Asymptotic limits of the Shallow Water equations

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Shallow Water equations:

\[
\begin{align*}
\partial_t h + \text{div}(hu) &= 0, \\
\partial_t (hu) + \text{div}(hu \otimes u) + gh\nabla h &= -gh\nabla b.
\end{align*}
\]

- \( u(t, x) \): flow velocity
- \( h(t, x) \): water height
- \( b(x) \): topography
- \( x \in \Omega \subset \mathbb{R}^2 \)
Shallow Water equations:

\[ \partial_t h + \text{div}(hu) = 0, \]

\[ \partial_t (hu) + \text{div}(hu \otimes u) + gh \nabla h = -gh \nabla b. \]
**Shallow Water equations**

**Dimensionless Shallow Water equations:**

\[
\frac{L}{t_{\text{ref}} u_{\text{ref}}} \partial_t h + \text{div}(hu) = 0,
\]
\[
\frac{L}{t_{\text{ref}} u_{\text{ref}}} \partial_t (hu) + \text{div}(hu \otimes u) + \frac{gh_{\text{ref}}}{u_{\text{ref}}^2} h \nabla h = -\frac{gh_{\text{ref}}}{u_{\text{ref}}^2} b_{\text{ref}} h \nabla b.
\]

---

$h(t, x)$: water height

$b(x)$: topography

$u(t, x)$: flow velocity

$b_{\text{ref}}$: reference topography

$h_{\text{ref}}$: reference water height

$L$: characteristic length

$x \in \Omega \subset \mathbb{R}^2$
Shallow Water equations

Dimensionless Shallow Water equations:

$$Sr \partial_t h + \text{div}(hu) = 0,$$

$$Sr \partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{Fr^2} h \nabla h = - \frac{1}{Fr^2} \beta h \nabla b.$$ 

with

$$Sr (= St) = \frac{L}{t_{ref} u_{ref}}$$ the Strouhal number (vortex),

$$Fr = \frac{u_{ref}}{\sqrt{gh_{ref}}}$$ the Froude number (flow vs gravity waves velocities)

and $$\beta = \frac{b_{ref}}{h_{ref}}.$$
Dimensionless Shallow Water equations:

\[ S_r \partial_t h + \text{div}(hu) = 0, \]

\[ S_r \partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{Fr^2} h \nabla h = -\frac{1}{Fr^2} \beta h \nabla b. \]

with

\[ S_r (= St) = \frac{L}{t_{ref} u_{ref}} \] the Strouhal number,

\[ Fr = \frac{u_{ref}}{\sqrt{gh_{ref}}} = \varepsilon^\alpha (\varepsilon \ll 1) \] the Froude number

and \[ \beta = \frac{b_{ref}}{h_{ref}} = 1. \]
Low Froude number flows:

velocities of the flow $<$ speed of the gravity waves

$\Rightarrow$ multiple length / time scales
(depending on initial and boundary conditions).
Low Froude number flows:

velocities of the flow $< \text{speed of the gravity waves}$

$\implies \text{multiple length / time scales}$

(depending on initial and boundary conditions).

During $t_{\text{ref}}$:

$$t_{\text{ref}} u_{\text{ref}} = \frac{L}{S\epsilon}$$

$$t_{\text{ref}} \sqrt{g h_{\text{ref}}} = \left( \frac{L}{S \epsilon} \right) / \epsilon^{\alpha}$$

distance of an advected particle $<$ distance of gravity waves.
Multiple scales in Shallow Water equations

Low Froude number flows:

velocities of the flow $< \text{speed of the gravity waves}$

$\implies$ multiple length / time scales
(depending on initial and boundary conditions).

During $t_{\text{ref}}$:

\[
t_{\text{ref}} u_{\text{ref}} = \frac{L}{Sr} \quad \text{distance of an advected particle} < \text{distance of gravity waves.}
\]

\[
t_{\text{ref}} \sqrt{gh_{\text{ref}}} = \frac{(L/Sr)}{\varepsilon^\alpha} \quad \text{time scales for advected particle} > \text{time scales for gravity waves.}
\]

In a $O(L)$ domain:

\[
\frac{L}{u_{\text{ref}}} = Sr t_{\text{ref}} \quad \text{time scales for advected particle} > \text{time scales for gravity waves.}
\]
Multiscale topography

\[(\varepsilon \ll 1) \quad x \quad x/\varepsilon : \text{quick variations}\]

\[\varepsilon x : \text{slow variations}\]
Multiscale topography

\( (\varepsilon \ll 1) \)

\( x \)

\( x/\varepsilon : \text{quick variations} \)

\( \varepsilon x : \text{slow variations} \)

\[ X = \frac{x}{\varepsilon} \quad \chi = \varepsilon x. \]
1. Balanced flow, topography at the ‘normal’ scale: $b = b(x)$

2. Balanced flow, topography with quick variations: $b = b(X, x)$
   - Weakly nonlinear regime
   - Fully nonlinear regime

3. Topography with long scale variations: $b = b(x, \chi)$

Formal derivations

D. Bresch, R. Klein, C. L., 2011
Balanced flow, topography at the ‘normal’ scale: $b = b(x)$

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   - Weakly nonlinear regime
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3. Topography with long scale variations: $b = b(x, \chi)$
Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)

\[ \begin{align*}
  b &= b(x), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\end{align*} \]

Flow on advective time scales.

Shallow Water equations:

\[ \begin{align*}
  \partial_t h + \text{div}(hu) &= 0, \\
  \partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h &= \frac{1}{\varepsilon^2} h \nabla b.
\end{align*} \]
Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)

\[
b = b(x), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
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Flow on advective time scales.

Shallow Water equations:

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\partial_t h + \text{div}(hu) = 0,
\]

\[
\partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.
\]

Asymptotic development:

\[
h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, x),
\]

\[
u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, x).
\]
Balanced flow, topography at the 'normal' scale: $b = b(x)$

\[ b = b(x), \; \text{Fr} = \varepsilon, \; \text{Sr} = 1 \]

\[ O(\varepsilon^{-2}) \]

\[ h^0 \nabla (h^0 + b) = 0, \]

\[ O(\varepsilon^{-1}) \]

\[ h^1 \nabla (h^0 + b) + h^0 \nabla h^1 = 0, \]

\[ O(\varepsilon^0) \]

\[ \partial_t h^0 + \text{div}(h^0 u^0) = 0, \]

\[ \partial_t (h^0 u^0) + \text{div}(h^0 u^0 \otimes u^0) + h^2 \nabla (h^0 + b) + h^1 \nabla h^1 + h^0 \nabla h^2 = 0. \]
Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)

\[
b = b(x), \quad Fr = \varepsilon, \quad Sr = 1
\]

\( O(\varepsilon^{-2}) \)

\[
h^0 \nabla (h^0 + b) = 0, \quad h^0 + b \equiv c^0(t)
\]

\( O(\varepsilon^{-1}) \)

\[
h^1 \nabla (h^0 + b) + h^0 \nabla h^1 = 0, \quad h^1 \equiv c^1(t)
\]

\( O(\varepsilon^0) \)

\[
\partial_t h^0 + \text{div}(h^0 u^0) = 0,
\]

\[
\partial_t (h^0 u^0) + \text{div}(h^0 u^0 \otimes u^0) + h^2 \nabla (h^0 + b) + h^1 \nabla h^1 + h^0 \nabla h^2 = 0.
\]

\[
\frac{\partial b}{\partial t} = 0: \quad \text{div}(h^0 u^0) = - \frac{d}{dt} c^0(t), \quad \frac{dc^0}{dt} = - \frac{1}{|\Omega|} \int_{\Omega} h^0 u^0 \cdot n \, d\sigma
\]

\[
\partial_t (h^0 u^0) + \text{div}(h^0 u^0 \otimes u^0) + h^0 \nabla h^2 = 0.
\]
Balanced flow, topography at the 'normal' scale: $b = b(x)$

$$b = b(x), \ Fr = \varepsilon, \ Sr = 1$$

Shallow Water limit when $b = b(x), \ Fr = \varepsilon, \ Sr = 1$:

Lake equations

\[
\begin{align*}
\partial_t (h^0 u^0) + \text{div}(h^0 u^0 \otimes u^0) + h^0 \nabla h^2 &= 0, \\
\frac{dc^0}{dt} = \frac{dh^0}{dt} &= -\frac{1}{|\Omega|} \oint_{\Omega} h^0 u^0 \cdot n \, d\sigma
\end{align*}
\]

+ initial / boundary conditions on $h^0, c^0$.

see D. Bresch, G. Métivier, AMRX, 2010
for a rigorous justification of the limit.
Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)

2. Balanced flow, topography with quick variations: \( b = b(X, x) \)
   - Weakly nonlinear regime
   - Fully nonlinear regime

3. Topography with long scale variations: \( b = b(x, \chi) \)
Balanced flow, topography with quick variations: $b = b(X, x)$

$$b = b(X, x)$$

$X \in \mathbb{T}^2$
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

Characteristic lengths too short to support gravity waves. Weakly nonlinear regime.

Shallow Water equations:

\[
\begin{align*}
    \partial_t h + \varepsilon \text{div}(hu) &= 0, \\
    \partial_t (hu) + \varepsilon \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h &= \frac{1}{\varepsilon^2} h \nabla b.
\end{align*}
\]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

Weakly nonlinear regime

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\begin{align*}
\partial_t h + \varepsilon \text{div}(hu) &= 0, \\
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\end{align*}
\]

**Asymptotic development:**

\[
\begin{align*}
h(t, x, \varepsilon) &= \sum_i \varepsilon^i h^i(t, X, x), \\
u(t, x, \varepsilon) &= \sum_i \varepsilon^i u^i(t, X, x).
\end{align*}
\]
Balanced flow, topography with quick variations: $b = b(X, x)$

Weakly nonlinear regime

$$b = b(X, x), \quad Fr = \varepsilon^{3/2}, \quad Sr = \varepsilon^{-1}$$

**$O(\varepsilon^{-3})$**

$$h^0 \nabla_X (h^0 + b) = 0,$$

**$O(\varepsilon^{-2})$**

$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0,$$
Balanced flow, topography with quick variations: $b = b(X, x)$

**Weakly nonlinear regime**

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

\[ O(\varepsilon^{-3}) \]

\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\[ O(\varepsilon^{-2}) \]

\[ h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0, \]

\[ \bar{h}^0 \nabla_x (\bar{h}^0 + b)^X + \bar{h}^0 \nabla_X h^1 X = \bar{h}^0 \nabla_x (\bar{h}^0 + b)^X = 0 : c(t, x) = c(t). \]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

Weakly nonlinear regime

\[ b = b(X, x), \quad Fr = \varepsilon^{3/2}, \quad Sr = \varepsilon^{-1} \]

\[ O(\varepsilon^{-3}) \]

\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\[ O(\varepsilon^{-2}) \]

\[ h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0, \]

\[ \overline{h^0} \nabla_x (h^0 + b)^X + h^0 \nabla_X h^1^X = \overline{h^0} \nabla_x (h^0 + b)^X = 0 : c(t, x) = c(t). \]

\[ \cdots \]

\[ O(\varepsilon^0) \]

\[ \partial_t h^0 + \text{div}_X (h^0 u^0) = 0, \]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

Weakly nonlinear regime

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

\( O(\varepsilon^{-3}) \)

\[ h^0 \nabla_X (h^0 + b) = 0 , \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\( O(\varepsilon^{-2}) \)

\[ h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0 , \]

\[ \frac{h^0 \nabla_x (h^0 + b)^X}{X} + \frac{h^0 \nabla_X h^1^X}{X} = \frac{h^0 \nabla_x (h^0 + b)^X}{X} = 0 : c(t, x) = c(t). \]

\[ \mathrm{...} \]

\( O(\varepsilon^0) \)

\[ \partial_t h^0 + \text{div}_X (h^0 u^0) = 0 , \quad \frac{\partial_t (h^0 + b)^X}{X} = 0 : c(t) = c , \]

\[ h^0(X, x) = -b(X, x) + c. \]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

\( O(\varepsilon^{-3}) \)

\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\( O(\varepsilon^{-2}) \)

\[
\begin{align*}
    h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 &= 0, \\
    \overline{h^0 \nabla_x (h^0 + b)^X} + \overline{h^0 \nabla_X h^1}^X &= \overline{h^0 \nabla_x (h^0 + b)^X} = 0 : c(t, x) = c(t).
\end{align*}
\]

\[ h^0 \nabla_X h^1 = 0 : \quad h^1(t, X, x) = h^1(t, x) \quad O(\varepsilon^{-1}) \quad \rightarrow \quad h^1(t, x) = h^1(t). \]

\[ \ldots \]

\( O(\varepsilon^0) \)

\[ \partial_t h^0 + \text{div}_X (h^0 u^0) = 0, \quad \overline{\partial_t (h^0 + b)^X} = 0 : c(t) = c, \]
Balanced flow, topography with quick variations: $b = b(X, x)$

We are in the weakly nonlinear regime:

$$b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}$$

Asymptotic limits of the Shallow Water equations $O(\varepsilon^1)$

\[
\partial_t h^1 + \text{div}_x (h^0 u^0) + \text{div}_x (h^1 u^0) + \text{div}_x (h^0 u^1) = 0.
\]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[
b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}
\]

Weakly nonlinear regime

\[
b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}
\]

Asymptotic limits of the Shallow Water equations

\[
O(\varepsilon^1)
\]

\[
\partial_t h^1 + \text{div}_x(h^0 u^0) + \text{div}_x(h^1 u^0) + \text{div}_x(h^0 u^1) = 0.
\]

\[
\int_{\Omega} O(\varepsilon^1)^X \, dx : \quad \frac{dh^1}{dt} = -\frac{1}{|\Omega|} \int_{\Omega} h^0 u^0 X \cdot n \, d\sigma
\]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

Weakly nonlinear regime

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

\[ O(\varepsilon^1) \]

\[ \partial_t h^1 + \text{div}_x (h^0 u^0) + \text{div}_X (h^1 u^0) + \text{div}_X (h^0 u^1) = 0. \]

\[ \int_\Omega O(\varepsilon^1)^X dx : \quad \frac{dh^1}{dt} = -\frac{1}{|\Omega|} \int_\Omega h^0 u^0 X \cdot n \, d\sigma \]

We assume rigid vertical walls on \( \partial \Omega \): \( h^1 = \text{cst} = 0 \).
Balanced flow, topography with quick variations: $b = b(X, x)$

Weakly nonlinear regime

$$b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}$$

$$O(\varepsilon^1)$$

$$\partial_t h^1 + \text{div}_x (h^0 u^0) + \text{div}_X (h^1 u^0) + \text{div}_X (h^0 u^1) = 0.$$ 

$$\int \Omega O(\varepsilon^1)^X dx : \quad \frac{dh^1}{dt} = -\frac{1}{|\Omega|} \oint_{\Omega} h^0 u^0 X \cdot n \, d\sigma$$

We assume rigid vertical walls on $\partial \Omega$: $$h^1 = \text{cst} = 0.$$ 

$$\text{div}_x (h^0 u^0)^X = 0.$$
Balanced flow, topography with quick variations: $b = b(X, x)$

$\mathbf{Weakly\ linear\ regime}$

$$b = b(X, x), \ Fr = \varepsilon^{3/2}, \ Sr = \varepsilon^{-1}$$

Using each equation we obtain:

\[
\begin{align*}
\partial_t (h^0 u^0) + \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_X h^3 &= 0 \\
\text{div}_X (h^0 u^0) &= 0 \\
\text{div}_X h^0 u^0 &= 0 \\
\nabla_X h^2 &= 0
\end{align*}
\]

with $h^0(X, x) = c - b(X, x)$. 
Balanced flow, topography with quick variations: \( b = b(X, x) \)

Weakly nonlinear regime

\[ b = b(X, x), \quad Fr = \varepsilon^{3/2}, \quad Sr = \varepsilon^{-1} \]

Using each equation we obtain:

\[
\begin{align*}
\partial_t (h^0 u^0) + \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_X h^3 &= 0 \\
\text{div}_X (h^0 u^0) &= 0 \\
\text{div}_x h^0 u^0 X &= 0 \\
\nabla_X h^2 &= 0
\end{align*}
\]

\( \checkmark \) energy principle

with \( h^0(X, x) = c - b(X, x) \).
Balanced flow, topography with quick variations: $b = b(X, x)$

$$b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}$$

Using each equation we obtain:

\[
\begin{align*}
\partial_t (h^0 u^0) + \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_X h^3 &= 0 \\
\text{div}_X (h^0 u^0) &= 0 \\
\text{div}_X h^0 u^0_X &= 0 \\
\nabla_X h^2 &= 0
\end{align*}
\]

✓ energy principle

with $h^0(X, x) = c - b(X, x)$.

→ large scale ? Average in $X$.
→ small scale ? $\tilde{h} = h - \overline{h}^X$. 

Carine Lucas (MAPMO - Orléans)  Asymptotic limits of the Shallow Water equations  6 nov. 2015 - 15 / 29
Balanced flow, topography with quick variations: $b = b(X, x)$

Weakly nonlinear regime

$$b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}$$

At large scale:

\[
\begin{cases}
\partial_t (h^0 u^0)^X + h^0 X \nabla_x h^2 = -h^3 \nabla_X b^X \\
\text{div}_x h^0 u^0 X = 0
\end{cases}
\]

- response of the leading-order large-scale flow to accumulated small-scale pressure forces on the topography,
- the second order $h^2$ acts like a Lagrangian multiplier.
Balanced flow, topography with quick variations: $b = b(X, x)$

Weakly nonlinear regime

$$b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1}$$

At small scale:

\[
\begin{align*}
\partial_t (\tilde{h}^0 u^0) + \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_X h^3 &= \tilde{b} \nabla_x h^2 \\
\text{div}_X (\tilde{h}^0 u^0) &= 0
\end{align*}
\]

- interactions between small and large scales,
- $\nabla_x h^2$ acts on the fluctuations of the topography to drive the small scale flow.
Balanced flow, topography with quick variations: $b = b(X, x)$

Weakly nonlinear regime

\[ b = b(X, x), \quad \text{Fr} = \varepsilon^{3/2}, \quad \text{Sr} = \varepsilon^{-1} \]

At small scale:

\[
\begin{align*}
\partial_t \overline{h^0 u^0} + \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_X h^3 &= \tilde{b} \nabla_x h^2 \\
\text{div}_X \overline{h^0 u^0} &= 0
\end{align*}
\]

- interactions between small and large scales,
- $\nabla_x h^2$ acts on the fluctuations of the topography to drive the small scale flow.

Weakly nonlinear limit version of the lake equations with oscillatory topography.

D. Bresch, D. Gérard-Varet, AML, 2007
Balanced flow, topography with quick variations: $b = b(X, x)$

$$b = b(X, x), \; Fr = \varepsilon, \; Sr = 1$$

Characteristic lengths too short to support gravity waves.
Fully nonlinear regime.

Shallow Water equations:

$$\partial_t h + \text{div}(hu) = 0,$$

$$\partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.$$
Balanced flow, topography with quick variations: $b = b(X, x)$

$$b = b(X, x), \ Fr = \varepsilon, \ Sr = 1$$

Characteristic lengths too short to support gravity waves. Fully nonlinear regime.

Shallow Water equations:

$$\partial_t h + \text{div}(hu) = 0,$$

$$\partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.$$

Asymptotic development:

$$h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, X, x),$$

$$u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, X, x).$$
Balanced flow, topography with quick variations: $b = b(X, x)$

\[ b = b(X, x), \ Fr = \varepsilon, \ Sr = 1 \]

\[ O(\varepsilon^{-3}) \]

\[ h^0 \nabla_X (h^0 + b) = 0, \]
\[ b = b(X, x), \ Fr = \varepsilon, \ Sr = 1 \]

\[ O(\varepsilon^{-3}) \]
\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]
Balanced flow, topography with quick variations: $b = b(X, x)$

Fully nonlinear regime

$$b = b(X, x), \ Fr = \varepsilon, \ Sr = 1$$

\[ O(\varepsilon^{-3}) \]
\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\[ O(\varepsilon^{-2}) \]
\[ h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0, \]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[ \begin{align*}
\text{Fully nonlinear regime} \\
&= b(X, x), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1 \\
\end{align*} \]

\[ O(\varepsilon^{-3}) \]

\[ h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x) \]

\[ O(\varepsilon^{-2}) \]

\[ h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0, \]

\[ h^0(t, X, x) + b(X, x) \equiv c(t) \]
Balanced flow, topography with quick variations: $b = b(X, x)$

**Fully nonlinear regime**

$$b = b(X, x), \ Fr = \varepsilon, \ Sr = 1$$

\[ O(\varepsilon^{-3}) \]

$$h^0 \nabla_X (h^0 + b) = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t, x)$$

\[ O(\varepsilon^{-2}) \]

$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0, \quad h^0(t, X, x) + b(X, x) \equiv c(t)$$

\[ O(\varepsilon^{-1}) \]

$$\text{div}_X (h^0 u^0) = 0, \quad \text{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^1 + h^0 \nabla_X h^2 = 0.$$
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[
b = b(X, x), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\]

\[
u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_X h^1.
\]

Small scale:
Balanced flow, topography with quick variations: $b = b(X, x)$

$\mathbf{Fr} = \varepsilon, \mathbf{Sr} = 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$  

Small scale:

Taking the curl ($\zeta = \text{curl} u = -\partial_{X_2} u_1 + \partial_{X_1} u_2$):

$$u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \text{div}_X u^0 = \text{div}_X (\zeta^0 u^0) = 0,$$

as $\text{div}_X (h^0 u^0) = 0$, it reads $h^0 u^0 \cdot \nabla_X (\zeta^0 / h^0) = 0$.  

Carine Lucas (MAPMO - Orléans)

Asymptotic limits of the Shallow Water equations 6 nov. 2015 - 20 / 29
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[ b = b(X, x), \; \text{Fr} = \varepsilon, \; \text{Sr} = 1 \]

\[ u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1. \]

Small scale:

Taking the curl (\( \zeta = \text{curl} u = -\partial_{X_2} u_1 + \partial_{X_1} u_2 \)):

\[ u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \text{div}_X u^0 = \text{div}_X (\zeta^0 u^0) = 0, \]

as \( \text{div}_X (h^0 u^0) = 0 \), it reads \( h^0 u^0 \cdot \nabla_X (\zeta^0 / h^0) = 0. \)

\[ \zeta^0 = H^0 Q(\psi^*, 0, x, t), \]

if \( Q \) is a potential vorticity distribution function,

if \( \psi^*, 0 \) is a stream function for \( h^0 u^0 \),

\[ \psi^*, 0 = \psi^0 + X^\perp \cdot \underbrace{h^0 u^0}_{X} \quad \text{with} \quad h^0 u^0 = \nabla^X \psi^*, 0, \]

\[ h^0 \nabla^2_X \psi^0 - \nabla_X h^0 \cdot \nabla_X \psi^0 = (h^0)^3 Q(\psi^*, 0, x, t) - \nabla_X h^0 \cdot \overline{h^0 u^0}_{X^\perp}. \]

Cell problem for a stationary vortical flow over variable topography.
{$b = b(X, x), \ Fr = \varepsilon, \ Sr = 1$}

{$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1$}.

Small scale:

Taking the curl ($\zeta = \text{curl} u = -\partial_{X_2} u_1 + \partial_{X_1} u_2$):

{$u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \text{div}_X u^0 = \text{div}_X (\zeta^0 u^0) = 0$},

as $\text{div}_X (h^0 u^0) = 0$, it reads $h^0 u^0 \cdot \nabla_X (\zeta^0 / h^0) = 0$.

{$\zeta^0 = H^0 Q(\psi^*, 0, x, t)$},

if $Q$ is a potential vorticity distribution function,

if $\psi^*, 0$ is a stream function for $h^0 u^0$,

{$\psi^*, 0 = \psi^0 + X \perp \cdot h^0 u^0 X$ with $h^0 u^0 = \nabla_X^\perp \psi^*, 0$},

{$h^0 \nabla_X^2 \psi^0 - \nabla_X h^0 \cdot \nabla_X \psi^0 = (h^0)^3 Q(\psi^*, 0, x, t) - \nabla_X h^0 \cdot \frac{\nabla_X h^0 X}{h^0 u^0 X \perp}$}.

Cell problem for a stationary vortical flow over variable topography.
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[ b = b(X, x), \ Fr = \varepsilon, \ Sr = 1 \]

\[ u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1. \]

Large scale:
Balanced flow, topography with quick variations: \( b = b(X, x) \)

\[
b = b(X, x), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\]

\[
\begin{align*}
    u^0 \cdot \nabla_X u^0 + \nabla_X h^2 &= -\nabla_x h^1. \\

    \text{Large scale (average in } X): \\
    U \cdot T + \nabla_x h^1 &= -q,
\end{align*}
\]

with

\[
\begin{align*}
    U &= h^0 u^0 X, \\
    \tilde{u} &= u^0 - \frac{1}{h^0} \nabla^0 u^0 X = \frac{1}{h^0} \nabla^1_X \psi^0, \\
    T &= \frac{1}{h^0} \nabla_X \tilde{u}^X \\
    \text{and } q &= \tilde{u} \cdot \nabla_X \tilde{u}^X.
\end{align*}
\]
Balanced flow, topography with quick variations: \( b = b(X, x) \)

**Fully nonlinear regime**

\[
b = b(X, x), \ Fr = \varepsilon, \ Sr = 1
\]

\[
u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.
\]

Large scale (average in \( X \)):

\[
U \cdot T + \nabla_x h^1 = -q, \quad \text{Darcy type problem}
\]

with

\[
U = \overline{h^0 u^0_X}, \ \tilde{u} = u^0 - \frac{1}{h^0} \overline{h^0 u^0_X} = \frac{1}{h^0} \nabla_X \psi^0,
\]

\[
T = \frac{1}{h^0} \nabla_X \tilde{u}^X \quad \text{and} \quad q = \frac{\tilde{u}}{h^0} \cdot \nabla_X \tilde{u}^X \quad \text{(small scale viscous forces)}.
\]

\[
O(\varepsilon^0)^X:
\]

\[
div_x U = -div_x ((\nabla_x h^1 + q) \cdot T^{-1}) = -\frac{d\overline{h^0_X}}{dt}.
\]
Outline

1. Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)

2. Balanced flow, topography with quick variations: \( b = b(X, x) \)
   - Weakly nonlinear regime
   - Fully nonlinear regime

3. Topography with long scale variations: \( b = b(x, \chi) \)
Topography with long scale variations: $b = b(x, \chi)$

\[ b = b(x, \chi), \quad Fr = \varepsilon, \quad Sr = 1 \]
Topography with long scale variations: 
\[ b = b(x, \chi) \]

Advective times for the normal scale \( L \),
with gravity wave dynamics on a large scale \( L/\epsilon \).

Shallow Water equations:

\[
\frac{\partial}{\partial t} h + \text{div}(hu) = 0,
\]

\[
\frac{\partial}{\partial t}(hu) + \text{div}(hu \otimes u) + \frac{1}{\epsilon^2} h \nabla h = \frac{1}{\epsilon^2} h \nabla b.
\]
Topography with long scale variations: \( b = b(x, \chi) \)

\[ b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1 \]

Advective times for the normal scale \( L \),
with gravity wave dynamics on a large scale \( L/\epsilon \).

Shallow Water equations:

\[
\partial_t h + \text{div}(hu) = 0,
\]

\[
\partial_t (hu) + \text{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.
\]

Asymptotic development:

\[
h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, x, \chi),
\]

\[
u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, x, \chi).
\]
Topography with long scale variations: \( b = b(x, \chi) \)

\[ b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1 \]

We get:

- \( h^0 + b = c = c(t, \xi, \chi) \)
- \( \text{div}_x(h^0 u^0) = 0 \)
- \( h^1 = h^1(t, \chi) \)
- \[
\begin{aligned}
\partial_t (h^0 u^0) + \text{div}_x(h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_\chi h^1 &= 0, \\
\partial_t h^1 + \text{div}_x(h^0 u^1) + \text{div}_x(h^1 u^0) + \text{div}_\chi(h^0 u^0) &= 0.
\end{aligned}
\]

→ average in \( x \): long-wave equations

→ study of the small scale flow
Topography with long scale variations: \( b = b(x, \chi) \)

\[
b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\]

We get:

- \( h^0 + b = c = c(t, \xi, \chi) \)
- \( \text{div}_x(h^0 u^0) = 0 \)
- \( h^1 = h^1(t, \chi) \)
- \[
\begin{align*}
\partial_t(h^0 u^0) + \text{div}_x(h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_\chi h^1 &= 0, \\
\partial_t h^1 + \text{div}_x(h^0 u^1) + \text{div}_x(h^1 u^0) + \text{div}_\chi(h^0 u^0) &= 0.
\end{align*}
\]

\[\rightarrow\] average in \( x \): long-wave equations
\[\rightarrow\] study of the small scale flow
Topography with long scale variations: $b = b(x, \chi)$

$$b = b(x, \chi), \ Fr = \varepsilon, \ Sr = 1$$

Long wave:

$$\begin{cases} 
\partial_t \left( h^0 u^0_x \right) + h^0_x \nabla \chi h^1 = h^2 \nabla_x h^0_x \\
\partial_t h^1 + \text{div}_\chi \left( h^0 u^0_x \right) = 0. 
\end{cases}$$
Topography with long scale variations: $b = b(x, \chi)$

\[ b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1 \]

Long wave:

\[
\begin{align*}
\partial_t \left( \overline{h^0 u^0_x} \right) + \overline{h^0_x} \nabla \chi h^1 &= \overline{h^2 \nabla_x h^0 x} \\
\partial_t h^1 + \text{div} \chi \left( \overline{h^0 u^0_x} \right) &= 0.
\end{align*}
\]

\[ \approx \text{standard linearized shallow water equations} \]

\[ \overline{h^2 \nabla_x h^0 x} \] (from $h^0 \nabla_x h^2$): net resistance

(smaller-scale flow through the rough topography).
Topography with long scale variations: $b = b(x, \chi)$

$$b = b(x, \chi), \quad Fr = \varepsilon, \quad Sr = 1$$

Small scale:

$$\begin{cases} 
\partial_t \tilde{h}^0 u^0 + \text{div}_x (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 = -\tilde{h}^0 \nabla \chi h^1, \\
\text{div}_x h^0 u^0 = 0,
\end{cases}$$

where $\varphi = \bar{\varphi}^x + \tilde{\varphi}.$
Topography with long scale variations: \( b = b(x, \chi) \)

\[
b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\]

Small scale:

\[
\begin{align*}
\partial_t \tilde{h}_0 u^0 + \text{div}_x (h_0 u^0 \otimes u^0) + h_0 \nabla_x h^2 &= -\tilde{h}_0 \nabla_\chi h^1, \\
\text{div}_x h_0 u^0 &= 0,
\end{align*}
\]

where \( \varphi = \varphi^x + \tilde{\varphi} \).

- divergence free,
- \( h^2 \): Lagrangian multiplier,
- small-scale flow driven by the long-wave unbalanced part of the large-scale height gradient.

R. Klein, JCP, 1995 (low Mach number)
Topography with long scale variations: \( b = b(x, \chi) \)

\[
b = b(x, \chi), \quad \text{Fr} = \varepsilon, \quad \text{Sr} = 1
\]

If \( b(x, \chi) = b(\chi) \):

\[
\implies \text{wave equation with spatially varying signal speed for } h^1:
\]

\[
\partial_t^2 h^1 - \text{div}_\chi ((c - b(\chi)) \nabla_\chi h^1) = 0,
\]

where \( c = b + h^0 \equiv \text{const.} \).
Concluding remarks

1. Balanced flow, topography at the ‘normal’ scale: $b = b(x)$

2. Balanced flow, topography with quick variations: $b = b(X, x)$
   - Weakly nonlinear regime
   - Fully nonlinear regime

3. Topography with long scale variations: $b = b(x, \chi)$
Concluding remarks

1. Balanced flow, topography at the ‘normal’ scale: $b = b(x)$
   $Fr = \varepsilon, Sr = 1$: Lake equations.

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   - Weakly nonlinear regime
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Concluding remarks

1 Balanced flow, topography at the ‘normal’ scale: $b = b(x)$
   $Fr = \varepsilon$, $Sr = 1$: Lake equations.

2 Balanced flow, topography with quick variations: $b = b(X, x)$
   - Weakly nonlinear regime
     $Fr = \varepsilon^{3/2}$, $Sr = \varepsilon^{-1}$: The large-scale accumulation of net pressure forces drives the large-scale balanced flow; the large-scale height gradients produce small-scale forces driving the small-scale flow.
   - Fully nonlinear regime

3 Topography with long scale variations: $b = b(x, \chi)$
Balanced flow, topography at the ‘normal’ scale: \( b = b(x) \)
Fr = \( \varepsilon \), Sr = 1: Lake equations.

Balanced flow, topography with quick variations: \( b = b(X, x) \)
- Weakly nonlinear regime
  Fr = \( \varepsilon^{3/2} \), Sr = \( \varepsilon^{-1} \): The large-scale accumulation of net pressure forces drives the large-scale balanced flow; the large-scale height gradients produce small-scale forces driving the small-scale flow.
- Fully nonlinear regime
  Fr = \( \varepsilon \), Sr = 1: Darcy-type equation with accumulation of small-scale forces; the small-scale flow is driven by the large-scale mean height gradients (vorticity).

Topography with long scale variations: \( b = b(x, \chi) \)
Balanced flow, topography at the ‘normal’ scale: $b = b(x)$
Fr = $\varepsilon$, Sr = 1: Lake equations.

Balanced flow, topography with quick variations: $b = b(X, x)$

- Weakly nonlinear regime
  Fr = $\varepsilon^{3/2}$, Sr = $\varepsilon^{-1}$: The large-scale accumulation of net pressure forces drives the large-scale balanced flow; the large-scale height gradients produce small-scale forces driving the small-scale flow.

- Fully nonlinear regime
  Fr = $\varepsilon$, Sr = 1: Darcy-type equation with accumulation of small-scale forces; the small-scale flow is driven by the large-scale mean height gradients (vorticity).

Topography with long scale variations: $b = b(x, \chi)$
Fr = $\varepsilon$, Sr = 1: as for the weakly nonlinear case, but the large-scale flow involves non-balanced terms.