Asymptotic limits of the Shallow Water equations

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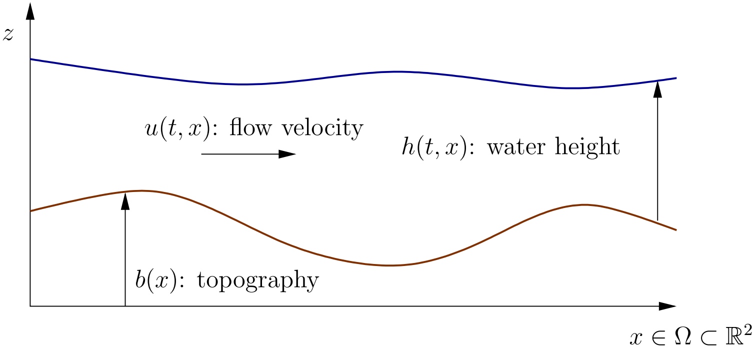
Shallow Water equations

Shallow Water equations:

$$\partial_t h + \operatorname{div}(hu) = 0,$$
$$\partial_t (hu) + \operatorname{div}(hu \otimes u) + gh\nabla h = -gh\nabla b.$$



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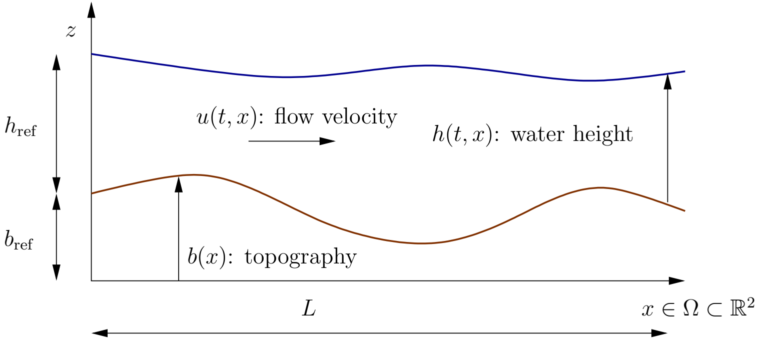
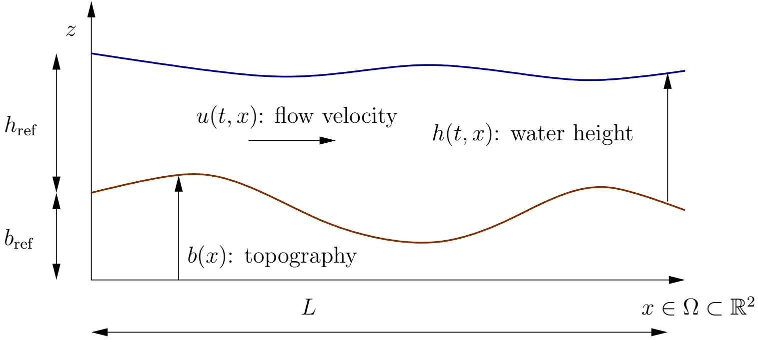


Shallow Water equations

Dimensionless Shallow Water equations:

$$\frac{L}{t_{\rm ref} u_{\rm ref}} \partial_t h + \mathrm{div}(hu) = 0,$$
$$\frac{L}{t_{\rm ref} u_{\rm ref}} \partial_t (hu) + \mathrm{div}(hu \otimes u) + \frac{g h_{\rm ref}}{u_{\rm ref}^2} h \nabla h = -\frac{g h_{\rm ref}}{u_{\rm ref}^2} \frac{b_{\rm ref}}{h_{\rm ref}} h \nabla b.$$



Shallow Water equations

Dimensionless Shallow Water equations:

$$\mathsf{Sr}\partial_t h + \mathrm{div}(hu) = 0,$$

$$\mathsf{Sr}\partial_t (hu) + \mathrm{div}(hu \otimes u) + \frac{1}{\mathsf{Fr}^2} h\nabla h = -\frac{1}{\mathsf{Fr}^2}\beta h \nabla b.$$

with

$\mathsf{Sr}(= \mathsf{St}) = \dfrac{L}{t_{\mathrm{ref}} u_{\mathrm{ref}}}$ the Strouhal number (~~vortex~~),

$\mathsf{Fr} = \dfrac{u_{\mathrm{ref}}}{\sqrt{g h_{\mathrm{ref}}}}$ the Froude number (flow vs gravity waves velocities)

and $\beta = \dfrac{b_{\mathrm{ref}}}{h_{\mathrm{ref}}}$.

Shallow Water equations

Dimensionless Shallow Water equations:

$$\mathsf{Sr}\partial_t h + \operatorname{div}(hu) = 0,$$

$$\mathsf{Sr}\partial_t (hu) + \operatorname{div}(hu \otimes u) + \frac{1}{\mathsf{Fr}^2} h\nabla h = -\frac{1}{\mathsf{Fr}^2}\beta h \nabla b.$$

with

$\mathsf{Sr}(=\mathsf{St}) = \dfrac{L}{t_{\mathrm{ref}} u_{\mathrm{ref}}}$ the Strouhal number,

$\mathsf{Fr} = \dfrac{u_{\mathrm{ref}}}{\sqrt{g h_{\mathrm{ref}}}} = \varepsilon^\alpha \ (\varepsilon \ll 1)$ the Froude number

and $\beta = \dfrac{b_{\mathrm{ref}}}{h_{\mathrm{ref}}} = 1$.

Multiple scales in Shallow Water equations

Low Froude number flows:

velocities of the flow $<$ speed of the gravity waves

$\Longrightarrow$ multiple length / time scales
(depending on initial and boundary conditions).

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During t_{ref}:

$t_{\mathrm{ref}} u_{\mathrm{ref}} = L/\mathrm{Sr}$		$t_{\mathrm{ref}} \sqrt{g h_{\mathrm{ref}}} = (L/\mathrm{Sr})/\varepsilon^{\alpha}$
distance of an advected particle	$<$	distance of gravity waves.

Multiple scales in Shallow Water equations

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(depending on initial and boundary conditions).

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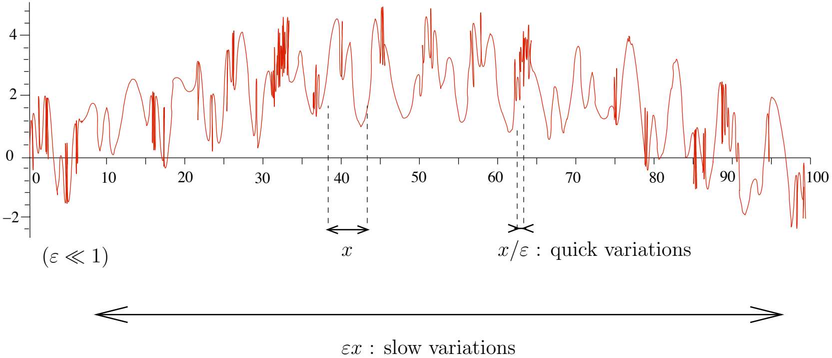
distance of an advected particle	$<$	distance of gravity waves.

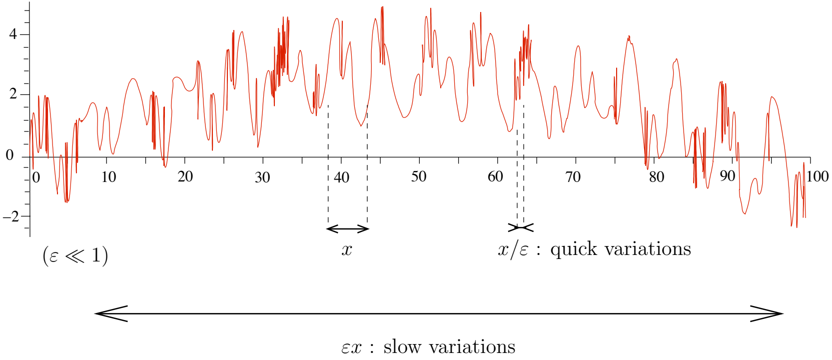
In a $O(L)$ domain:

$L/u_{\mathrm{ref}} = \mathrm{Sr}\, t_{\mathrm{ref}}$		$L/\sqrt{h_{\mathrm{ref}}} = \varepsilon^{\alpha}\, \mathrm{Sr}\, t_{\mathrm{ref}}$
time scales for advected particle	$>$	time scales for gravity waves.

Multiscale topography



Multiscale topography



$$X = \frac{x}{\varepsilon} \qquad \chi = \varepsilon x.$$

Outline

1. Balanced flow, topography at the ‘normal’ scale: $b = b(x)$
2. Balanced flow, topography with quick variations: $b = b(X, x)$
   - Weakly nonlinear regime
   - Fully nonlinear regime
3. Topography with long scale variations: $b = b(x, \chi)$

Formal derivations

D. Bresch, R. Klein, C. L., 2011

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$b = b(x)$, Fr $= \varepsilon$, Sr $= 1$

Flow on advective time scales.

Shallow Water equations:

$$\partial_t h + \mathrm{div}(hu) = 0,$$

$$\partial_t (hu) + \mathrm{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.$$

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Asymptotic development:

$$h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, x),$$
$$u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, x).$$

$b = b(x)$, Fr $= \varepsilon$, Sr $= 1$

$O(\varepsilon^{-2})$

$$h^0 \nabla (h^0 + b) = 0\,,$$

$O(\varepsilon^{-1})$

$$h^1 \nabla (h^0 + b) + h^0 \nabla h^1 = 0\,,$$

$O(\varepsilon^0)$

$$\begin{aligned} \partial_t h^0 + \operatorname{div}(h^0 u^0) &= 0, \\ \partial_t (h^0 u^0) + \operatorname{div}(h^0 u^0 \otimes u^0) + \\ h^2 \nabla (h^0 + b) + h^1 \nabla h^1 + h^0 \nabla h^2 &= 0. \end{aligned}$$

$b = b(x)$, Fr $= \varepsilon$, Sr $= 1$

$O(\varepsilon^{-2})$

$$h^0 \nabla (h^0 + b) = 0\,, \qquad h^0 + b \equiv c^0(t)$$

$O(\varepsilon^{-1})$

$$h^1 \nabla (h^0 + b) + h^0 \nabla h^1 = 0\,, \qquad h^1 \equiv c^1(t)$$

$O(\varepsilon^{0})$

$$\begin{aligned} \partial_t h^0 + \operatorname{div}(h^0 u^0) &= 0, \\ \partial_t (h^0 u^0) + \operatorname{div}(h^0 u^0 \otimes u^0) + \\ h^2 \nabla (h^0 + b) + h^1 \nabla h^1 + h^0 \nabla h^2 &= 0. \end{aligned}$$

$$\frac{\partial b}{\partial t} = 0 : \quad \operatorname{div}(h^0 u^0) = -\frac{d}{dt} c^0(t), \quad \frac{dc^0}{dt} = -\frac{1}{|\Omega|} \int_\Omega h^0 u^0 \cdot n \, d\sigma$$

$$\partial_t (h^0 u^0) + \operatorname{div}(h^0 u^0 \otimes u^0) + h^0 \nabla h^2 = 0.$$

$b = b(x)$, Fr $= \varepsilon$, Sr $= 1$

Shallow Water limit when $b = b(x)$, Fr $= \varepsilon$, Sr $= 1$:

Lake equations

$$\begin{aligned} \partial_t(h^0 u^0) + \mathrm{div}(h^0 u^0 \otimes u^0) + h^0 \nabla h^2 &= 0, \\ \frac{dc^0}{dt} = \frac{dh^0}{dt} &= -\frac{1}{|\Omega|} \oint_{\Omega} h^0 u^0 \cdot n \, d\sigma \end{aligned}$$

+ initial / boundary conditions on h^0, c^0.

see D. Bresch, G. Métivier, AMRX, 2010
for a rigorous justification of the limit.

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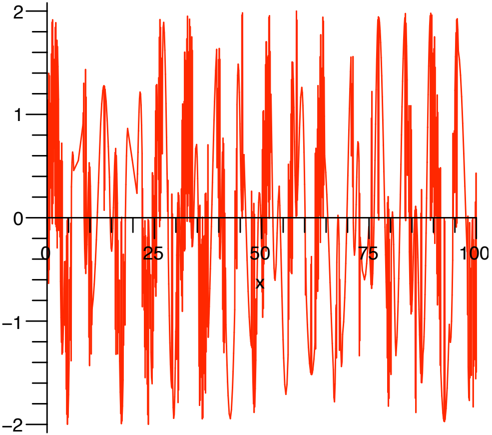

$X \in \mathbb{T}^2$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

Characteristic lengths too short to support gravity waves.
Weakly nonlinear regime.

Shallow Water equations:

$$\partial_t h + \varepsilon \mathrm{div}(hu) = 0,$$

$$\partial_t (hu) + \varepsilon \mathrm{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.$$

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Asymptotic development:

$$h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, X, x),$$

$$u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, X, x).$$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

$O(\varepsilon^{-3})$

$$h^0 \nabla_X (h^0 + b) = 0\,,$$

$O(\varepsilon^{-2})$

$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0\,,$$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

$O(\varepsilon^{-3})$

$$h^0 \nabla_X (h^0 + b) = 0\,, \qquad h^0(t, X, x) + b(X, x) \equiv c(t, x)$$

$O(\varepsilon^{-2})$

$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0\,,$$

$$\overline{h^0 \nabla_x (h^0 + b)}^X + \overline{h^0 \nabla_X h^1}^X = \overline{h^0 \nabla_x (h^0 + b)}^X = 0 \ : c(t, x) = c(t).$$

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...

$O(\varepsilon^0)$

$$\partial_t h^0 + \mathrm{div_X}(h^0 u^0) = 0,$$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

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$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0\,,$$

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...

$O(\varepsilon^0)$

$$\partial_t h^0 + \mathrm{div}_X (h^0 u^0) = 0, \qquad \overline{\partial_t (h^0 + b)}^X = 0 : c(t) = c,$$

$$h^0(X, x) = -b(X, x) + c.$$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

$O(\varepsilon^{-3})$

$$h^0 \nabla_X (h^0 + b) = 0\,, \qquad h^0(t, X, x) + b(X, x) \equiv c(t, x)$$

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$$h^0 \nabla_x (h^0 + b) + h^1 \nabla_X (h^0 + b) + h^0 \nabla_X h^1 = 0\,,$$

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$$h^0 \nabla_X h^1 = 0 : \quad h^1(t, X, x) = h^1(t, x) \quad \xrightarrow{O(\varepsilon^{-1})} h^1(t, x) = h^1(t).$$

...

$O(\varepsilon^0)$

$$\partial_t h^0 + \mathrm{div_X}(h^0 u^0) = 0, \qquad \overline{\partial_t (h^0 + b)}^X = 0 : c(t) = c,$$

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$O(\varepsilon^1)$

$$\partial_t h^1 + \mathrm{div}_x(h^0 u^0) + \mathrm{div}_X(h^1 u^0) + \mathrm{div}_X(h^0 u^1) = 0.$$

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$\int_\Omega \overline{O(\varepsilon^1)}^X dx :$ $\qquad \dfrac{dh^1}{dt} = -\dfrac{1}{|\Omega|} \oint_\Omega \overline{h^0 u^0}^X \cdot n \, d\sigma$

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We assume rigid vertical walls on $\partial\Omega$: $\qquad h^1 = cst = 0.$

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We assume rigid vertical walls on $\partial\Omega$: $\qquad h^1 = cst = 0.$

$$\mathrm{div}_x \overline{(h^0 u^0)}^X = 0.$$

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

Using each equation we obtain:

$$\begin{cases} \partial_t(h^0u^0) + \mathrm{div}_X(h^0u^0 \otimes u^0) + h^0\nabla_x h^2 + h^0\nabla_X h^3 = 0 \\ \mathrm{div}_X(h^0u^0) = 0 \\ \mathrm{div}_x \overline{h^0u^0}^X = 0 \\ \nabla_X h^2 = 0 \end{cases}$$

with $h^0(X, x) = c - b(X, x)$.

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☑ energy principle

with $h^0(X, x) = c - b(X, x)$.

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☑ energy principle

with $h^0(X, x) = c - b(X, x)$.

$\longrightarrow$ large scale ? Average in X.
$\longrightarrow$ small scale ? $\widetilde{h} = h - \overline{h}^X$.

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

At large scale:

$$\begin{cases} \partial_t \overline{(h^0 u^0)}^X + \overline{h^0}^X \nabla_x h^2 = -\overline{h^3 \nabla_X b}^X \\ \mathrm{div}_x \overline{h^0 u^0}^X = 0 \end{cases}$$

- response of the leading-order large-scale flow to accumulated small-scale pressure forces on the topography,
- the second order h^2 acts like a Lagrangian multiplier.

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

At small scale:

$$\begin{cases} \partial_t \widetilde{(h^0 u^0)} + \mathrm{div}_X (h^0 u^0 \otimes u^0) + h^0 \widetilde{\nabla_X h^3} = \widetilde{b} \nabla_x h^2 \\ \mathrm{div}_X \widetilde{(h^0 u^0)} = 0 \end{cases}$$

- interactions between small and large scales,
- $\nabla_x h^2$ acts on the fluctuations of the topography to drive the small scale flow.

$b = b(X, x)$, Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$

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Weakly nonlinear limit version of the lake equations with oscillatory topography.

D. Bresch, D. Gérard-Varet, AML, 2007

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

Characteristic lengths too short to support gravity waves.
Fully nonlinear regime.

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Asymptotic development:

$$h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, X, x),$$

$$u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, X, x).$$

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$O(\varepsilon^{-3})$

$$h^0 \nabla_X (h^0 + b) = 0\,,$$

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$O(\varepsilon^{-3})$

$$h^0 \nabla_X (h^0 + b) = 0\,, \qquad h^0(t, X, x) + b(X, x) \equiv c(t, x)$$

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$O(\varepsilon^{-1})$

$$\mathrm{div}_X (h^0 u^0) = 0\,,$$

$$\mathrm{div}_X (h^0 u^0 \otimes u^0) + h^0 \nabla_x h^1 + h^0 \nabla_X h^2 = 0.$$

$\longrightarrow$ small scale ?
$\longrightarrow$ large scale ?

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Small scale:

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Small scale:

Taking the curl ($\zeta = \mathrm{curl} u = -\partial_{X_2} u_1 + \partial_{X_1} u_2$):

$$u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \mathrm{div}_X u^0 = \mathrm{div}_X(\zeta^0 u^0) = 0\,,$$

as $\mathrm{div}_X(h^0 u^0) = 0$, it reads $h^0 u^0 \cdot \nabla_X(\zeta^0 / h^0) = 0$.

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Small scale:

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as $\mathrm{div}_X(h^0 u^0) = 0$, it reads $h^0 u^0 \cdot \nabla_X(\zeta^0 / h^0) = 0$.

$$\zeta^0 = H^0 Q(\psi^{*,0}, x, t)\,,$$

if Q is a potential vorticity distribution function,
if $\psi^{*,0}$ is a stream function for $h^0 u^0$,

$$\psi^{*,0} = \psi^0 + X^\perp \cdot \overline{h^0 u^0}^X \text{ with } h^0 u^0 = \nabla_X^\perp \psi^{*,0}\,,$$

$$h^0 \nabla_X^2 \psi^0 - \nabla_X h^0 \cdot \nabla_X \psi^0 = (h^0)^3 Q(\psi^{*,0}, x, t) - \nabla_X h^0 \cdot \overline{h^0 u^0}^{X\perp}\,.$$

Cell problem for a stationary vortical flow over variable topography.

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Small scale:

Taking the curl ($\zeta = \mathrm{curl} u = -\partial_{X_2} u_1 + \partial_{X_1} u_2$):

$$u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \mathrm{div}_X u^0 = \mathrm{div}_X(\zeta^0 u^0) = 0\,,$$

as $\mathrm{div}_X(h^0 u^0) = 0$, it reads $h^0 u^0 \cdot \nabla_X(\zeta^0 / h^0) = 0$.

$$\zeta^0 = H^0 Q(\psi^{*,0}, x, t)\,,$$

if Q is a potential vorticity distribution function,
if $\psi^{*,0}$ is a stream function for $h^0 u^0$,

$$\psi^{*,0} = \psi^0 + X^\perp \cdot \overline{h^0 u^0}^X \text{ with } h^0 u^0 = \nabla_X^\perp \psi^{*,0}\,,$$

$$h^0 \nabla_X^2 \psi^0 - \nabla_X h^0 \cdot \nabla_X \psi^0 = (h^0)^3 Q(\psi^{*,0}, x, t) - \nabla_X h^0 \cdot \boxed{\overline{h^0 u^0}^{X\perp}}\,.$$

Cell problem for a stationary vortical flow over variable topography.

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Large scale:

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Large scale (average in X):

$$U \cdot T + \nabla_x h^1 = -q,$$

with

$$U = \overline{h^0 u^0}^X,\ \widetilde{u} = u^0 - \frac{1}{h^0}\overline{h^0 u^0}^X = \frac{1}{h^0}\nabla_X^\perp \psi^0\,,$$

$$T = \overline{\frac{1}{h^0}\nabla_X \widetilde{u}}^X \text{ and } q = \overline{\widetilde{u} \cdot \nabla_X \widetilde{u}}^X\ .$$

$b = b(X, x)$, Fr $= \varepsilon$, Sr $= 1$

$$u^0 \cdot \nabla_X u^0 + \nabla_X h^2 = -\nabla_x h^1.$$

Large scale (average in X):

$$U \cdot T + \nabla_x h^1 = -q, \qquad \text{Darcy type problem}$$

with $U = \overline{h^0 u^0}^X$, $\widetilde{u} = u^0 - \frac{1}{h^0}\overline{h^0 u^0}^X = \frac{1}{h^0}\nabla_X^\perp \psi^0$,

$T = \overline{\frac{1}{h^0}\nabla_X \widetilde{u}}^X$ and $q = \overline{\widetilde{u} \cdot \nabla_X \widetilde{u}}^X$ (small scale ~~viscous forces~~).

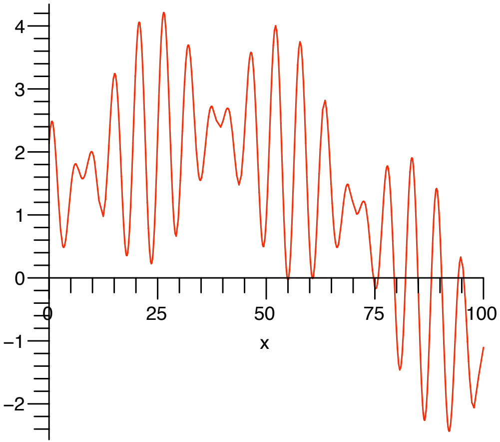
$\overline{O(\varepsilon^0)}^X$:

$$\mathrm{div}_x U = -\mathrm{div}_x \left((\nabla_x h^1 + q) \cdot T^{-1} \right) = -\frac{d\overline{h^0}^X}{dt}.$$

Outline

1. Balanced flow, topography at the ‘normal’ scale: $b = b(x)$
2. Balanced flow, topography with quick variations: $b = b(X, x)$
   - Weakly nonlinear regime
   - Fully nonlinear regime
3. Topography with long scale variations: $b = b(x, \chi)$

$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$



$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$

Advective times for the normal scale L,
with gravity wave dynamics on a large scale L/ϵ.

Shallow Water equations:

$$\partial_t h + \mathrm{div}(hu) = 0,$$
$$\partial_t (hu) + \mathrm{div}(hu \otimes u) + \frac{1}{\varepsilon^2} h \nabla h = \frac{1}{\varepsilon^2} h \nabla b.$$

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Asymptotic development:

$$h(t, x, \varepsilon) = \sum_i \varepsilon^i h^i(t, x, \chi),$$
$$u(t, x, \varepsilon) = \sum_i \varepsilon^i u^i(t, x, \chi).$$

$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$

We get:

- $h^0 + b = c = c(\cancel{t}, \cancel{x}, \cancel{\chi})$
- $\mathrm{div}_x(h^0 u^0) = 0$
- $h^1 = h^1(t, \chi)$
- $$\begin{cases} \partial_t(h^0 u^0) + \mathrm{div}_x(h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 + h^0 \nabla_\chi h^1 = 0, \\ \partial_t h^1 + \mathrm{div}_x(h^0 u^1) + \mathrm{div}_x(h^1 u^0) + \mathrm{div}_\chi(h^0 u^0) = 0. \end{cases}$$

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$\longrightarrow$ average in x: long-wave equations

$\longrightarrow$ study of the small scale flow

$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$

Long wave:

$$\begin{cases} \partial_t \left(\overline{h^0 u^0}^x \right) + \overline{h^0}^x \nabla_\chi h^1 = \overline{h^2 \nabla_x h^0}^x \\ \partial_t h^1 + \mathrm{div}_\chi \left(\overline{h^0 u^0}^x \right) = 0. \end{cases}$$

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$\approx$ standard linearized shallow water equations

$\overline{h^2 \nabla_x h^0}^x$ (from $h^0 \nabla_x h^2$): net resistance
(small-scale flow through the rough topography).

$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$

Small scale:

$$\begin{cases} \partial_t \widetilde{\widetilde{h^0 u^0}} + \mathrm{div}_x(h^0 u^0 \otimes u^0) + h^0 \nabla_x h^2 = -\widetilde{\widetilde{h^0}} \nabla_\chi h^1, \\ \mathrm{div}_x \widetilde{\widetilde{h^0 u^0}} = 0, \end{cases}$$

where $\varphi = \overline{\varphi}^x + \widetilde{\widetilde{\varphi}}$.

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where $\varphi = \overline{\varphi}^x + \widetilde{\widetilde{\varphi}}$.

- divergence free,
- h^2: Lagrangian multiplier,
- small-scale flow driven by the long-wave unbalanced part of the large-scale height gradient.

R. Klein, JCP, 1995 (low Mach number)

$b = b(x, \chi)$, Fr $= \varepsilon$, Sr $= 1$

If $b(x, \chi) = b(\chi)$:

$\Longrightarrow$ wave equation with spatially varying signal speed for h^1:

$$\partial_t^2 h^1 - \mathrm{div}_\chi \left((c - b(\chi)) \nabla_\chi h^1 \right) = 0\,, \tag{1}$$

where $c = b + h^0 \equiv \mathrm{const.}$

Concluding remarks

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   - Fully nonlinear regime

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     Fr $= \varepsilon^{3/2}$, Sr $= \varepsilon^{-1}$: The large-scale accumulation of net pressure forces drives the large-scale balanced flow; the large-scale height gradients produce small-scale forces driving the small-scale flow.
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3. Topography with long scale variations: $b = b(x, \chi)$
   Fr $= \varepsilon$, Sr $= 1$: as for the weakly nonlinear case, but the large-scale flow involves non-balanced terms.