On some models for bi-fluid flows: "incompressible/compressible" systems

Didier Bresch

LAMA UMR5127 CNRS

E-mail: didier.bresch@univ-smb.fr

Based on joint works with:
B. Desjardins (FMJH, Moma group) & C.K. Lin (NCTU, Taiwan)
V. Giovangigli (Ecole Polytechnique) & E. Zatorska (Polish Academy of Sciences)
F. Couderc, P. Noble & J.–P. Vila (Insa Toulouse).

Thanks to S. Dellacherie, G. Faccanoni, B. Grec, F. Lagoutière, Y. Penel for the invitation

Workshop Low Velocity Flows, Paris Descartes, November 05-06, 2015

The part of France, I'm born: Auvergne

Puy de Dôme $\Longrightarrow$ volcanic rocks

Exposition Courtadon: Jardin du Palais Royal 1/09–23/11 2015
http://www.unepierredansmonjardin.com

Main objectives of my talk

- Understand the structure of the systems used in modelization
  Show that playing with the equations may give original results
- Go as closed as possible to weakest regularity (Energy regularity)
  This asks to be precise in the choice of the unknowns and in estimates
- Understand continuous level to hope to enrich the discrete level.

Let us play first on the blackboard !!

Simple calculations showing some interesting features !!

Let us now focus on:

A mixture model/low-mach system with large heat release.

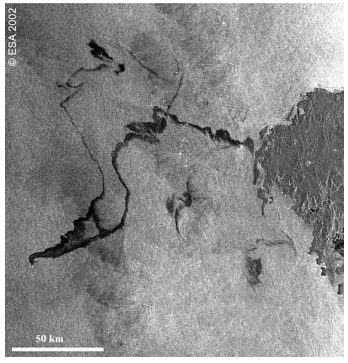
A system which encodes incompressible/compressible features.

What I want to show you:

- The importance of viscosity even if it is small,
- The presence of viscosity effect in dispersive term.
- The existence of two velocities even if the model seems to have only one.

Examples related to the mixture model

Powder-snow avalanche



Spreading of pollutant in water

Mixture system

Consider the following system in periodic box:

$$
\begin{gathered}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\
[M-NS] \quad \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\
\operatorname{div} \mathbf{u} = -2\kappa \triangle \varphi(\varrho).
\end{gathered}
$$

where $D(\mathbf{u}) = (\nabla u + \nabla^t u)/2$ or equivalently

$$
\begin{gathered}
\partial_t \varrho + \nabla \varrho \cdot (\mathbf{u} + 2\kappa \nabla \varphi(\varrho)) - 2\kappa \operatorname{div}(\varrho \nabla \varphi(\varrho)) = 0, \\
\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\
\operatorname{div} \mathbf{u} = -2\kappa \triangle \varphi(\varrho).
\end{gathered}
$$

Note here κ const

Physical literature

Such system:

- 1) Low mach number limit from
  Heat-conducting compressible Navier-Stokes eq. with large heat release.
  See the book by P.–L. Lions.
- 2) Formally obtained as mixture equations with Fick law to close the system. See the book by Rajagopal and Tao.

Some special cases and possible extension:

- 1) For $\varphi(\varrho) = -1/\varrho$ we recover combustion model.
  See works by Embid, Majda, Lions, Lafitte, Dellacherie, Penel...
- 2) For $\varphi(\varrho) = \log \varrho$ we recover pollutant model.
  See works by Graffi, Straughan, Antonsev, Kazhikhov, Monakov...
- 3) See extension to other nonlinearities and to two-fluid system
  by Dellacherie, Faccanoni, Grec, Penel, Lafitte etc... for other cases.

$\kappa = 0 \implies$ Non-homogeneous incompressible Navier–Stokes equations.

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$

$$[NH - INS] \quad \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) + \nabla \Pi = \mathbf{0},$$

$$\operatorname{div} \mathbf{u} = 0.$$

Global well posedness: A. Kazhikhov '77, J. Simon '87, P.–L. Lions '98.

Mathematical literature on the mixture system

- Local strong solutions

  Beirão Da Veiga '82, Secchi '82,
  Danchin & Liao '12 (in critical Besov spaces).

- Global in time solutions

  Kazhikov & Smagulov '77: Modified conv. term, constraint on c_0
  existence of generalized solution which is unique in 2d,
  Lions '98: 2d weak solutions ($\varphi = -1/\varrho$), small perturb. const. ρ_0,
  Secchi '88: 2d unique solution for small c_0
  Danchin & Liao '12: Small perturb. const. ρ + small initial velocity.

- No smallness assumption

  B., Essoufi & Sy '07, for special relation

  $$\varphi'(s) = \mu'(s)/s, \quad \kappa = 1 \quad \Longrightarrow \quad \text{Kazhikhov-Smagulov type system}$$

  Cai, Liao & Sun '12: Uniqueness in 2d,
  Liao '14: Global strong solution in 2d, critical Besov spaces.

Numerical literature

- J. Etienne, E. Hopfinger, P. Saramito.
  Numerical simulations of high density ratio lock-exchange flows.
  No change of variable.
  Finite element + characteristic method with mesh refinements.
- C. Acary-Robert, D. Bresch, D. Dutykh.
  Numerical simulation of powder-snow avalanche interaction with obstacle.
  Numerical test using Open-Foam,
  change of variable + relation between μ and φ
  Discussion around a new entropy encountered in a theoretical paper.
- C. Calgaro, E. Creusé, T. Goudon.
  Simulation of Mixture Flows: Pollution Spreading and Avalanches.
  Change of variable + get ride of high-order terms
  (Kazhikhov-Smagulov type system).
  Numerical schemes: hybrid Finite Volume/Finite Element method.
  Test and comparison.

Goal of this part on this powder snow avalanches system:
A two-velociy hydrodynamics in this model

The case $\mu'(s) = s\varphi'(s)$:

$\Longrightarrow$ A non-linear hypocoercivity property!
$\Longrightarrow$ A two-velocity hydrodynamic in the spirit of H. Brenner but.....
.... with two different velocities:
not volume and mass velocities as in H. Brenner's work
That means not $\mathbf{u}$ and $\mathbf{u} + 2\kappa\nabla\varphi(\rho)$ but two others specified later on.
$\Longrightarrow$ Global existence of weak solutions for a wide range of coefficient.
$\Longrightarrow$ An answer to an open question in P.–L. Lions's book.
$\Longrightarrow$ An interesting numerical scheme
(work in progress with P. Noble, J.–P. Vila).

The case $\mu'(s) \neq s\varphi'(s)$:

A conclusion under some inequalities constraints.
$\Longrightarrow$ An answer to an other open question in P.–L. Lions's book.

Special case where φ and μ are related: Two velocity hydrodynamics

Let us remark

$$\int \nabla \Pi_1 \cdot \mathbf{u} = 2\,\kappa \int \Pi_1 \Delta \varphi(\varrho)$$

and

$$\int \nabla \Pi_1 \cdot (\mathbf{u} + 2\nabla\varphi(\varrho)) = -2\,(1-\kappa) \int \Pi_1 \Delta \varphi(\varrho)$$

Thus

$$\int \nabla \Pi_1 \cdot ((1-\kappa)\mathbf{u}) + \int \nabla \Pi_1 \cdot (\kappa(\mathbf{u} + 2\nabla\varphi(\varrho))) = 0$$

Momentum equation on $\mathbf{u}$:

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho)D(\mathbf{u})) = -\nabla\Pi_1$$

Momentum equation on $\mathbf{v} = \mathbf{u} + 2\nabla\varphi(\varrho)$:

$$\partial_t(\varrho\mathbf{v}) + \operatorname{div}(\varrho\mathbf{v} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho)A(\mathbf{v})) - 2\nabla\Big((\mu'(\varrho)\varrho - \mu(\varrho))\operatorname{div}\mathbf{u}\Big) = -\nabla\Pi_1$$

where $A(\mathbf{v}) = (\nabla\mathbf{v} - \nabla^t\mathbf{v})/2$.

Additional entropy equality

Testing Eq on $\mathbf{u}$ by $(1-\kappa)\mathbf{u}$ and Eq on $\mathbf{v}$ by $\kappa\mathbf{v}$ and adding we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\left((1-\kappa)\frac{|\mathbf{u}|^2}{2}+\kappa\frac{|\mathbf{v}|^2}{2}\right)\,\mathrm{d}x$$
$$+2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x+2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x$$
$$+2(1-\kappa)\kappa^2\int_\Omega (\mu'(\varrho)\varrho-\mu(\varrho))|2\triangle\varphi|^2\,\mathrm{d}x=0,$$

which generalizes the "B-D" entropy to the M-NS system.

Two-velocity hydrodynamic: joint velocity and drift velocity

Remark that:

$$(1-\kappa)|\mathbf{u}|^2 + \kappa|\mathbf{u} + 2\nabla\varphi|^2 = |\mathbf{w}|^2 + (1-\kappa)\kappa|2\nabla\varphi|^2.$$

$\Longrightarrow$ See two-velocity hydrodynamics papers by S.C. Shugrin and S. Gavrilyuk
Defining a new velocity vector field (joint velocity)

$$\mathbf{w} = \mathbf{u} + \kappa\nabla\varphi(\varrho), \quad \text{we see that} \quad \operatorname{div}\mathbf{w} = 0$$

Note that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ is called the drift velocity.

Appropriate unknowns: $\mathbf{w}$ and $\sqrt{(1-\kappa)\kappa}\mathbf{v_1}$.

If $\kappa = 1$, then we get the following system on $(\varrho, \mathbf{v})$:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) - 2\Delta\mu(\varrho) = 0,$$
$$[KS] \quad \partial_t (\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) A(\mathbf{u})) + \nabla \Pi_1 = \mathbf{0},$$
$$\operatorname{div} \mathbf{v} = 0$$

with $\mathbf{u} = \mathbf{v} - 2\nabla\varphi(\varrho)$ (Note that $\mathbf{v}$ is divergence free).

$\Longrightarrow$ Kazhikhov-Smagulov type system

Global well posedness without asking any size constraint on the initial density !!

Proved by D.B., E. Hassan Essoufi, M. Sy '07

With the κ-entropy type estimate: More general results!!

Special case where φ and μ are related

For $0 < T < \infty$, $\Omega = \mathbb{T}^3$ and the low Mach number system

$$[M-NS] \quad \begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi &= \mathbf{0}, \\ \operatorname{div} \mathbf{u} &= -2\kappa \triangle \varphi(\varrho), \end{aligned}$$

with $\varphi'(s) = \mu'(s)/s$, $0 < \kappa < 1$ we have:

T1: For the initial conditions satisfying

$$\sqrt{(1-\kappa)\kappa}\varrho^{\mathbf{0}} \in H^{\mathbf{1}}(\Omega), \quad 0 < r \leq \varrho^{\mathbf{0}} \leq R < \infty, \quad \mathbf{u^0} + 2\kappa \nabla \varphi(\rho^{\mathbf{0}}) \in H,$$

$\mu(\varrho)$ such that $\mu(\varrho) \in C^{\mathbf{1}}([r, R])$, $\mu'(\varrho) > 0$, $\mu \geq c > 0$ on $[r, R]$, and

$$\left(\frac{1-d}{d} \mu(\varrho) + \mu'(\varrho)\varrho \right) \geq c > 0.$$

There exists a global in time weak solution* to [M-NS].

T2: For $\kappa \to 0$ this solution converges to the weak solution of the non-homogenous incompressible N-S equations; for $\kappa \to 1$ (and $\kappa \varrho_\kappa^{\mathbf{0}} \in H^{\mathbf{1}}(\Omega)$) it converges to the weak solutions of the Kazhikhov-Smagulov system.

D.B., V. Giovangigli, E. Zatorska '15: JMPA

General case

For $T < \infty$, $\Omega = \mathbb{T}^3$ and the General low Mach number system

$$[M - NS - G] \quad \begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi &= \mathbf{0}, \\ \operatorname{div} \mathbf{u} &= -2 \triangle \tilde{\varphi}(\varrho), \end{aligned}$$

with $\tilde{\varphi}'(s) = \tilde{\mu}'(s)/s$, we have:

T3: Under previous assumptions on the data and, for $\mu(\cdot) \in C^1([r, R])$, $\mu'(\cdot) > 0$, $\mu \geq c > 0$ on $[r, R]$ and $\tilde{\varphi}(\cdot) \in C^1([r, R])$ and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ related by

$$c \leq \min_{\varrho \in [r,R]} \left(\mu(\varrho) - \tilde{\mu}(\varrho) \right),$$

$$\max_{\varrho \in [r,R]} \frac{\left(\mu(\varrho) - \tilde{\mu}(\varrho) - \xi \tilde{\mu}(\varrho) \right)^2}{2 \left(\mu(\varrho) - \tilde{\mu}(\varrho) \right)} \leq \xi \min_{\varrho \in [r,R]} \left(\tilde{\mu}'(\varrho) \varrho + \frac{1 - d}{d} \tilde{\mu}(\varrho) \right).$$

for some positive constants c, ξ. There exists global weak solution to [M-NS-G].

D.B., V. Giovangigli, E. Zatorska '15: JMPA

Remarks

- For $\mu(\varrho) = \varrho^\alpha$ the exponent α from T1 is

$$\alpha > 1 - \frac{1}{d}.$$

  In particular, it does not depend on κ.

- Assume that all assumptions of T3 are satisfied and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ are replaced by

$$\mu(\varrho) = \varrho, \quad \tilde{\mu}(\varrho) = \log \varrho \quad (\textit{i.e.} \ \ \tilde{\varphi}(\varrho) = -1/\varrho).$$

  Then there exist a non-empty interval $[\tilde{r}, \tilde{R}]$ such that if

$$0 < \tilde{r} \leq \varrho^0 \leq \tilde{R} < 0,$$

  then the weak solution to [M-NS-G] exists globally in time, which corresponds to the dense gas approximation:

S. Chapman and T.G. Cowling:
The mathematical theory of non-uniform gases, 1970.

$\Longrightarrow$ Generalization of P.–L. Lions's result to the 3d case!

Construction of solution

We consider the augmented regularized system with three unknowns $(\varrho, \mathbf{w}, \mathbf{v_1})$:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{w}) - 2\kappa \triangle \mu(\varrho) = 0,$$

$$\partial_t (\varrho \mathbf{w}) + \operatorname{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{w}) - 2(1-\kappa) \operatorname{div}(\mu(\varrho) D(\mathbf{w}))$$
$$- 2\kappa \operatorname{div}(\mu(\varrho) A(\mathbf{w})) + \nabla \Pi_{\mathbf{1}} + \varepsilon \triangle^{\mathbf{2}} \mathbf{w} = -2\kappa(1-\kappa) \operatorname{div}(\mu(\varrho) \nabla \mathbf{v_1}),$$

$$\partial_t (\varrho \mathbf{v_1}) + \operatorname{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{v_1}) - 2\kappa \operatorname{div}(\mu(\varrho) \nabla \mathbf{v_1})$$
$$- 2\kappa \nabla((\mu'(\varrho)\varrho - \mu(\varrho)) \operatorname{div} \mathbf{v_1}) = -2 \operatorname{div}(\mu(\varrho) \nabla^t \mathbf{w}),$$
$$\operatorname{div} \mathbf{w} = 0.$$

Of course, we have to prove that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ to solve the initial system.
To do that Important property: $\operatorname{div} \mathbf{w} = 0$.

Augmented system in other topics: numerical schemes (D. Jamet *et al.*)

For compressible NS equations (see later-on): an extra integrability needed
This is the term $-\varepsilon \operatorname{div}\left(|\nabla \mathbf{w}|^2 \nabla \mathbf{w}\right)$:
An hyper diffusive term introduced by O.A. Ladhyzenskaya.

Extensions to density dependent viscosities compressible NS equations

$$[CNS] \quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla P(\varrho) = \mathbf{0} \end{aligned}$$

with $\lambda(\varrho) = 2(\mu'(\varrho)\varrho - \mu(\varrho))$ (algebraic relation found by D.B., B. Desjardins).

$$P(\rho) = \rho^2/2, \qquad \mu(\varrho) = \mu\varrho \implies \text{Viscous shallow-water type system}$$

Let us introduce an arbitrary coefficient κ such that $0 < \kappa < 1$.

Extensions to density dependent viscosities compressible NS equations

For this compressible barotropic system we have generalized κ- entropy:

$$\int_\Omega \varrho \left(\frac{|\mathbf{w}|^2}{2} + (1-\kappa)\kappa \frac{|2\nabla\varphi(\varrho)|^2}{2} \right)(T)\,\mathrm{d}x + \int_\Omega \varrho e(\varrho)\,\mathrm{d}x$$

$$+ 2(1-\kappa)\int_0^T\!\!\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}t + 2(1-\kappa)\int_0^T\!\!\int_\Omega (\mu'(\varrho)\varrho - \mu(\varrho))(\mathrm{div}\,\mathbf{u})^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+ 2\kappa\int_0^T\!\!\int_\Omega \mu(\varrho)|A(\mathbf{w})|^2\,\mathrm{d}x\,\mathrm{d}t + 2\kappa\int_0^T\!\!\int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$F \leq \int_\Omega \varrho_0 \left(\frac{|\mathbf{w}_0|^2}{2} + (1-\kappa)\kappa \frac{|2\nabla\varphi(\varrho_0)|^2}{2} \right)\,\mathrm{d}x + \int_\Omega \varrho_0 e(\varrho_0)\,\mathrm{d}x,$$

where we have introduced $e(\varrho)$ defined as

$$\frac{\varrho^2 \mathrm{d}e(\varrho)}{\mathrm{d}\varrho} = p(\varrho).$$

κ-entropy may be used to construction of κ-entropy solutions with a simple construction scheme for the compressible system with extra terms (singular pressure or drag terms).

D.B., B. Desjardins, E. Zatorska '15: JMPA

Extensions to density dependent viscosities compressible NS equations

- Global κ-entropy weak solution of the CNS through the κ-entropy:
  Work D.B., B. Desjardins, E. Zatorska (JMPA 2015)
  Non-linear extension of hypocoercivity property known for linearized CNS.
- Interesting framework for asymptotics and numerics (κ relative entropy):
  CRAS 2015 with P. Noble and J.–P. Vila
  (various asymptotics through relative entropy)
  a) Inviscid shallow-water equation
  from the degenerate viscous shallow-water
  b) Low mach number and high Reynolds limit
  Appropriate schemes based on the continuous study: In progress.

Better rate in the asymptotic limits than with constant viscosities.

Relative κ entropy (joint work: D.B., P. Noble, J.–P. Vila)

An appropriate tool for asymptotic limits: The κ relative entropy.

Relative κ entropy (CRAS 2015 D.B, P. Noble, J.–P. Vila)

A useful tool to measure distance between quantities!!

Idea on a simple energy: Assume $e(u,\rho) = e_1(u) + e_2(\rho)$, calculate

$$E(u,\rho) := e(u,\rho) - e(U,r) - \nabla e_1(u)\cdot(u-U) - e_2'(r)(\rho - r)$$

If global strict-convexity then control of

$$|u-U|^2 + |\rho - r|^2$$

See for instance:
C. Dafermos, R. Di Perna, H.T. Yau, Y. Brenier, C. Bardos, F. Otto, A. Tzavaras, L. St Raymond.....

See R. Herbin, T. Galloüet, D. Maltese, A. Novotny, E. Feireisl for the compressible Navier-Stokes with constant viscosities, relative entropy and comparision continuous/discrete solutions..

May be complicated depending on the study !!

If pressure law = viscosity dependency : 1st result by B. Haspot in the 1d case.

For compressible Navier-Stokes with degenerate viscosities in multi-d case with $\mu(\varrho) = \varrho$ and general pressure $p(\varrho)$. Let

$$E(\rho, \mathbf{v}, \mathbf{w}|r, V, W) = \frac{1}{2}\int_\Omega \varrho(|\mathbf{w}-W|^2+|\mathbf{v}-V|^2)\mathrm{d}x+\int_\Omega (F(\varrho)-F(r)-F'(r)(\varrho-r))\mathrm{d}x$$

We calculate

$$I = E(\rho, v, w|r, V, W)(\tau) - E(\rho, v, w|r, V, W)(0)$$
$$+2\kappa\mu\int_0^\tau\int_\Omega \varrho[A(\mathbf{v}-V)|^2 + 2\mu\int_0^\tau\int_\Omega \varrho|D(\sqrt{(1-\kappa)}(\mathbf{v}-V)-\sqrt{\kappa}(\mathbf{w}-W))|^2$$
$$+2\kappa\mu\int_0^\tau\int_\Omega \varrho\Big[p'(\varrho)\nabla\log\varrho - p'(r)\nabla\log r\Big]\cdot\Big[\nabla\log\varrho-\nabla\log r\Big]$$

for all $\tau \in [0, T]$ and for any pair of test functions

$$r \in C^1([0, T]\times\overline{\Omega}), \qquad r > 0, \qquad V, W \in C^1([0, T]\times\overline{\Omega}).$$

$$
\begin{aligned}
I \leq & \int_0^\tau \int_\Omega \varrho\Big(((\mathbf{v}-\sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w})\cdot\nabla W)\cdot(W-\mathbf{w}) \\
& \qquad +((\mathbf{v}-\sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w})\cdot\nabla V)\cdot(V-\mathbf{v})\Big) \\
& + \int_0^\tau \int_\Omega \varrho\Big(\partial_t W\cdot(W-\mathbf{w})+\partial_t V\cdot(V-\mathbf{v})\Big) + \int_0^\tau \int_\Omega \partial_t F'(r)(r-\varrho) \\
& - \int_0^\tau \int_\Omega \nabla F'(r)\cdot\Big[\varrho(\mathbf{v}-\sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w}) - r(V-\sqrt{\frac{\kappa}{(1-\kappa)}}W)\Big] \\
& + \int_0^\tau \int_\Omega (p(r)-p(\varrho))\,\mathrm{div}(V-\sqrt{\frac{\kappa}{(1-\kappa)}}W) \\
& -\kappa\int_0^\tau \int_\Omega p'(\varrho)\nabla\varrho\cdot[2\mu\frac{\nabla r}{r}-\frac{1}{\sqrt{(1-\kappa)\kappa}}W] \\
& +2\mu\int_0^\tau \int_\Omega \varrho\Big(D(\sqrt{(1-\kappa)}V)-\nabla(\sqrt{\kappa}W)\Big): \qquad (1) \\
& \qquad \Big(D(\sqrt{(1-\kappa)}(V-\mathbf{v}))-\nabla(\sqrt{\kappa}(W-\mathbf{w}))\Big) \\
& +2\kappa\mu\int_0^\tau \int_\Omega \varrho A(V):A(V-\mathbf{v}) + 2\kappa\mu\int_0^\tau \int_\Omega \frac{\varrho}{r}p'(r)\nabla r\cdot(\frac{\nabla r}{r}-\frac{\nabla\varrho}{\varrho}) \\
& +2\sqrt{\kappa(1-\kappa)}\mu\int_0^\tau \int_\Omega \varrho\Big[A(W):A(\mathbf{v}-V)-A(\mathbf{w}-W):A(V)\Big]
\end{aligned}
$$

Using that

$$\begin{aligned}
&\varrho[p'(\varrho)\nabla \log \varrho - p'(r)\nabla \log r] \cdot [\nabla \log \varrho - \nabla \log r] = \\
&\varrho p'(\varrho)|\nabla \log \varrho - \nabla \log r|^2 + \nabla[p(\varrho) - p(r) - p'(r)(\varrho - r)] \cdot \nabla \log r \\
&-\left[\varrho(p'(\varrho) - p'(r)) - p''(r)(\varrho - r)r\right]|\nabla \log r|^2,
\end{aligned}$$

Let (r, V, W) strong solution with $r > 0$ then

$$\begin{aligned}
&E(\rho, v, w/r, V, W)(\tau) - E(\rho, v, w/r, V, W)(0) \\
&\leq \int_0^\tau \Big[A(r, V, W)(E(\rho, v, w/r, V, W)\Big]
\end{aligned}$$

where

$$A(r, V, W)(= \|\nabla V\|_{L^\infty(\Omega)} + \|\nabla W\|_{L^\infty(\Omega)} + \|\nabla \log r\|^2_{L^\infty(\Omega)} + \|\Delta \log \rho\|_{L^\infty(\Omega)})$$

$\Longrightarrow$ Weak-strong uniqueness.

$\Longrightarrow$ useful to compare continuous to discrete solution.

An other application of the relative entropy:

Definition. The pair $(\overline{\varrho}, \overline{u})$ is a dissipative solution of the compressible Euler equations if and only if $(\overline{\varrho}, \overline{u})$ satisfies the relative energy inequality

$$E(\overline{\varrho}, \overline{u}, 0 \big| r, U, 0)(t) \leq E(\overline{\varrho}, \overline{u}, 0 \big| r, U, 0)(0) \exp\Big[c_0(r) \int_0^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)} d\tau\Big]$$

$$+ \int_0^t \exp\Big[c_0(r) \int_s^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)}\Big] \int_\Omega \varrho E(r, U) \cdot (U - \overline{u})\, dxds$$

for all smooth test functions (r, U) defined on $[0, T] \times \overline{\Omega})$ so that r is bounded above and below away from zero and (r, w) solves

$$\partial_t r + \mathrm{div}(rU) = 0,$$
$$\partial_t U + U \cdot \nabla U + \nabla F'(r) = E(r, U)$$

for some residual $E(r, U)$.

For constant viscosity: See C. Bardos and T. Nguyen.

For incompressible Euler: See P.–L. Lions' book.

For degenerate viscosity, using the κ-relative entropy
Convergence of global κ-entropy solutions of NS to dissipative solution of Euler
See work with P. Noble and J.-P. Vila.

Relative entropy may be used for singular limit behavior:

- Low Mach Number limit
- High-rotating fluid
- Inviscid limit
- .....

and may me used also for numerical purposes.

See book E. Feireisl, A. Novotny in the constant viscosity case. See numerical results by E. Feireisl, R. Herbin, D. Maltese, A. Novotny.

Euler-Korteweg systems (joint work with F. Couderc, P. Noble, J.–P. Vila)

An other example (Euler-Korteweg) where playing with the structure may give:

- Interesting formula
- Efficient numerical scheme

Euler-Korteweg systems (CRAS 2015 D.B., F. Couderc, P. Noble, J.–P. Vila)

$$[E-K] \quad \begin{array}{c} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) = \mathrm{div}\mathbf{K} \end{array}$$

with

$$\mathbf{K} = \Big(\rho \mathrm{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\Big)\mathrm{Id} - K(\rho)\nabla\rho \otimes \nabla\rho$$

where $K(\rho)$ is the capillary coefficient. Note that

$$\mathrm{div}\,\mathbf{K} = \rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)$$

$$= \mathrm{div}\Big(F(\rho)\nabla\nabla\varphi(\rho)\Big) + \nabla\Big((F'(\rho)\rho - F(\rho))\Delta\varphi(\rho)\Big)$$

with $\sqrt{\rho}\varphi'(\rho) = \sqrt{K(\rho)}$, $F'(\rho) = \sqrt{F(\rho)\rho}$.

$\Longrightarrow$ extended formulation of the Euler-Korteweg system with $w = \nabla\varphi(\rho)$.
In the same spirit than for the two-velocity hydrodynamics before.

$\Longrightarrow$ Stable schemes under hyperbolic CFL condition.

Numerical simulations

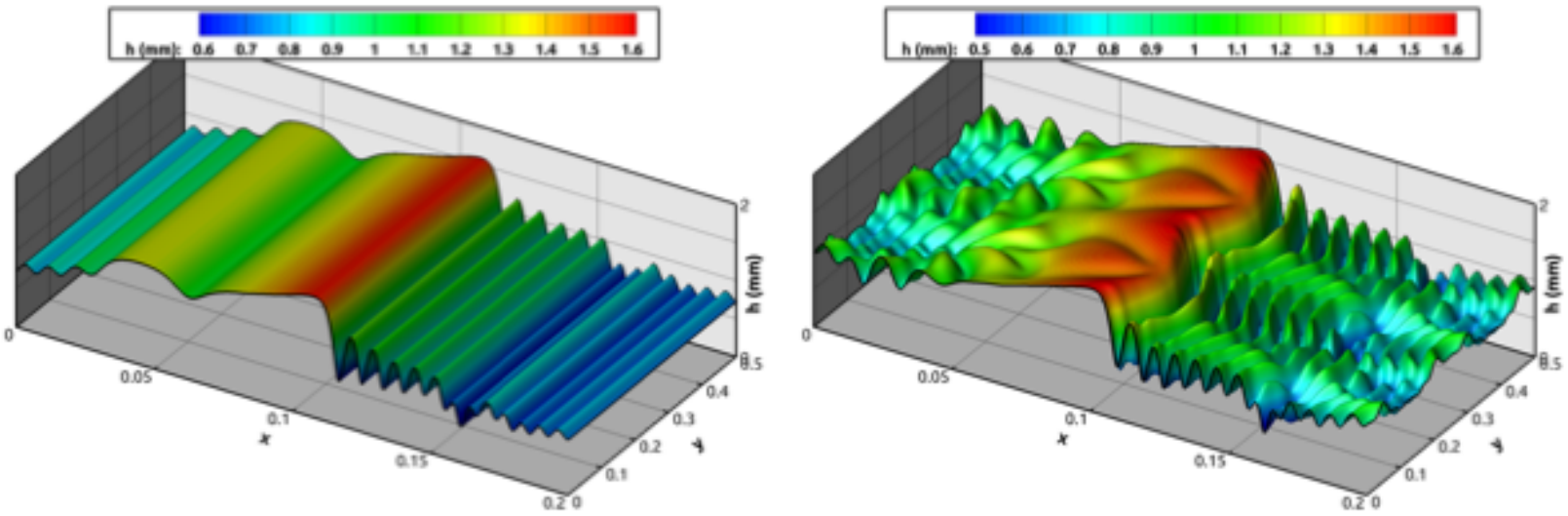


Figure 1. Numerical simulation of a roll-wave in presence of surface tension. On the left: one dimensional roll-wave without transverse perturbations. On the right: a two-dimensional roll-wave