

## Steenrod operations and algebraic classes

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$X$  smooth proj over  $k = \overline{k}$ ,  $\ell$  invertible in  $k$ .

Th (Kawai 77 / C, Gliwitz - Thelde & Szawiel 10) :

Quick 11

$$H_{\text{ét}}^{2*}(X, \mathbb{F}_\ell^{\otimes *})_{\text{deg}} := \text{Im} \left[ d : CH^*(X) \rightarrow H_{\text{ét}}^{2*}(X, \mathbb{F}_\ell^{\otimes *}) \right]$$

is stable by mod.  $\ell$  Steenrod operations.

The mod 2 Steenrod algebra is a graded  $\mathbb{Z}/2$ -algebra generated by Steenrod squares  $(S_q^i)_{i \geq 0}$  subject to the Adem relations. It acts functorially on the mod 2 cohomology of any space  $X$ .

$$S_q^i : H^k(X, \mathbb{Z}/2) \rightarrow H^{k+i}(X, \mathbb{Z}/2).$$

$$S_q^0 = \text{Id}, S_q^1 = \text{Bockstein}, S_q^k : H^k \xrightarrow{x \mapsto x^2} H^{2k}, \dots$$

Analogous mod  $\ell$  Steenrod algebra for odd prime  $\ell$ .

Consequence: Odd degree Steenrod operations kill algebraic classes.

Kasai thus recovers the first c-ex to the integral Hodge conj  
(due to Atiyah & Hirzebruch 62)

CTN Saito'sly construct the first c-ex to the integral Tate conj  $\overline{\mathbb{F}_p}$ .

Today, many other c-ex by degeneration methods (Kollar, ...)

Degeneration methods fail.  
Few c-ex known. For instance,  
all have torsion in  $\mathbb{P}$ -adic cohomology.

Today: Two extensions of these constructions to degeneracy

- ① Non-closed fields
- ② Interaction with Hodge / Tate classes

# ① Non-closed fields $k$ ( $\ell = 2 \in k^*$ , to satisfy)

In general, algebraic classes are not preserved by Steenrod operations.  
killed by odd degree Steenrod op.

$$\text{Namely, } d: CH^c(X)/_2 \longrightarrow H_{\text{ét}}^{2c}(X, \mathbb{Z}_2^{\otimes c})$$

↗ (  
 Pontryagin's Steenrod  
operations      ↗  
not  
Sq-equivalent  
in general.

↑  
Steenrod operations

The correct statement is:

th:  $Sq(d(x)) = \sum_{i \geq 0} (1 + \omega)^{c-i} d(Sq^{2i}(x)),$

where  $\omega = (-1) \in k^*/(k^*)^2 = H^1(k, \mathbb{Z}/2).$

Corollary: If  $-1$  is a square in  $k$ , then  $d$  is Sq-equivalent.

Hence, algebraic classes are preserved by  $Sq$  (and killed by odd degree Steenrod operations).

$$\text{Corollary: } S_q^3 + (c+1)\omega S_q^2 + \frac{c(c-1)}{2}\omega^3$$

kills  $H_{\text{ét}}^{2c}(X, \mathbb{P}_z^{\otimes c})_{\text{alg}}$ .

Over  $k \neq \overline{k}$ , it is particularly interesting to construct non-algebraic classes that are geometrically algebraic (even geometrically trivial!), as this is then a purely arithmetic phenomena.

Application 1 (finite field)  $p \neq l$  prime numbers

$\mathbb{F}$  finite field of char  $p$ .

There exists  $X/\mathbb{F}$  smooth proj. geom connected and

$x \in \ker \left[ H_{\text{ét}}^4(X, \mathbb{Z}_l(2)) \rightarrow H_{\text{ét}}^4(X_{\mathbb{F}}, \mathbb{Z}_l(2)) \right]$  non-algebraic.

Remarks: When  $\mathbb{F}$  contains a primitive  $l^2$ -th root of unity, due to Scavia & Suzuki (2022): first such examples! They rely on the vanishing of odd degree Steenrod operations.

- We use algebraic presentation of classifying space of a adic necessarily contractible finite étale group scheme ( $\mu_l \times \Gamma_l$ ).

Aplicatia 2 (redu). There exists  $X/R$ , smooth proj. with

$X(R) = \emptyset$  and  $x \in \text{Ker} \left[ H^4_{G(\mathbb{C}/R)}(X(\mathbb{C}), \bar{\epsilon}(2)) \rightarrow H^4(X(\mathbb{C}), \bar{\epsilon}(1)) \right]$   
non-algebraic

Remark: •  $X = \bigcup E \times Q$ , anisotropic quartic 3fold  
elliptic curve with  $E(R)$  non-connected.

•  $X_{\mathbb{C}}$  satisfies the integral Hodge conjecture, but  
 $X$  fails the "red integral Hodge conjecture" defined by B.-Mittal  
Previously, only examples in dimension  $\geq 2$ .  
Are there examples in dim. 3?

Proof of Th: Use "relative Wu theorem" describing the

behaviour of  $Sq := Sq^0 + Sq^1 + \dots$  with respect to  $f_*$ ,

for:  $f: Y \rightarrow X$  proper morphism of smooth var /  $k$ :

\* in Chow [Borel]: for  $\alpha \in CH^c(Y)/\mathbb{Z}$ ,

$$Sq(f_* \alpha) = f_* (Sq \alpha \cdot w(N_f))$$

↑  
Stiefel-Whitney class

$$N_f := [f^* T_X] - [T_Y]$$

\* in étale cohomology [Saito - Suzuki for  $p=2$  and  $k=\bar{k}$ ]:

$$f_* \alpha \in H_{\text{ét}}^q(Y, \mathbb{F}_p), \quad Sq(f_* \alpha) = f_* (Sq \alpha \cdot w^{\text{ét}}(N_f))$$

↑  
étale Stiefel-Whitney class

But  $w(N_f)$  and  $w^{\text{ét}}(N_f)$  are not compatible!

If the  $\lambda_i$  are the Chern roots of  $N_f$  in Chow theory mod. 2,

$$\begin{cases} w(N_f) = \prod_i (1 + \lambda_i) \\ w^{\text{ét}}(N_f) = \prod_i (1 + \omega + \text{cl}(\lambda_i)) \end{cases}$$

Applying the Wu-theorem for  $\alpha = 1$ , one then computes  $\begin{cases} \text{cl}(Sq^x) \\ Sq(\text{cl}x) \end{cases}$

and one sees how they differ!

## ② Interaction with Hodge & Tate classes ( $k = \bar{k}$ )

Another consequence of Kurihara's theorem:

Even degree Steenrod operations send algebraic classes to  
 selectrics mod.  $\mathbb{F}_p$  of  $\begin{cases} \text{Hodge classes } (\mathbb{C}) \\ \text{Tate classes } (\mathbb{F}_p) \end{cases}$

This really may distract the integral Hodge & Tate conjectures, because  
 selectrics mod.  $\mathbb{F}_p$  of Hodge & Tate classes have no reason to be  
 stabilized by Steenrod operations!

Th:  $\ell \in \{2, 3, 5\}$  There exists  $X$  moduli of  $\dim 2\ell+2 / \mathbb{F}_p$   
 with  $H_{\text{ét}}^*(X, \mathbb{Z}_\ell)$  torsion-free and  $\alpha \in H_{\text{ét}}^{2\ell}(X, \mathbb{Z}_\ell(2))$  Tate  
 but not algebraic.

Remarks: • All previous c-ex to the integral Tate conj  $/ \mathbb{F}_p$  in the  
 literature relied on topological obstructions that failed to arise  
 in  $H_{\text{ét}}^*(X, \mathbb{Z}_\ell)$ .

- Hodge analogue (less interesting).
- $X$  is a well-known  $(2\ell+2)$ -dimensional Poincaré  
 section in an "algebraic approximation" of the classifying space of  $E_8$ .