Low-velocity scheme for hyperbolic conservation laws with constrains

Low Velocity Flows 05-06/11/15

Ernst Mach
(1836-1916)

Cindy Guichard (Upmc),
Martin Parisot (Inria),
Jacques Sainte-Marie (Cerema),
Fabien Wahl (Upmc)

William Froude
(1810-1879)

Many flows are partially in charge: **Groundwater flow**

Many flows are partially in charge: **Flow under floe**

Many flows are partially in charge: **Urban flood through sewer**

Many flows are partially in charge: **Iceberg drift**

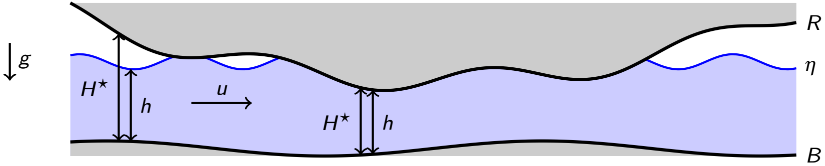
Many flows are partially in charge: **Wave generator / wave energy converter**

Many flows are partially in charge: **Floating tidal turbin**

Partially free surface model with friction and viscosity:

Starting from Navier-Stokes equations with gravity and **roof** and following [Gerbeau, Perthame'00], we get

$$\begin{cases} \partial_t h + \nabla\cdot(hu) = 0 \\ \partial_t(hu) + \nabla\cdot\left(hu\otimes u + \frac{g}{2}h^2 \mathrm{I_d}\right) = -h\nabla(gB+p) - \dfrac{\kappa_b + \kappa_r \mathbb{1}_{h>H^\star}}{1+\frac{\kappa_b+\kappa_r\mathbb{1}_{h>H^\star}}{3\mu}h} u + 4\mu\nabla\cdot(h\nabla u) \\ h \le H^\star, \qquad (H^\star - h)(p - P_a) = 0. \end{cases}$$



Unknowns:

$h(t,x)$	water depth
$u(t,x)$	averaged horizontal velocity
$p(t,x)$	pressure at surface $\eta = h + z_b$

Given parameters:

$P_a(t,x)$	atmospheric pressure
$B(t,x) \le R(t,x)$	bottom and roof level
$H^\star(t,x) = R - B$	maximum water depth

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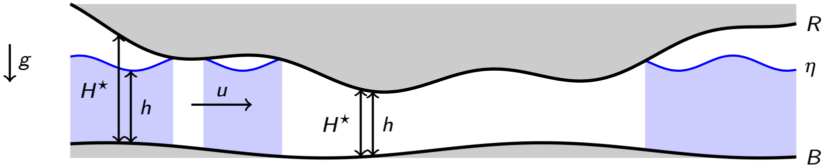

⚠: **Coupling approach**

- In the part with free surface ⇒ Shallow water equations. **hyperbolic**

Partially free surface model:

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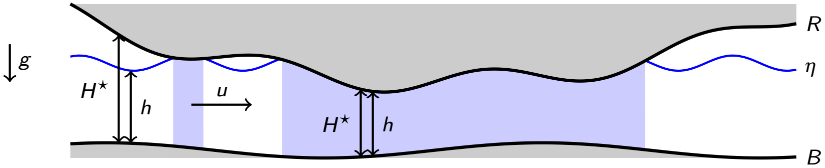
$$\begin{cases} \nabla\cdot\left(H^\star u\right) = -\partial_t H^\star \\ \partial_t u + u\cdot\nabla u + \nabla p = -g\nabla R \\ h = H^\star, \qquad \left(H^\star - h\right)\left(p - P_a\right) = 0. \end{cases}$$



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- In the part in charge ⇒ Lake equations **not hyperbolic**

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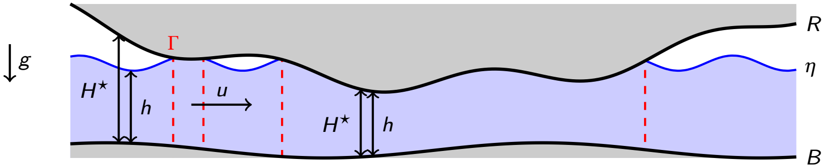
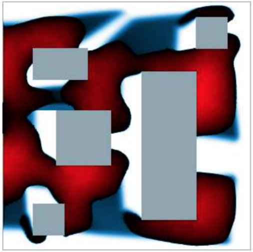


⚠: **Coupling approach**

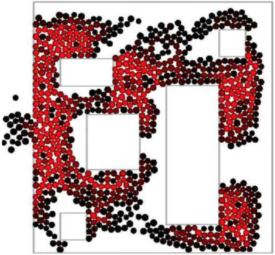
- In the part with free surface ⇒ Shallow water equations. **hyperbolic**
- In the part in charge ⇒ Lake equations **not hyperbolic**
- What boundary condition at the interface λ ? Dynamic of the interface ?

Crowd motion model: [Maury, Roudneff-Chupin, Samtambrogio'12]

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ u + \nabla p = U \\ \rho \leq 1 \qquad (1-\rho)\,p = 0 \end{cases}$$



Macro model (A. Roudneff-Chupin'11)



Micro model (J. Venel'08)

Unknowns:

$\rho(t,x)$	density of pedestrian
$u(t,x)$	real velocity
$p(t,x)$	pressure

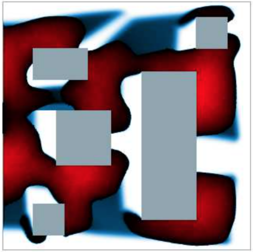
Given parameter:

$U(t,x)$	wished velocity
$R^\star = 1$	Maximum density

Crowd motion model: [Maury, Roudneff-Chupin, Samtambrogio'12]

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0 \\ u = P_{C_\rho}(U) \\ \nabla p = P_{C_\rho^\perp}(U) \end{cases}$$

with $C_\rho = \left\{ v \in L^2(\Omega),\ \forall q \in H^1_\rho,\ \int v\cdot\nabla q \leq 0 \right\}$
and $H^1_\rho = \left\{ q \in H^1(\Omega),\ \text{and } q\left(\mathbb{1}_{\rho<1}\right) = 0 \right\}$



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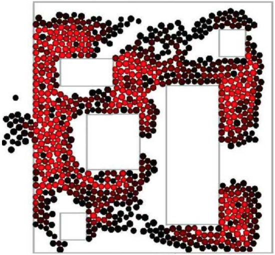

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⚠: **Numerical strategy**

- **Prediction step:** numerical scheme without the constrain
- **Correction step:** projection on the set of admissible velocity

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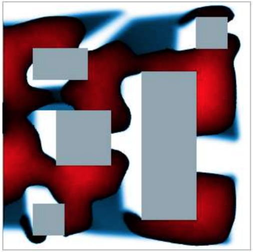

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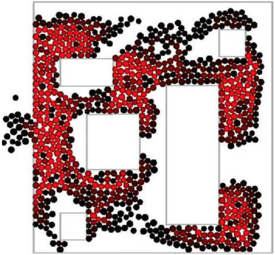


Micro model (J. Venel'08)

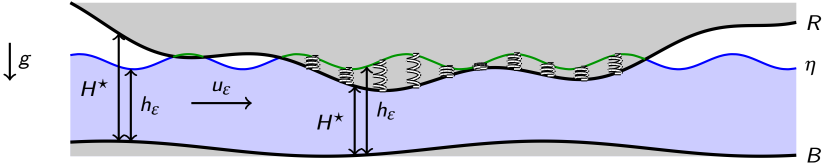
⚠: **Numerical strategy**

- **Prediction step:** numerical scheme without the constrain
- **Correction step:** projection on the set of admissible velocity
  - **expensive** numerical scheme
  - **not-easily adaptable** for **space-variable** upper bound

Relaxed partially free surface model:

Let us consider for any parameter $\varepsilon > 0$ the **relaxed** model:

$$(SW_\varepsilon^\star)\begin{cases} \partial_t h_\varepsilon + \nabla\cdot(h_\varepsilon u_\varepsilon) = 0 \\ \partial_t (h_\varepsilon u_\varepsilon) + \nabla\cdot\left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon + \frac{g}{2} h_\varepsilon^2 \mathrm{I_d}\right) = -h_\varepsilon \nabla(gB + p_\varepsilon) \\ p_\varepsilon = P_a + \frac{g(h_\varepsilon - H^\star)_+}{\varepsilon^2} \end{cases}$$



Formal convergence: $(SW_\varepsilon^\star) \underset{\varepsilon\to 0}{\longrightarrow} (SW^\star)$

- In the part **in charge** $h_\varepsilon > H^\star$:
  - The **main term** of momentum leads to the **constrain** $h_\varepsilon = H^\star + O\left(\varepsilon^2\right)$
  - The **main term** of the mass $\nabla\cdot(H^\star u_\varepsilon) = -\partial_t H^\star + O\left(\varepsilon^2\right)$
    and the **second order term** of the momentum leads to the **Lake equations**
    $$\partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla(p_\varepsilon) = -g\nabla R + O\left(\varepsilon^2\right)$$

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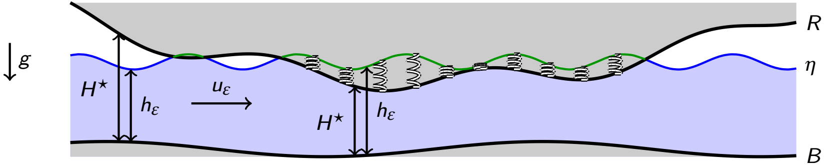

Formal convergence: $(SW_\varepsilon^\star) \underset{\varepsilon\to 0}{\longrightarrow} (SW^\star)$

- In the part **in charge** $h_\varepsilon > H^\star$: $\underset{\varepsilon\to 0}{\longrightarrow}$ **Lake**
- In the part with **free surface** $h_\varepsilon < H^\star$:
  - We have $p_\varepsilon = P_a$ then for any ε we solve locally the **Shallow Water equations**

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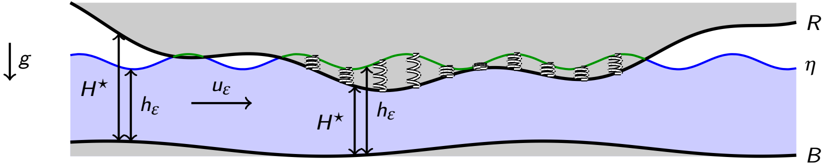

Formal convergence: $(SW_\varepsilon^\star) \xrightarrow[\varepsilon\to 0]{} (SW^\star)$

- In the part **in charge** $h_\varepsilon > H^\star$: $\xrightarrow[\varepsilon\to 0]{}$ **Lake**
- In the part with **free surface** $h_\varepsilon < H^\star$: $\xrightarrow[\varepsilon\to 0]{}$ **Shallow Water**
- At the **interface** λ_ε:
  - The flux are equals: $h_\varepsilon = H^\star$ and $p_\varepsilon = P_a$
  - Unknown is the position $\lambda_\varepsilon(t)$.

Hyperbolicity of the relaxed model $(SW_\varepsilon^\star)$:

The relaxed model is an hyperbolic model with source term.

Eigenvalues: $$u_\varepsilon \pm \sqrt{\left(1+\frac{1}{\varepsilon^2}\right) g h_\varepsilon}$$

⚠: Hyperbolic equation with **stiff potential force** ~ **low-Mach regime**

- Modeling error, i.e. $h_\varepsilon = h + O\left(\varepsilon^2\right)$

On a cartesian grid, the explicit Godunov-type solver

- Large numerical diffusion, i.e. $h_{\mathrm{d}x} = h_\varepsilon + O\left(\frac{\mathrm{d}x}{\varepsilon}\right)$
- Restrictive CFL condition $\left(|u_\varepsilon| + \sqrt{\left(1+\frac{1}{\varepsilon^2}\right) g h_\varepsilon}\right) \mathrm{d}t \leq C\, \mathrm{d}x$

Theoretical optimal setting: $\varepsilon = O\left(\mathrm{d}x^{\frac{1}{3}}\right), \qquad h_{\mathrm{d}x} = h + O\left(\mathrm{d}x^{\frac{2}{3}}\right), \qquad \mathrm{d}t = O\left(\mathrm{d}x^{\frac{4}{3}}\right).$

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⚠: Potential wave celerity **not fast enough**.

Q. How produce a **more efficient** numerical scheme ?

a) **first order** accurate: $h_{\mathrm{d}x} = h + O(\mathrm{d}x)$

b) modeling error **smaller than** numerical error: $\varepsilon^2 < \mathrm{d}x$

c) stable under **hyperbolic** CFL condition: $\mathrm{d}t = O(\mathrm{d}x)$

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A. Use an **asymptotic preserving low-Mach scheme**

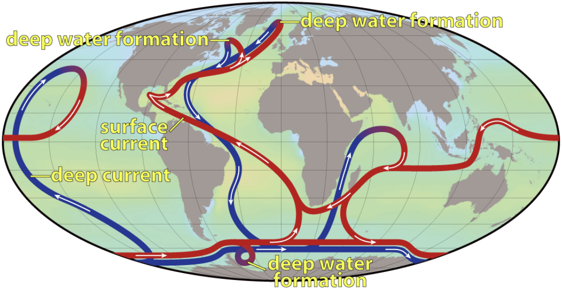
Bilayers Shallow Water model: [Gill'82]

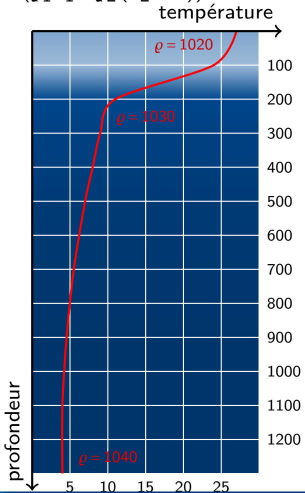
$$\left(SW_{\frac{\delta}{Fr},\delta}\right)\begin{cases}\partial_t(\varrho_1 h_1) + \nabla\cdot(\varrho_1 h_1 u_1) = G\\ \partial_t(\varrho_1 h_1 u_1) + \nabla\cdot(\varrho_1 h_1 u_1\otimes u_1) = G u_\eta - \dfrac{\varrho_1 h_1}{F_r^2}\nabla(h_1+h_2+B)\\ \partial_t(\varrho_2 h_2) + \nabla\cdot(\varrho_2 h_2 u_2) = -G\\ \partial_t(\varrho_2 h_2 u_2) + \nabla\cdot(\varrho_2 h_2 u_2\otimes u_2) = -G u_\eta - \dfrac{h_2}{F_r^2}\nabla(\varrho_1 h_1 + \varrho_2(h_2+B))\end{cases}$$

- **Low-Froude number**
  - mean depth $H > 3000m$
  - current speed $V \approx 1m/s$

  $\Big\} \Rightarrow F_r = \dfrac{V}{\sqrt{gH}} \approx 10^{-3}$
- **Low-density stratification**
  - heating from above
  - hot water lighter

  $\Big\} \Rightarrow 0 < 1 - \dfrac{\varrho_1}{\varrho_2} = \delta \approx 10^{-3}$





Bilayers Shallow Water model: [Gill'82]

$$\left(SW_{\frac{\delta}{Fr},\delta}\right) \to (SW_{1,0}) \begin{cases} \partial_t(h_1) & + \nabla\cdot(h_1 u_1) & = G \\ \partial_t(h_1 u_1) & + \nabla\cdot(h_1 u_1 \otimes u_1) & = G u_\eta - h_1 \nabla p \\ \partial_t(h_2) & + \nabla\cdot(h_2 u_2) & = -G \\ \partial_t(h_2 u_2) & + \nabla\cdot(h_2 u_2 \otimes u_2) & = -G u_\eta - h_2 \nabla(p - h_1) \\ & h_1 + h_2 + B & = 0 \end{cases}$$

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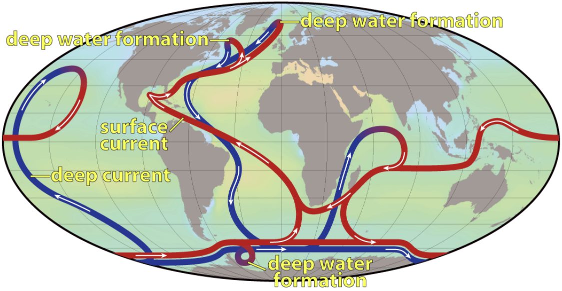
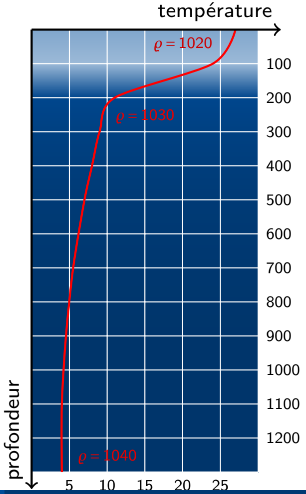




A few references on asymptotic preserving low-Mach scheme:

[Liou, Steffen'93]: **Advection Upstream Splitting Method** (AUMS)
Splitting of the equation into the advection part and the potential forces (pressure).

[Dellacherie'10]: **Consistency with the asymptotic waves equation**
Centered discretization of the potential forces (at least when $\varepsilon \to 0$).

[Grenier, Vila, Villedieu'13]: **Entropic stabilisation for linear potential**
Regularization using the gradient of the mass

[Parisot, Vila'15]: **Generalization for multi-layers (multi-phasics) and non-conservative products** Regularization using the gradient of the potential $\phi(h,x)$

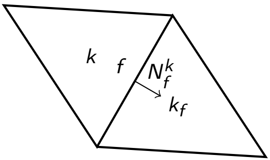
Qb) How to get a first order (independently of ε) **space discretization** error?

Numerical variables:

in cell: $$\psi_k = \frac{1}{|V_k|}\int_{V_k} \psi \,\mathrm{d}x$$

at edge: $$2\,(\psi)_f = \psi_k + \psi_{k_f} \quad \text{and} \quad 2\,[\psi]_k^{k_f} = \psi_{k_f} - \psi_k$$

parameters: $$\ell_k = \frac{|V_k|}{|\partial V_k|} \quad \text{and} \quad \mu_f^k = \frac{|f|}{|\partial V_k|}$$



Step 1 Use an **AUSM** based scheme

$$\partial_t \begin{pmatrix} h_k \\ h_k u_k \end{pmatrix} + \frac{1}{\ell_k}\sum_{f\in\mathbb{F}_k} \begin{pmatrix} \mathscr{F}_f^h \\ \mathscr{F}_f^{hu} \end{pmatrix} \cdot N_f^k \mu_f^k = \begin{pmatrix} 0 \\ \mathscr{Q}_k \end{pmatrix}$$

with $\mathscr{F}_f^h = \int_f h_\varepsilon u_\varepsilon \,\mathrm{d}\sigma$, $\quad \mathscr{F}_f^{hu} = \int_f h_\varepsilon u_\varepsilon \otimes u_\varepsilon \,\mathrm{d}\sigma \quad$ and $\quad \mathscr{Q}_k = -\frac{1}{|V_k|}\int_{V_k} g h_\varepsilon \nabla \phi_\varepsilon \,\mathrm{d}x.$

Advect the velocity with an **up-wind** scheme

$$\partial_t\left(h_k u_k\right) + \frac{1}{\ell_k}\sum_{f\in\mathbb{F}_k}\left(u_k\left(\mathscr{F}_f^h\cdot N_f^k\right)_+ - u_{k_f}\left(\mathscr{F}_f^h\cdot N_f^k\right)_-\right)\mu_f^k = \mathscr{Q}_k$$

- Ensure the **dissipation of the discrete kinetic energy** (as a pollutant)

Step 2 Use a **centered discretization** of the potential for any $\varepsilon > 0$

$$\mathcal{Q}_k \approx -\frac{h_k}{\ell_k} \sum_{f \in \mathbb{F}_k} \phi_f N_f^k |f| = -\frac{h_k}{\ell_k} \sum_{f \in \mathbb{F}_k} [\phi]_k^{k_f} N_f^k |f|$$

- Leads to a **consistent** numerical scheme when ε goes to 0.

Step 3 **Regularization** using the potential jump (τ : time scale; λ : regu. param.)

$$\mathcal{F}_f^h = (hu)_f - \lambda\tau \left(\frac{h}{\ell}\right)_f [\phi]_k^{k_f} N_f^k$$

- Ensure :
  - the stability of the **steady state at rest** ($\phi = Cst$ and $u = 0$)
  - the **dissipation of the discrete potential energy**

Sketch of proof:

The mass scheme can be formally interpreted as a discretization of:

Mass : $\partial_t h + \nabla \cdot (h(u - \lambda\tau\nabla\phi)) = 0$

Multiplying the mass conservation by ϕ:

Pot. energy : $\partial_t \mathcal{E} + \nabla \cdot (h\phi(u - \lambda\tau\nabla\phi)) = -hu\nabla\phi - \lambda\tau h \left|\nabla\phi\right|^2$

Qc) How to get a stable scheme **independently of** ε?

A Using an **IMEX** scheme: implicit for the water level h / explicit for the velocity u

Step 1 **water level: implicit** scheme of type **non-linear** advection-diffusion.

$$h_k^{n+1} - h_k^n + \frac{\mathrm{d}t}{\ell_k} \sum_{f \in \mathbb{F}_k} \underbrace{\left(\left(h^{n+1} u^n \right)_f \cdot N_f^k - \lambda \, \mathrm{d}t \left(\frac{h^{n+1}}{\ell} \right)_f \left[\phi^{n+1} \right]_k^{k_f} \right)}_{\mathscr{F}_f^{n+1} \cdot N_f^k} \mu_f^k = 0$$

Step 2 **velocity: explicit** upwind scheme with source term.

$$h_k^{n+1} u_k^{n+1} - h_k^n u_k^n + \frac{\mathrm{d}t}{\ell_k} \sum_{f \in \mathbb{F}_k} \left(u_k^n \left(\mathscr{F}_f^{n+1} \cdot N_f^k \right)^+ - u_{k_f}^n \left(\mathscr{F}_f^{n+1} \cdot N_f^k \right)^- \right) \mu_f^k = -\frac{\mathrm{d}t}{\ell_k} h_k^{n+1} \sum_{f \in \mathbb{F}_k} \left[\phi^{n+1} \right]_k^{k_f} N$$

THERE OREM: Entropy dissipation

Let $\lambda \geq 1$ and assume the following **CFL-like condition** is satisfies

$$\left(\left| u_f^n \cdot N_f^k \right| + \sqrt{\frac{\lambda}{2}} \sqrt{\left| \left[\phi^{n+1} \right]_k^{k_f} \right|} \right) \mathrm{d}t \leq \frac{\min\left(h_k^{n+1}, h_{k_f}^{n+1} \right)}{h_k^{n+1} + h_{k_f}^{n+1}} \min\left(\ell_k, \ell_{k_f} \right)$$

then the discrete mechanic energy is **decreasing**.

Non-linear fixed point :

from $h_k^{n,0} = h_k^n$

- we compute the time step

$$\left(\left|u_f^{n} \cdot N_f^k\right| + \sqrt{\frac{\lambda}{2}}\sqrt{\left|[\phi^{n,q}]_k^{k_f}\right|}\right) \mathrm{d}t \leq \frac{\min\left(h_k^{n,q}, h_{k_f}^{n,q}\right)}{h_k^{n,q} + h_{k_f}^{n,q}} \min\left(\ell_k, \ell_{k_f}\right)$$

- we compute implicitly the scheme for $h_k^{n,q+1}$

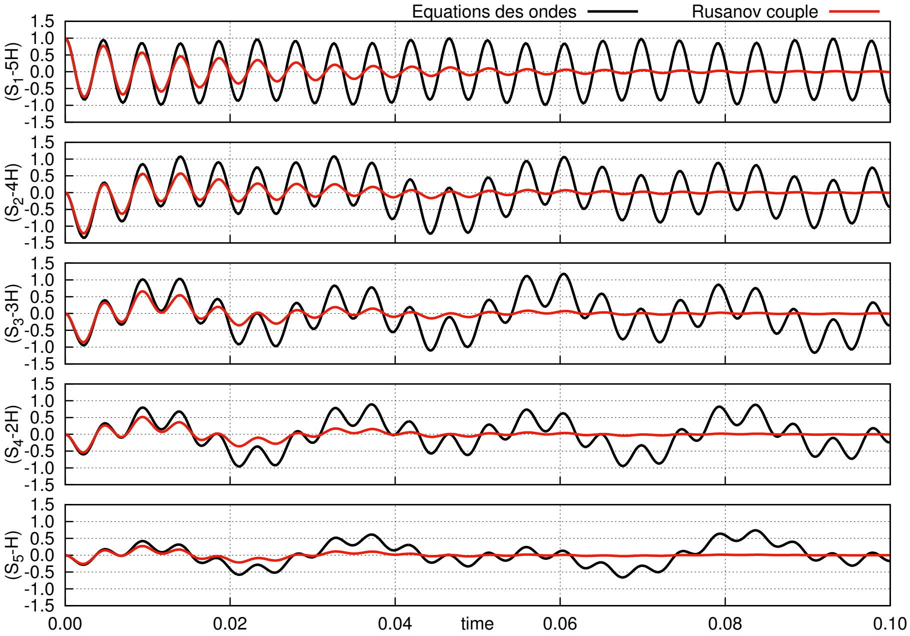
$$h_k^{n,q+1} - h_k^n + \frac{\mathrm{d}t}{\ell_k} \sum_{f \in \mathbb{F}_k} \left(\left(h^{n,q+1} u^n\right)_f \cdot N_f^k - \lambda\, \mathrm{d}t \left(\frac{h^{n,q}}{\ell}\right)_f \left[\phi^{n,q+1}\right]_k^{k_f}\right) \mu_f^k = 0$$

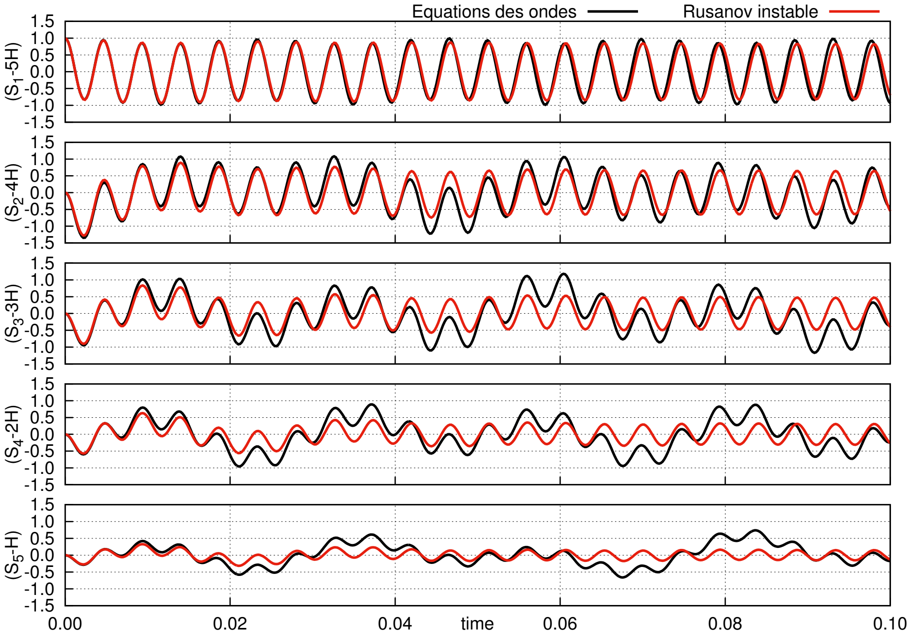
- we compute explicitly the scheme for $u_{i,k}^{n,q+1}$
- we estimate the variation of entropy $E_k^{n,q+1} = f\left(h_k^{n,q+1}, u_k^{n,q+1}\right)$
- we test $\left(E_k^{n,q+1} \leq E_k^n - \mathrm{d}t\ \text{flux}\right)$ — *a posteriori* **stop test**

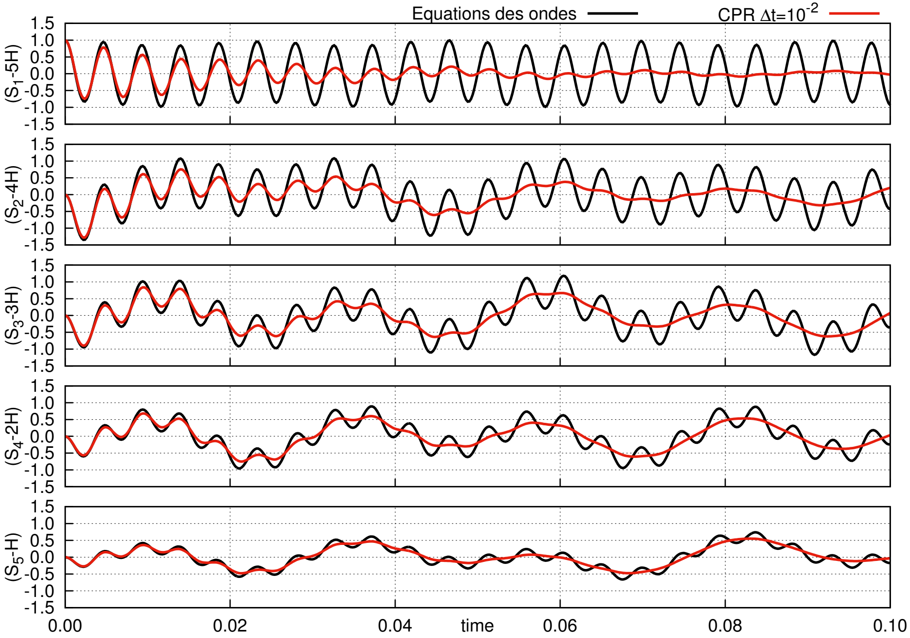
if not — **new iteration**

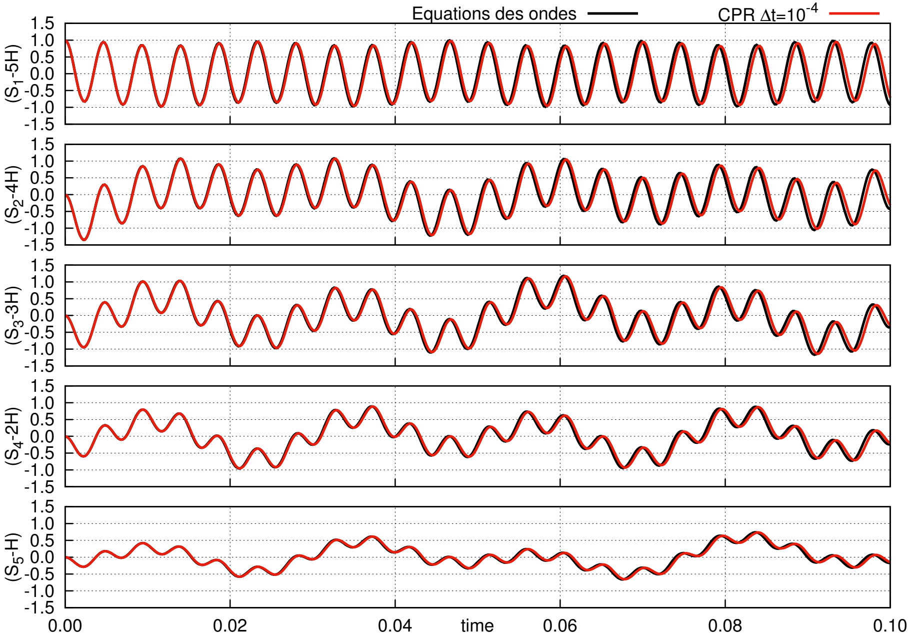
if yes, we set $\left(h_k^{n+1}, u_k^{n+1}\right) = \left(h_k^{n,q+1}, u_k^{n,q+1}\right)$ — **next time step**

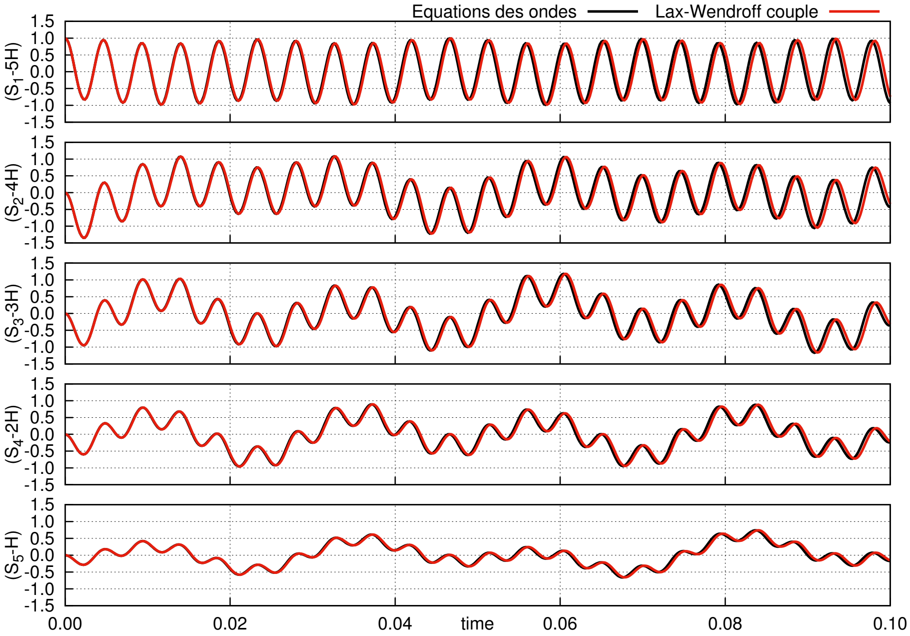
Converge with **few iterations** when the potential is **regular** enough.

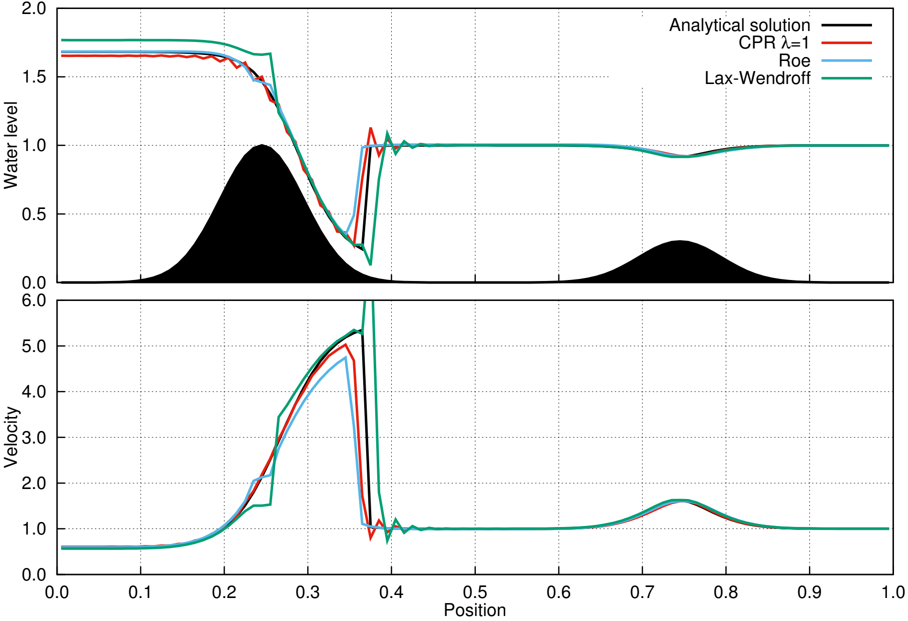

Equations des ondes
Rusanov couple
(S1-5H)
(S2-4H)
(S3-3H)
(S4-2H)
(S5-H)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10

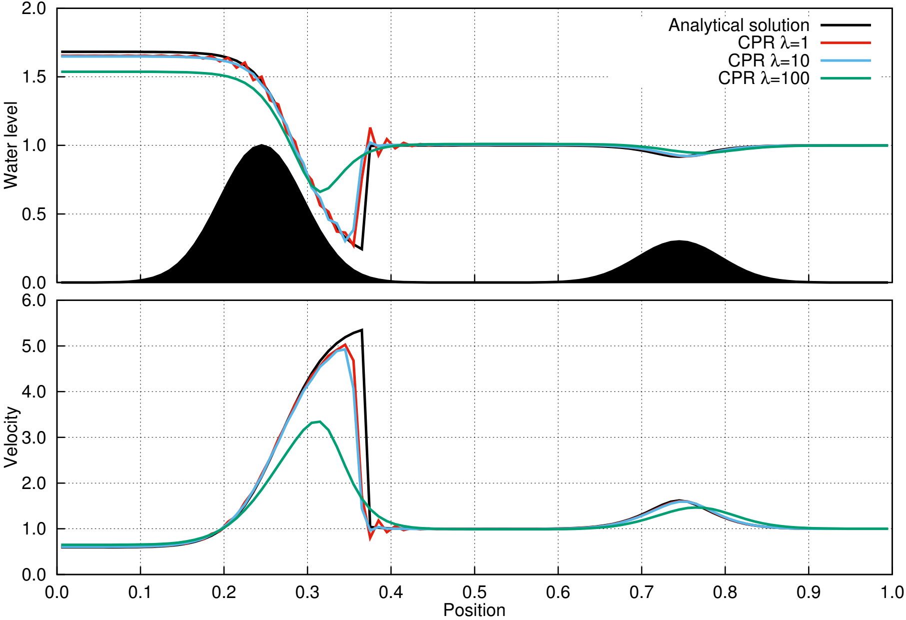

Equations des ondes
Rusanov instable
$(S_1\text{-}5H)$
$(S_2\text{-}4H)$
$(S_3\text{-}3H)$
$(S_4\text{-}2H)$
$(S_5\text{-}H)$
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10

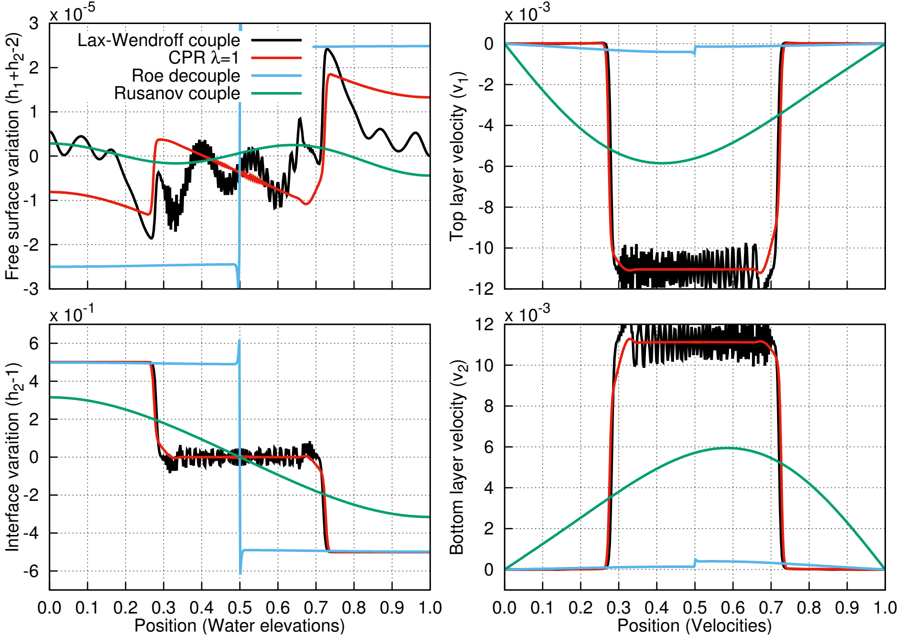

Equations des ondes
CPR Δt=10^-2
(S1-5H)
(S2-4H)
(S3-3H)
(S4-2H)
(S5-H)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10


Equations des ondes
CPR Δt=10^{-4}
(S$_1$-5H)
(S$_2$-4H)
(S$_3$-3H)
(S$_4$-2H)
(S$_5$-H)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10


Equations des ondes
Lax-Wendroff couple
(S1-5H)
(S2-4H)
(S3-3H)
(S4-2H)
(S5-H)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0.00
0.02
0.04
time
0.06
0.08
0.10


Water level
Velocity
Position
Analytical solution
CPR λ=1
Roe
Lax-Wendroff
0.0
0.5
1.0
1.5
2.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0


Analytical solution
CPR λ=1
CPR λ=10
CPR λ=100
Water level
2.0
1.5
1.0
0.5
0.0
Velocity
6.0
5.0
4.0
3.0
2.0
1.0
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Position


x 10^-5
Free surface variation (h1+h2-2)
Lax-Wendroff couple
CPR λ=1
Roe decouple
Rusanov couple
x 10^-3
Top layer velocity (v1)
x 10^-1
Interface varaition (h2-1)
x 10^-3
Bottom layer velocity (v2)
Position (Water elevations)
Position (Velocities)

PROPERTY: **Energy conservation** for $(SW^\star)$ (steady bottom $\partial_t B = 0$)

For smooth enough solution,
the mechanic energy $E = \mathscr{E} + \mathscr{K}$ of the partially free surface model $(SW^\star)$ satisfies

$$\partial_t E + \nabla \cdot ((\mathscr{K} + h(g(h+B)+p))u) = -(p-P_a)\partial_t H^\star$$

with the kinetic energy $\mathscr{K} = \frac{1}{2} h|u|^2$
and the potential energy $\mathscr{E} = \frac{g}{2}(h+B)^2$.

Sketch of proof:

Multiplying the momentum balance by u with the potential $\phi = g(h+B)$
Momentum : $\partial_t(hu) + \nabla \cdot (hu \otimes u) = -h\nabla(\phi + p)$
Kin. energy : $\partial_t \mathscr{K} + \nabla \cdot (\mathscr{K} u) = hu\nabla(\phi + p)$

PROPERTY: **Energy conservation** for $(SW^\star)$ (steady bottom $\partial_t B = 0$)

For smooth enough solution,
the mechanic energy $E = \mathcal{E} + \mathcal{K}$ of the partially free surface model $(SW^\star)$ satisfies

$$\partial_t E + \nabla \cdot \left(\left(\mathcal{K} + h \left(g \left(h + B \right) + p \right) \right) u \right) = -\left(p - P_a \right) \partial_t H^\star$$

with the kinetic energy $\mathcal{K} = \frac{1}{2} h |u|^2$
and the potential energy $\mathcal{E} = \frac{g}{2} \left(h + B \right)^2$.

Sketch of proof:

Multiplying the momentum balance by u with the potential $\phi = g \left(h + B \right)$
Momentum : $\partial_t \left(hu \right) + \nabla \cdot \left(hu \otimes u \right) = -h \nabla \left(\phi + p \right)$
Kin. energy : $\partial_t \mathcal{K} + \nabla \cdot \left(\mathcal{K} u \right) = hu \nabla \left(\phi + p \right)$

Multiplying the mass conservation by $\left(\phi + p - P_a \right)$:
Mass : $\partial_t h + \nabla \cdot \left(hu \right) = 0$
Pot. energy : $\partial_t \mathcal{E} + \nabla \cdot \left(h \left(\phi + p - P_a \right) u \right) = -hu \nabla \left(\phi + p \right) - \left(p - P_a \right) \partial_t H^\star - \left(p - P_a \right) \partial_t \left(h - H^\star \right)$

PROPERTY: **Energy conservation** for $(SW^\star)$ (steady bottom $\partial_t B = 0$)

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the mechanic energy $E = \mathscr{E} + \mathscr{K}$ of the partially free surface model $(SW^\star)$ satisfies

$$\partial_t E + \nabla \cdot ((\mathscr{K} + h(g(h+B)+p))u) = -(p-P_a)\partial_t H^\star$$

with the kinetic energy $\mathscr{K} = \frac{1}{2} h|u|^2$
and the potential energy $\mathscr{E} = \frac{g}{2}(h+B)^2$.

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Multiplying the mass conservation by $(\phi + p - P_a)$:
Mass : $\partial_t h + \nabla \cdot (hu) = 0$
Pot. energy : $\partial_t \mathscr{E} + \nabla \cdot (h(\phi + p - P_a)u) = -hu\nabla(\phi + p) - (p-P_a)\partial_t H^\star - (p-P_a)\partial_t (h - H^\star)$

Thanks to the condition: $(p-P_a)(h-H^\star) = 0$, we have
$(p-P_a)\partial_t (h-H^\star) = (H^\star - h)\partial_t (p-P_a)$ then $(p-P_a)\partial_t (h-H^\star) = 0$.

PROPERTY: **Energy conservation** for $(SW_\varepsilon^\star)$ (steady bottom $\partial_t B = 0$)

For smooth enough solution,
the mechanic energy $E_\varepsilon = \mathscr{E}_\varepsilon + \mathscr{K}_\varepsilon$ of the partially free surface model $(SW_\varepsilon^\star)$ satisfies

$$\partial_t E_\varepsilon + \nabla\cdot\left(\left(\mathscr{K}_\varepsilon + h_\varepsilon\left(g\left(h_\varepsilon + B\right) + p_\varepsilon\right)\right) u_\varepsilon\right) = -\left(p_\varepsilon - P_a\right)\partial_t H^\star$$

with the kinetic energy $\mathscr{K}_\varepsilon = \dfrac{1}{2} h_\varepsilon \left|u_\varepsilon\right|^2$

and the potential energy $\mathscr{E}_\varepsilon = \dfrac{g}{2}\left(\left(h_\varepsilon + B\right)^2 + \dfrac{\left(h_\varepsilon - H^\star\right)_+^2}{\varepsilon^2}\right)$.

Sketch of proof:

Multiplying the momentum balance by u_ε with the potential $\phi_\varepsilon = g\left(\left(h_\varepsilon + B\right) + \dfrac{\left(h_\varepsilon - H^\star\right)_+}{\varepsilon^2}\right)$

Momentum : $\partial_t\left(h_\varepsilon u_\varepsilon\right) + \nabla\cdot\left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon\right) = -h_\varepsilon \nabla\phi_\varepsilon$
Kin. energy : $\partial_t \mathscr{K}_\varepsilon + \nabla\cdot\left(\mathscr{K}_\varepsilon u_\varepsilon\right) = h_\varepsilon u_\varepsilon \nabla\phi_\varepsilon$

PROPERTY: **Energy conservation** for $(SW_\varepsilon^\star)$ (steady bottom $\partial_t B = 0$)

For smooth enough solution,
the mechanic energy $E_\varepsilon = \mathscr{E}_\varepsilon + \mathscr{K}_\varepsilon$ of the partially free surface model $(SW_\varepsilon^\star)$ satisfies

$$\partial_t E_\varepsilon + \nabla\cdot\left(\left(\mathscr{K}_\varepsilon + h_\varepsilon\left(g\left(h_\varepsilon + B\right) + p_\varepsilon\right)\right)u_\varepsilon\right) = -\left(p_\varepsilon - P_a\right)\partial_t H^\star$$

with the kinetic energy $\mathscr{K}_\varepsilon = \dfrac{1}{2} h_\varepsilon \left|u_\varepsilon\right|^2$
and the potential energy $\mathscr{E}_\varepsilon = \dfrac{g}{2}\left(\left(h_\varepsilon + B\right)^2 + \dfrac{\left(h_\varepsilon - H^\star\right)_+^2}{\varepsilon^2}\right)$.

Sketch of proof:

Multiplying the momentum balance by u_ε with the potential $\phi_\varepsilon = g\left(\left(h_\varepsilon + B\right) + \dfrac{\left(h_\varepsilon - H^\star\right)_+}{\varepsilon^2}\right)$

Momentum : $\partial_t\left(h_\varepsilon u_\varepsilon\right) + \nabla\cdot\left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon\right) = -h_\varepsilon \nabla\phi_\varepsilon$
Kin. energy : $\partial_t \mathscr{K}_\varepsilon + \nabla\cdot\left(\mathscr{K}_\varepsilon u_\varepsilon\right) = h_\varepsilon u_\varepsilon \nabla\phi_\varepsilon$

Multiplying the mass conservation by ϕ_ε:
Mass : $\partial_t h_\varepsilon + \nabla\cdot\left(h_\varepsilon u_\varepsilon\right) = 0$
Pot. energy : $\partial_t \mathscr{E}_\varepsilon + \nabla\cdot\left(h_\varepsilon \phi_\varepsilon u_\varepsilon\right) = -h_\varepsilon u_\varepsilon \nabla\phi_\varepsilon - \dfrac{g\left(h_\varepsilon - H^\star\right)_+}{\varepsilon^2}\partial_t H^\star$

PROPERTY: **Energy conservation** for $(SW_\varepsilon^\star)$ (steady bottom $\partial_t B = 0$)

For smooth enough solution,
the mechanic energy $E_\varepsilon = \mathscr{E}_\varepsilon + \mathscr{K}_\varepsilon$ of the partially free surface model $(SW_\varepsilon^\star)$ satisfies

$$\partial_t E_\varepsilon + \nabla\cdot\big(\big(\mathscr{K}_\varepsilon + h_\varepsilon\big(g\left(h_\varepsilon + B\right) + p_\varepsilon\big)\big)u_\varepsilon\big) = -\left(p_\varepsilon - P_a\right)\partial_t H^\star$$

with the kinetic energy $\mathscr{K}_\varepsilon = \dfrac{1}{2}h_\varepsilon\left|u_\varepsilon\right|^2$
and the potential energy $\mathscr{E}_\varepsilon = \dfrac{g}{2}\left((h_\varepsilon + B)^2 + \dfrac{(h_\varepsilon - H^\star)_+^2}{\varepsilon^2}\right)$.

Sketch of proof:

Multiplying the momentum balance by u_ε with the potential $\phi_\varepsilon = g\left((h_\varepsilon + B) + \dfrac{(h_\varepsilon - H^\star)_+}{\varepsilon^2}\right)$

Momentum : $\partial_t\left(h_\varepsilon u_\varepsilon\right) + \nabla\cdot\left(h_\varepsilon u_\varepsilon \otimes u_\varepsilon\right) = -h_\varepsilon \nabla\phi_\varepsilon$
Kin. energy : $\partial_t \mathscr{K}_\varepsilon + \nabla\cdot\left(\mathscr{K}_\varepsilon u_\varepsilon\right) = h_\varepsilon u_\varepsilon \nabla\phi_\varepsilon$

Multiplying the mass conservation by ϕ_ε:
Mass : $\partial_t h_\varepsilon + \nabla\cdot\left(h_\varepsilon u_\varepsilon\right) = 0$

Pot. energy : $\partial_t \mathscr{E}_\varepsilon + \nabla\cdot\left(h_\varepsilon\phi_\varepsilon u_\varepsilon\right) = -h_\varepsilon u_\varepsilon\nabla\phi_\varepsilon - \dfrac{g\left(h_\varepsilon - H^\star\right)_+}{\varepsilon^2}\partial_t H^\star$

By definition: $p_\varepsilon - P_a = \dfrac{g\left(h_\varepsilon - H^\star\right)_+}{\varepsilon^2}$.

HLL $\varepsilon = \mathrm{d}x^{\frac{1}{3}}$ | CPR $\varepsilon = \mathrm{d}x^2$

Centered-Potential regularization scheme:

- The CPR scheme is very **robust**:
  - large number of unknowns
  - non-conservative product
  - stiff conservative source term (low-Mach)
  - easily adaptable to several physics
- The CPR scheme is very **stable**:
  - entropic stability
  - well-balanced for steady state at rest
  - not restrictive CFL condition
  - weak numerical dissipation

Prospects:
- explicit version
- wet/dry transition
- non-conservative forces (Coriolis, surface tension...)

Partially free surface flows:

- Derivation of a shallow water type **model for partially free surface flows**
- **Formal analysis and numerical resolution** for regular solution

Prospects:
- reduce the oscillation of pressure and the CFL condition at the interface
- modeling of the pressure in bubbles
- coupling with the dynamics of a buoy
- submerged object

Thank you for your attention