Moritz Kerz, Universität Regensburg 2022-10-25

Picard-Lefschetz formula for normal crossing spaces

(joint with A. Beilinson, H. Esnault; work in progress)

Background

Classical Picard-Lefschetz formula: cohomological study of Lefschetz pencils.

Essential in Deligne's first proof of the Weil conjectures (1974).

Can monodromy weight conjecture be studied in terms of Lefschetz pencils?

Our Picard-Lefschetz formula in the normal crossing case gives a description of the local monodromy of a degenerated Lefschetz pencil.

The use of Hodge theory below can be replaced by a Frobenius weight argument over a finite field.

Semi-stable Lefschetz pencils (motivation)

$\mathcal{X} \subset \mathbb{P}^N_{\mathcal{O}}$ closed subscheme, semi-stable/$\mathcal{O}$ ($\mathcal{O}$ henselian DVR), rel.dim.$(\mathcal{X}/\mathcal{O}) = n$
$\mathcal{A} \subset \mathbb{P}^N_{\mathcal{O}}$ codim 2 generic linear subspace $\rightsquigarrow$ Lefschetz pencil

$$f\colon \tilde{\mathcal{X}} \to \mathbb{P}^1_{\mathcal{O}}$$

with $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{X}\cap\mathcal{A}}(\mathcal{X})$,
$f_{\bar\eta}\colon \tilde{\mathcal{X}}_{\bar\eta} \to \mathbb{P}^1_{\bar\eta}$ geometric generic fibre.

Fact(Deligne,Katz SGA 7/2). After Veronese embedding $R^a f_{\bar\eta *}\mathbb{Q}_\ell$ is constant sheaf on $\mathbb{P}^1_{\bar\eta}$ for $a \neq n-1$.
$\mathsf{F} = R^{n-1} f_{\bar\eta *}\mathbb{Q}_\ell[1] \in \mathrm{Perv}(\mathbb{P}^1_{\bar\eta})$ perverse sheaf of form

$$\mathsf{F} = \mathrm{const} \oplus j_{!*}(\text{irr. loc.system}).$$

Classical Picard-Lefschetz formula describes local monodromy of F. Our normal crossing PL-formula describes local monodromy of

$$\mathsf{G} = R\Psi_{\mathbb{P}^1/\mathcal{O}}(\mathsf{F}) \in \mathsf{Perv}(\mathbb{P}^1_{\bar{s}}).$$

The monodromy weight conjecture can be rephrased as the equality of monodromy filtrations

$$\mathsf{fil}^M_a H^0(\mathbb{P}^1_{\bar{s}}, \mathsf{G}) \overset{?}{=} \mathsf{im}[H^0(\mathbb{P}^1_{\bar{s}}, \mathsf{fil}^M_a \mathsf{G}) \to H^0(\mathbb{P}^1_{\bar{s}}, \mathsf{G})]$$

for $a \in \mathbb{Z}$.

Preliminaries

X complex analytic space, $\pi\colon X \to \{\star\}$
$\omega = \pi^!(\mathbb{Q})$ dualizing sheaf, $D(\mathsf{G}) = \underline{\mathsf{Hom}}(\mathsf{G}, \omega)$ dual of $\mathsf{G} \in D^b_c(X, \mathbb{Q})$.

Consider $\mathsf{F} \in \mathsf{Perv}(X) \subset D^b_c(X, \mathbb{Q})$ (middle) perverse sheaf, i.e.

$$\dim \mathrm{supp} \mathcal{H}^{-a}(\mathsf{F}) \leq a$$

for all $a \in \mathbb{Z}$ and same for $D(\mathsf{F})$.

Assume given a n-symmetric perfect pairing

$$\mathsf{F} \otimes \mathsf{F} \to \omega(-n)$$

i.e. $\mathsf{F} \to D(\mathsf{F})(-n)$ is an isomorphism.

Milnor fibre

$f\colon X \to \mathbb{C}$ analytic map. For $x \in X$ let $M_x = \mathsf{B}_\delta(x) \cap f^{-1}(t)$ with $0 < |t - f(x)| \ll \delta \ll 1$ be the Milnor fibre and

$$\mathsf{sp}_x(\mathsf{F})\colon \mathsf{F}_x \to \Gamma(M_x, \mathsf{F}|_{M_x})$$

the specialization morphism. If $\mathsf{sp}_x(\mathsf{F})$ is not an isomorphism we call x an F-critical point (wrt to f).

Picard-Lefschetz problem. Assume $f\colon X \to \mathbb{C}$ has only one F-critical point $x_0 \in X$, with $f(x_0) = 0$. Calculate variation map

$$\mathsf{var} = \log(T)\colon H^{-1}(M_{x_0}, \mathsf{F}|_{M_x}) \to H_c^{-1}(M_{x_0}, \mathsf{F}|_{M_{x_0}})$$

(in other degrees variation vanishes), where $T \in \pi_1(\mathbb{C}^\times)$ is loop around $0 \in \mathbb{C}$ assumed to "act unipotently".

Nearby cycle and vanishing cycle functor

$Y = f^{-1}(0) \xrightarrow{i} X$, Cartesian square

$$\begin{tikzcd}
\bar{X}^{\times} \arrow[r, "\bar{\jmath}"] \arrow[d] & X \arrow[d, "f"] \\
\mathbb{C} \arrow[r, "\exp"] & \mathbb{C}
\end{tikzcd}$$

For $\mathsf{G} \in D^b_c(X)$:

- $\psi(\mathsf{G}) = i^*\bar{\jmath}_*\bar{\jmath}^*\mathsf{G}[-1] \in D^b_c(Y, \mathbb{Q})$ (nearby cycle functor) with $T\colon \psi(\mathsf{G}) \to \psi(\mathsf{G})$ monodromy action.

- $\phi(\mathsf{G}) = \mathrm{cone}[i^*\mathsf{G}[-1] \to \psi(\mathsf{G})] \in D^b_c(Y, \mathbb{Q})$ (vanishing cycle functor) with $T\colon \phi(\mathsf{G}) \to \phi(\mathsf{G})$ monodromy action.

- $\Gamma(M_x, \mathsf{G})[-1] \simeq \psi(\mathsf{G})_x$ for all $x \in Y$.

- $u\colon \psi(\mathsf{G}) \to \phi(\mathsf{G})$ canonical map, for T unipotent $v\colon \phi(\mathsf{G}) \to \psi(\mathsf{G})(-1)$ induced by $N = \log(T)$. Then $uv = N$, $vu = N$.

- $x_0 \in Y$ only G-critical point in fibre $f^{-1}(0) \Leftrightarrow \phi(\mathsf{G})|_{Y\setminus\{x_0\}} = 0$.

Facts (Beilinson, Deligne, Gabber; early 1980s).
- ψ, ϕ perverse t-exact ($\Rightarrow \phi(\mathsf{F}) = i_{x_0}(V)[0]$ skyscraper in degree zero, $i_{x_0}\colon \{x_0\} \to X$),
- perfect pairing $\psi(\mathsf{G}) \otimes \psi(D(\mathsf{G})) \to \omega(1)$,
- for T unipotent we get perfect pairing

$$\phi(\mathsf{G}) \otimes \phi(D(\mathsf{G})) \to \omega,$$

such that u and v are adjoint.

Abstract Picard-Lefschetz formula

- $W_c = H^0_{x_0}(Y, \psi(\mathsf{F}))$, $W = \mathcal{H}^0(Y, \psi(\mathsf{F}))_{x_0}$ with perfect pairing $\langle -, - \rangle \colon W \otimes W_c \to \mathbb{Q}(1-n)$,

- $V_c = H^0_{x_0}(Y, \phi(\mathsf{F})) \xrightarrow{\sim} \mathcal{H}^0(Y, \phi(\mathsf{F}))_{x_0} = V$ (space of vanishing cycles) with perfect pairing $\langle -, - \rangle \colon V \otimes V_c \to \mathbb{Q}(-n)$,

- var $= [V_c \xrightarrow{v} W_c(-1)] \circ [V_c \to V]^{-1} \circ [W \xrightarrow{u} V] \colon W \to W_c(-1)$ (variation).

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[rr, bend left, "\mathrm{var}"] & V \arrow[r, "v"] & W_c(-1), \end{tikzcd} \qquad u^* = v$$

PL-problem. Find a basis (as canonical as possible) $\delta_0^\dagger, \dots, \delta_m^\dagger \in V$(Tate twists).

PL(1). Calculate monodromy operator $T\colon V \to V$ in this basis.
PL(2) for T unipotent. Calculate $(m+1) \times (m+1)$-matrix $(\langle \delta_j^\dagger, \delta_i^\dagger \rangle)_{i,j}$ with inverse $A = (a_{ij})_{i,j}$.

Picard-Lefschetz formula. For $\delta_i = v(\delta_i^\dagger) \in W_c$(twist) we have

$$\mathrm{var}(\xi) = \sum_{i,j=0}^{m} \langle \xi, \delta_j \rangle a_{ij} \delta_i$$

for all $\xi \in W$.

Classical Picard-Lefschetz formula

$X = D^n$ with $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, $n \geq 1$, $\mathsf{F} = \mathbb{Q}_X[n]$, $f\colon X \to \mathbb{C}$ submersive except at 0 with Hessian $(\partial_i \partial_j f(0))_{i,j}$ non-degenerate (Morse function).

Morse Lemma. Locally f of the form

$$f(z) = z_1^2 + \cdots + z_n^2.$$

Theorem (Picard $n \leq 2$ (1897), Lefschetz (1924))
dim$(V) = 1$, $\delta^\dagger \in V$ generator of lattice $\mathcal{H}^0(\phi(\mathbb{Z}[n]))_0$.

PL(1). $[T\colon V \to V] = \begin{cases} \mathrm{id}_V & \text{for } n \text{ even,} \\ -\mathrm{id}_V & \text{for } n \text{ odd.} \end{cases}$

PL(2) for n even. $\langle \delta^\dagger, \delta^\dagger \rangle = (-1)^{\frac{n(n+1)}{2}}$.

Picard-Lefschetz formula. For $\delta = v(\delta^\dagger) \in W_c$ we have for var: $W \to W_c(-1)$

$$\boxed{\mathrm{var}(\xi) = (-1)^{\frac{n(n+1)}{2}} \langle \xi, \delta \rangle \delta}$$

Steenbrink-Rapoport-Zink sheaves

A semi-stable degeneration of a Lefschetz pencil leads to a Steenbrink-Rapoport-Zink sheaf.

$\widehat{D}^m = \{z \in D^{m+1} \mid z_0 \cdots z_m = 0\}$, $X = \widehat{D}^m \times D^n$ (normal crossing) $m > 0$, $d = m + n$.

Constant sheaf. We will show $\mathbb{Q}_X[d] \in \mathrm{Perv}(X)$:
$q \colon \tilde{X} \to X$ normalization, $\mathsf{K}^{(0)} = q_*\mathbb{Q}_{\tilde{X}}$, $\mathsf{K}^{(a)} = \wedge^{a+1}\mathsf{K}^{(0)}$ for $a \geq 0$.
The unit $\mathbb{Q}_X \to q_*q^*\mathbb{Q}_X$ gives rise to Koszul resolution

$$\mathbb{Q}_X \simeq [\mathsf{K}^{(0)} \to \mathsf{K}^{(1)} \to \mathsf{K}^{(2)} \to \cdots].$$

Consider brutal filtration

$$\mathsf{fil}_S^a \mathbb{Q}_X \simeq [0 \to \cdots \to 0 \to \mathsf{K}^{(a)} \to \mathsf{K}^{(a+1)} \to \cdots]$$

with $\mathsf{gr}_S^a \mathbb{Q}_X[d] \simeq \mathsf{K}^{(a)}[d-a]$ perverse.

But $\mathbb{Q}_X[d]$ is not self-dual, so has no PL-formula!

SRZ-sheaf. $\mathsf{F} \in \mathsf{Perv}(X)$ or $\mathsf{MHM}(X)$ together with

$$\text{nilpotent } \bar{N} \colon \mathsf{F} \to \mathsf{F}(-1)$$

$$\text{and isomorphism } c \colon \mathbb{Q}_X[d] \xrightarrow{\sim} \ker(\bar{N})$$

is calls SRZ-sheaf if

- $c(\mathsf{fil}_S^a \mathbb{Q}_X[d]) = \mathrm{im}\bar{N}^a \cap \ker \bar{N}$ for $a \geq 0$,

- $(\mathsf{F}, \bar{N}, c)$ is polarizable: there exists d-symmetric perfect pairing $\mathsf{F} \otimes \mathsf{F} \to \omega(-d)$ such that
$\star$ $\bar{N}$ is skew-symmetric and
$\star$ the two obvious pairings on

$$\mathsf{K}^{(a)}[d-a] = \mathsf{gr}_S^a \mathbb{Q}_X[d] \overset{c}{\xrightarrow{\sim}} \mathrm{im}\bar{N}^a \cap \ker \bar{N} / \mathrm{im}\bar{N}^{a+1} \cap \ker \bar{N}$$

coincide.

Proposition(Rapoport-Zink pre-perverse (1982); M. Saito, Illusie, T. Saito perverse (1990–2003)).
For $g\colon D^{m+1} \times D^n \to \mathbb{C}$, $(z,x) \mapsto z_0 \cdots z_m$ we get SRZ-sheaf $\mathsf{F} = \psi_g(\mathbb{Q}[d+1])$, with $\bar{N} = \log \bar{T}$, c induced by $\mathbb{Q}_X[d] = i^*(\mathbb{Q}[d+1])[-1] \to \psi(\mathbb{Q}[d+1])$.

Facts.

- The map

$$\mathbb{Q}(1) \times \cdots \times \mathbb{Q}(m) \xrightarrow{\sim} \mathrm{Aut}(\mathsf{F}, \bar{N}, c),$$
$$(a_1, \ldots, a_m) \mapsto \exp(a_1 \bar{N}) \cdots \exp(a_m \bar{N}^m)$$

is an isomorphism,

- SRZ-sheaf (F, N, c) unique up to non-unique isomorphism in $\mathrm{Perv}(X)$ and up to unique isomorphism in $\mathrm{MHM}(X)$.

Picard-Lefschetz formula for SRZ-sheaves

$f\colon X \to \mathbb{C}$ is stratified Morse function with only critical point $0 \in X$ $:\Leftrightarrow$ $f|_{0\times D^n}$ is Morse function with only critical point 0 and $f|_{S\times D^n}$ submersive with $S \subset \widehat{D}^m$ any closure of a stratum with $\dim(S) > 0$.

Stratified Morse Lemma. Locally f of the form

$$f(z,x) = z_0 + \cdots + z_m + x_1^2 + \cdots + x_n^2.$$

$(\mathsf{F}, \bar{N}, c)$ SRZ-sheaf, $\bar{T} = \exp(\bar{N})$.

PL(1). V has a "Tate motive"-like mixed Hodge structure, $\dim(V) = m+1$

- Semi-simple geometric monodromy action on V as in classical case

$$\boxed{T^{\text{ss}} = \begin{cases} \text{id}_{\text{V}} & \text{for } n \text{ even,} \\ -\text{id}_{\text{V}} & \text{for } n \text{ odd.} \end{cases}}$$

- Unipotent geometric monodromy action runs $m+1$-times faster than the structural monodromy action

$$\boxed{T^{\text{unip}} = \bar{T}^{m+1}}$$

PL(2) for n even. One vanishing cycle $\delta_0^\dagger \in V(\frac{n}{2})$ canonically induced from classical vanishing cycle via

$$\mathbb{Q}_{0\times D^n}[n] \overset{\pm}{\cong} \mathsf{K}^{(m)}[n] \to \mathbb{Q}[d] \xrightarrow{c} \mathsf{F}.$$

$\mathsf{MT}(V) =$ Tannaka group of rigid $\otimes$-subcategory of $\mathbb{Q}$-mixed Hodge structures generated by V.

$$0 \to \mathsf{Rad}_{\mathsf{unip}}(\mathsf{MT}(V)) \to \mathsf{MT}(V) \to \mathbb{G}_m \to 0$$

Conjecture(Mumford-Tate style).

$$\mathsf{Lie}(\mathsf{Rad}_{\mathsf{unip}}(\mathsf{MT}(V))) = \mathbb{Q}\bar{N}.$$

Note that $N = (m+1)\bar{N}$.

Fixing a maximal torus in $\mathsf{MT}(V)$ we get a splitting by weight spaces under the maximal torus

$$V(\frac{n}{2}) \cong \mathbb{Q} \oplus \mathbb{Q}(-1) \oplus \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-m).$$

So for $0 \leq a \leq m$ there exists unique $\delta_a^\dagger \in V(\frac{n}{2}+a)$ in a weight space of the maximal torus with $\bar{N}^a \delta_a^\dagger = \delta_0^\dagger$.

- $\langle \delta_a^\dagger, \delta_b^\dagger \rangle = 0$ for $a + b \neq m$ by weight reasons,
- $\langle \delta_0^\dagger, \delta_m^\dagger \rangle = \pm 1$ by the classical Picard-Lefschetz formula.

Picard-Lefschetz formula. For $\delta_i = v(\delta_i^\dagger) \in W_c$ we have

$$\boxed{\mathrm{var}(\xi) = \sum_{i=0}^{m} (-1)^{\frac{n(n+1)}{2}+i} \langle \xi, \bar{N}^i \delta_m \rangle \bar{N}^{m-i} \delta_m}$$

for all $\xi \in W$.

Remarks. The Mumford-Tate style conjecture is motivated by the fact that in the ℓ-adic world the Zariski closure of the local Galois action has precisely this unipotent radical. The conjecture would imply that the basis of vanishing cycles $(\delta_i^\dagger)_{0\leq i\leq m}$ defines a canonical orbit under the action of $\pm\exp(\mathbb{Q}\bar{N})$. Can one give a geometric interpretation of this orbit?