm}\rangle}$$

M_{Voev} $X\setminus D$

§2 Modulus Kähler diff

k : admits res. sing

Def $\mathcal{X}=(X,D)$ is normal crossing if X is sm, D_{red} is normal crossing.

Def $\forall\mathcal{X}=(X,D)$: nc pair, $\forall q\geq 0$

$$M\Omega^q_{\mathcal{X}}:=\Omega^q_X(\log D_{\text{red}})\otimes_{\mathcal{O}_X}\mathcal{O}_X(D-D_{\text{red}})$$

This is a quasi-coh sheaf of $\mathcal{O}_X$-mod

$\exists!\ M\Omega^q:\mathrm{MSm}^{\text{op}}\to\mathrm{Ab}$ s.t.

$$M\Omega^q(\underset{\substack{\uparrow\\ \text{nc pair}}}{\mathcal{X}})=\Gamma(X,M\Omega_{\mathcal{X}})$$

$M\Omega^q\in\mathrm{Sh}(\mathrm{MSm})$

Rem $M\Omega^q(X,\emptyset)=\Gamma(X,\Omega^q)$

$M\Omega^q(X,D_{\text{red}})=\Gamma(X,\Omega^q_X(\log D_{\text{red}}))$

- $\operatorname*{colim}_{n\geq1} M\Omega^q(X,nD)=\Gamma(X\setminus D,\Omega^q)$

§3 Representation in $\mathrm{MDA}^{\text{eff}}$

Prop 1 $\forall\mathcal{X}\in\mathrm{MSm}$, set

$$H^{i,q}(\mathcal{X}):=H^i_{\underset{(\text{Zar})}{\text{ét}}}(X,M\Omega^q_{\mathcal{X}})$$

Then this depends only on isom. class of $\mathcal{X}\in\mathrm{MSm}$

hence $\exists H^{i,q}:\mathrm{MSm}^{\text{op}}\to\mathrm{Ab}$

Pf Working ét locally, reduced to the case

$X=\mathbb{A}^n$, $D=\{t_1^{r}\cdots t_i^{r_i}=0\}$

$p:Y\to X$: blow-up along $Z\subset|D|$

It suffices to show $R^ip_*M\Omega^q_{(Y,p^*D)}=0$ $\forall i>0$.

This is reduced to $R^ip_*\mathcal{O}(j)=0$ $\forall i>0$ $\forall j>-(n+1)$ $\square$

Prop 2 $H^{i,q}(-)$ is $(\mathbb{P}^1,\infty)$-invariant.

i.e. $H^{i,q}(\mathcal{X})\cong H^{i,q}(\mathcal{X}\otimes(\mathbb{P}^1,\infty))$

Pf May assume $X=\operatorname{Spec}A$, $D=\operatorname{Spec}A/(a)$.

By quasi-coherence of $M\Omega_{\mathcal{X}}$, this amounts to

$$0 \to \underbrace{M\Omega^q_{(A,d)}}_{M\Omega^q(\mathrm{Spec}A,\, \mathrm{Spec}A/(d))} \longrightarrow M\Omega^q_{(A[t],\, d)} \oplus M\Omega^q_{(A[\frac{1}{t}],\, d/t)} \to M\Omega^q_{(A[t,\frac{1}{t}],\, d)} \to 0$$

is a split exact seq of A-mod. $\square$

Corollary

$$\begin{tikzcd} MS_m^{op} \arrow[r, "H^{i,q}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Ab \\ MDA^{\cancel{\mathrm{eff}}\, op} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$$H^{i,q}(\mathcal{X}) \cong \mathrm{Hom}_{MDA^{\cancel{\mathrm{eff}}}}\left(\mathcal{X},\, M\Omega^q[i]\right)$$

Cor (1) Taking $\mathcal{X} = (X, \phi)$

$$H^{i,q}(\mathcal{X}) \underset{\text{by def}}{=} H^i(X, \Omega^q) \cong \mathrm{Hom}_{MDA^{\cancel{\mathrm{eff}}}}\left((X,\phi), M\Omega^q[i]\right)$$

(2) Taking $\mathcal{X} = (X, D_{red})$: normal crossing,

$$H^{i,q}(\mathcal{X}) = H^i(X, \Omega^q(\log D_{red}))$$
$$\cong \mathrm{Hom}_{MDA^{\cancel{\mathrm{eff}}}}\left((X, D_{red}), M\Omega^q[i]\right)$$

(3) $$H^i(\underset{= X\setminus D}{\underset{\sim}{U}}, \Omega^q) \cong \operatorname*{colim}_{n \geq 1} \mathrm{Hom}_{MDA^{\cancel{\mathrm{eff}}}}\left((X, nD), M\Omega^q[i]\right)$$