Finite volumes schemes preserving the low Mach number limit for the Euler system

M.-H. Vignal

INSTITUT de MATHEMATIQUES de TOULOUSE

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Giacomo Dimarco, Univ. de Ferrara, Italie

Raphael Loubere, IMT, CNRS, France

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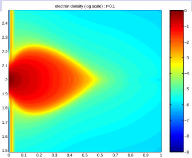
Outline

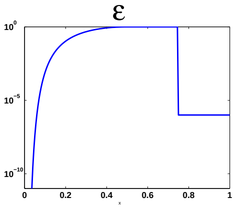
General context

Multiscale model : M_ε depends on a parameter ε



In the (space-time) domain ε can

- be small compared to the reference scale
- be of same order as the reference scale
- take intermediate values



When ε is small : $M_0 = \lim_{\varepsilon \to 0} M_\varepsilon$ asympt. model

Difficulties :

- Classical explicit schemes for M_ε : stable and consistent iff the mesh resolved all the scales of ε $\Rightarrow$ huge cost
- Schemes for M_0 with meshes independent of ε

  But : ➠ M_0 not valid everywhere, needs $\varepsilon \ll 1$

  ➠ location of the interface, moving interface

Principle of AP schemes

A possible solution : AP schemes

- Use the multi-scale model M_ε where you want.
- Discretize it with a scheme preserving the limit $\varepsilon \to 0$

➠ The mesh is independent of ε : **Asymptotic stability**.

➠ You recover an approximate solution of M_0 when $\varepsilon \to 0$:

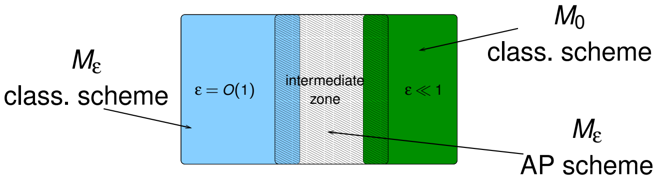
Asymptotic consistency

Asymptotically stable and consistent scheme

$\Rightarrow$ Asymptotic preserving scheme (AP)

([S.Jin] kinetic $\to$ hydro)

- You can use the AP scheme only to reconnect M_ε and M_0



Outline

The multi-scale model and its asymptotic limit

➠ **Isentropic Euler system in scaled variables** $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$

$$(M_\varepsilon) \begin{cases} \partial_t \rho + \nabla \cdot (\rho\, u) = 0, & (1)_\varepsilon \\ \partial_t (\rho\, u) + \nabla \cdot (\rho\, u \otimes u) + \dfrac{1}{\varepsilon} \nabla p(\rho) = 0, & (2)_\varepsilon \end{cases}$$

Parameter : $\varepsilon = M^2 = m|\overline{u}|^2 / (\gamma p(\overline{\rho})/\overline{\rho})$, $M =$Mach number

Boundary and initial conditions :

$$u \cdot n = 0, \text{ on } \partial\Omega, \quad \text{and} \quad \begin{cases} \rho(x,0) = \rho_0 + \varepsilon\, \tilde{\rho}_0(x), \\ u(x,0) = u_0(x) + \varepsilon\, \tilde{u}_0(x), \ \text{with } \nabla \cdot u_0 = 0. \end{cases}$$

The formal limit $\varepsilon \to 0$

$$(2)_\varepsilon \ \Rightarrow \ \nabla p(\rho) = 0, \ \Rightarrow \ \rho(x,t) = \rho(t).$$

$$(1)_\varepsilon \ \Rightarrow \ |\Omega|\, \rho'(t) + \rho(t) \int_{\partial\Omega} u \cdot n = 0, \ \Rightarrow \ \rho(t) = \rho(0) = \rho_0, \ \Rightarrow \ \nabla \cdot u = 0$$

The multi-scale model and its asymptotic limit

The asymptotic model : Rigorous limit [Klainerman, Majda, 81]

$$(M_0)\begin{cases} \rho = cste = \rho_0, \\ \rho_0 \nabla \cdot u = 0, & (1)_0 \\ \rho_0 \partial_t u + \rho_0 \nabla \cdot (u \otimes u) + \nabla \pi_1 = 0, & (2)_0 \end{cases}$$

where

$$\pi_1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(p(\rho) - p(\rho_0)\Big).$$

Explicit eq. for π_1 $\partial_t (1)_0 - \nabla \cdot (2)_0 \quad \Rightarrow \quad -\Delta \pi_1 = \rho_0 \nabla^2 : (u \otimes u).$

The pressure wave eq. from M_ε :

$$\partial_t (1)_\varepsilon - \nabla \cdot (2)_\varepsilon \quad \Rightarrow \quad \partial_{tt} \rho - \frac{1}{\varepsilon} \Delta p(\rho) = \nabla^2 : (\rho\, u \otimes u) \qquad (3)_\varepsilon$$

From a numerical point of view

- Explicite treatment of $(3)_\varepsilon \Rightarrow$ conditional stability $\Delta t \leq \sqrt{\varepsilon}\, \Delta x$
- Implicite treatment of $(3)_\varepsilon \Rightarrow$ uniform stability with respect to ε

Bibliography

All speed schemes

- **Preconditioning methods :** [Chorin 65], [Choi, Merkle 85], [Turkel, 87], [Van Leer, Lee, Roe, 91], [Li,Gu 08,10], · · ·
- **Splitting and pressure correction :** [Harlow, Amsden, 68,71], [Karki, Patankar, 89], [Bijl, Wesseling, 98], [Sewall, Tafti, 08], [Klein, Botta, Schneider, Munz, Roller 08], [Guillard, Murrone, Viozat 99, 04, 06] [Herbin, Kheriji, Latché 12,13], · · ·
  - **Asymptotic preserving schemes** [Degond, Deluzet, Sangam, V, 09], [Degond, Tang 11], [Cordier, Degond, Kumbaro 12], [Grenier, Vila, Villedieu 13] [Dellacherie, Omnes, Raviart,13], [Noelle, Bispen, Arun, Lukacova, Munz,14], [Chalons, Girardin, Kokh,15] [Dimarco, Loubère, V, in prep.]

Outline

Classical scheme in the low Mach number limit

If ρ^n and u^n are known at time t^n

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^n = 0, & (1)_n \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u \otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^n) = 0. & (2) \end{cases}$$

Reformulation $\dfrac{(1)_{n+1}-(1)_n}{\Delta t} - \nabla\cdot(2) \;\Rightarrow$

$$\frac{\rho^{n+2}-2\,\rho^{n+1}+\rho^n}{\Delta t^2} - \frac{1}{\varepsilon}\Delta p(\rho^n) = \nabla^2 : (\rho\, u\otimes u)^n$$

- Explicit treatment $\Rightarrow$ conditional stability $\Delta t \leq \sqrt{\varepsilon}\,\Delta x$
- $\varepsilon \to 0$ gives $\nabla p(\rho^n) = 0 \Rightarrow$ consistency lost at the limit

AP scheme in the low Mach number limit

Goal : Build an AP scheme for the multi-scale model M_ε

- Uniform stability in $\varepsilon \Rightarrow \Delta t = \mathcal{O}(1)$
- Consistency for all $\varepsilon \geq 0$: $\varepsilon \to 0 \Rightarrow$ scheme for the limit model M_0

Basic ideas : ➠ Implicit treatment of the right terms
➠ Reformulation to make appear the limit model

AP scheme, Euler-Lorentz, low Mach number and drift limits

[P. Degond, F. Deluzet, A. Sangam, MHV, JCP 2009]

AP scheme for Euler, low Mach number limit

[P. Degond, M. Tang, Commun. Comput. Phys, 2011]

- Parameter α fixed by the user, depends on the considered problem
  ➠ If $\varepsilon \ll 1$ then α small ➠ If $\varepsilon = O(1)$ then α sufficiently large
- Extension to the full Euler and Navier-Stokes systems :

[F. Cordier, P. Degond, A. Kumbaro, JCP 2012]

AP scheme in the low Mach number limit

Our AP scheme : If ρ^n and u^n are known at time t^n

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^{n+1} = 0, & (1)_n \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u\otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. & (2) \end{cases}$$

Reformulation $\dfrac{(1)_n-(1)_{n-1}}{\Delta t} - \nabla\cdot(2) \Rightarrow$

$$\frac{\rho^{n+1}-2\rho^n+\rho^{n-1}}{\Delta t^2} - \frac{1}{\varepsilon}\Delta p(\rho^{n+1}) = \nabla^2 : (\rho\, u\otimes u)^n$$

- Implicit treatment $\Rightarrow$ uniform stability in ε
- $\varepsilon \to 0$ gives $\nabla p(\rho^{n+1}) = 0 \Rightarrow$ consistency at the limit

Uncoupled formulation of the AP scheme

Take the divergence of the momentum eq. (2)

$$\nabla\cdot(2) \quad \Rightarrow \quad \nabla\cdot(\rho\, u)^{n+1} = \nabla\cdot(\rho\, u)^{n} - \Delta t\, \nabla^2 : (\rho\, u \otimes u)^{n} - \frac{\Delta t}{\varepsilon}\Delta p(\rho^{n+1})$$

Insert it in the mass eq. (1)

$$\frac{\rho^{n+1}-\rho^{n}}{\Delta t} + \nabla\cdot(\rho\, u)^{n} - \frac{\Delta t}{\varepsilon}\Delta p(\rho^{n+1}) - \Delta t\, \nabla^2 : (\rho\, u \otimes u)^{n} = 0.$$

Uncoupled formulation of the AP scheme

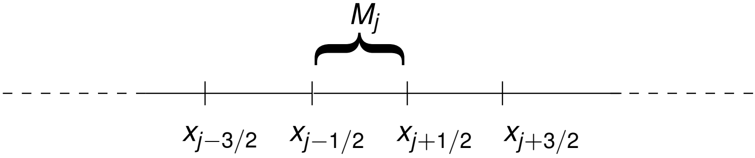
$$\begin{cases} \dfrac{\rho^{n+1}-\rho^{n}}{\Delta t} + \nabla\cdot(\rho\, u)^{n} - \dfrac{\Delta t}{\varepsilon}\Delta p(\rho^{n+1}) - \Delta t\, \nabla^2 : (\rho\, u \otimes u)^{n} = 0, \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^{n}}{\Delta t} + \nabla\cdot(\rho\, u \otimes u)^{n} + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. \end{cases}$$

Space discretization : Need a correct choice of the numerical viscosity

1-D space discretization for clarity

Splitting :

$$\frac{W^{n+1}-W^n}{\Delta t} + \partial_x \overbrace{F_e(W^n)}^{(0,\rho u^2)^n} + \partial_x \overbrace{F_i(W^{n+1})}^{(\rho u,\ p(\rho)/\varepsilon)^{n+1}} = 0$$



$$\frac{W_j^{n+1}-W_j^n}{\Delta t} + \frac{f_e(W_j^n, W_{j+1}^n) - f_e(W_{j-1}^n, W_j^n)}{\Delta x} + \frac{f_i(W_j^{n+1}, W_{j+1}^{n+1}) - f_i(W_{j-1}^{n+1}, W_j^{n+1})}{\Delta x} = 0$$

1-D space discretization for clarity

Rusanov fluxes

$$f_e(W_j^n, W_{j+1}^n) = \frac{F_e(W_j^n) + F_e(W_{j+1}^n)}{2} - D_e^n (W_j^n - W_{j+1}^n).$$

$$f_i(W_j^{n+1}, W_{j+1}^{n+1}) = \frac{F_i(W_j^{n+1}) + F_i(W_{j+1}^{n+1})}{2} - D_i^{n+1} (W_j^{n+1} - W_{j+1}^{n+1}).$$

Jacobian matrices

$$DF_e(W) = \begin{pmatrix} 0 & 0 \\ -u^2 & 2\,u \end{pmatrix} \quad \text{and} \quad DF_i(W) = \begin{pmatrix} 0 & 1 \\ c^2/\varepsilon & 0 \end{pmatrix}$$

Eigen values $\quad 0, \quad 2\,u \qquad\qquad -c/\sqrt{\varepsilon}, \quad c/\sqrt{\varepsilon}$

with $c^2 = p'(\rho)$

Stability of the linearized system around (ρ_0, u_0)

Under the CFL

$$\Delta t \leq \frac{\Delta x}{2\,|u_0|}$$

Explicit viscosity

$$2\,D_e^n = 2\,|u_0|$$

Implicit viscosity

- If $D_i^{n+1} = 0$ $\Rightarrow$ L^2 stability result
- If $2\,D_i^{n+1} = c_0/\sqrt{\varepsilon}$ $\Rightarrow$ L^∞ stability result

Uncoupled scheme in the non linear case

Reformulation :

$$\frac{W^{n+1}-W^n}{\Delta t}+\partial_x \overbrace{F_e(W^n)}^{(0,(\rho\, u^2)^n)}+\partial_x \overbrace{F_i(W^{n+1/2})}^{((\rho\, u)^n\,,\ p(\rho^{n+1})/\varepsilon)}$$
$$-\Delta t\,\Delta\begin{pmatrix} \frac{p(\rho^{n+1})}{\varepsilon}+(\rho\, u^2)^n \\ 0 \end{pmatrix}=0$$

Step 1 Determine ρ_j^{n+1} knowing $W^n=(\rho^n,(\rho\, u)^n)$

$$\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+\frac{(\rho\, u)_j^n+(\rho\, u)_{j+1}^n}{2\,\Delta x}-\frac{D_e^n+D_i^n}{\Delta x}\overbrace{(\rho_{j+1}^n-2\,\rho_j^n+\rho_{j-1}^n)}^{\Delta_{dis}(\rho^n)}$$
$$-\frac{\Delta t}{\varepsilon}\Delta_{dis}p(\rho^{n+1})-\Delta t\,\Delta_{dis}(\rho\, u^2)^n=0.$$

Uncoupled scheme in the non linear case

Step 2 Determine $(\rho\, u)_j^{n+1}$ knowing $V^{n+1/2} = (W^n, \rho^{n+1})$

$$\frac{(\rho\, u)_j^{n+1} - (\rho\, u)_j^n}{\Delta t} + \frac{g(V_j^{n+1/2}, V_{j+1}^{n+1/2}) - g(V_{j-1}^{n+1/2}, V_j^n)}{\Delta x} = 0$$

Fluxes given by

$$g(V_j^{n+1/2}, V_{j+1}^{n+1/2}) = \frac{(\rho\, u^2)_j^n + p(\rho_j^{n+1})/\varepsilon + (\rho\, u^2)_{j+1}^n + p(\rho_{j+1}^{n+1})/\varepsilon}{2}$$

$$-(D_e^n + D_i^n)\,((\rho\, u)_j^n - (\rho\, u)_{j+1}^n).$$

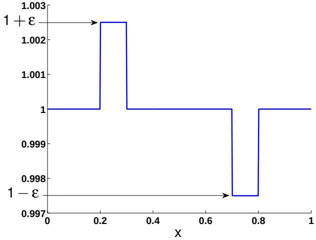
C.F.L Condition :

AP scheme stable if $\quad \Delta t \leq \dfrac{\Delta x}{\max(|2\, u_j^n|)} = O(1)$

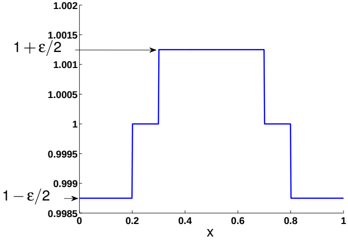
Outline

1-D test case

- Domain $[0,1]$, $\Delta x = 1/500$ or $1/300$
- Final time : $T = 0.05$,
- Pressure law $p(\rho) = \rho^\gamma$, $\gamma = 2$



Initial density



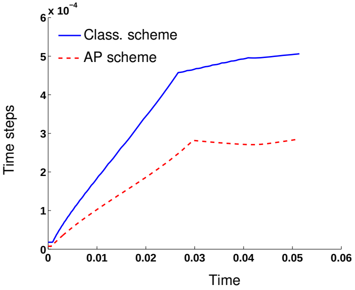
Initial momentum

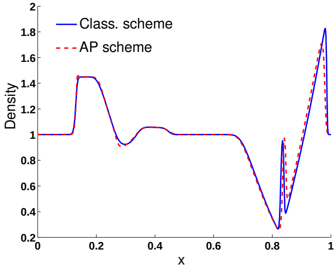
Non linear case $\gamma = 2$ and $\varepsilon = O(1)$, $\Delta x = 1/500$

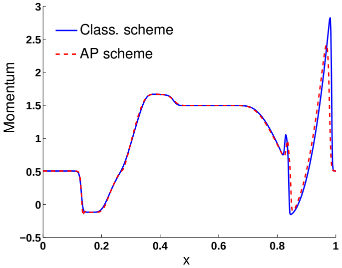
$\varepsilon = 0.99, \quad D_i = 0$

Class : 273 loops
CPU time 0.07

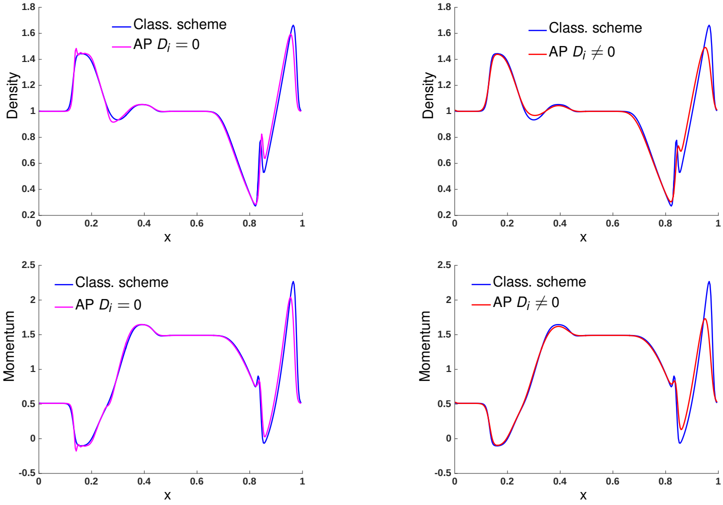
AP : 510 loops
CPU time 1.46







Non linear case $\gamma = 2$ and $\varepsilon = 0.99$, $\Delta x = 1/300$



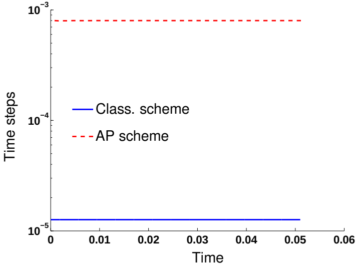
$D_i = 0$ centered implicit scheme $\quad$ $D_i \neq 0$ upwind implicit scheme

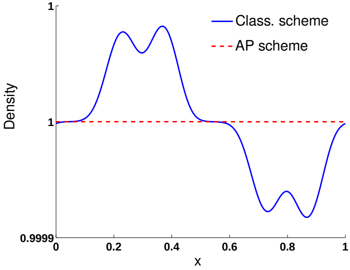
Non linear case $\gamma = 2$ and $\varepsilon \ll 1$

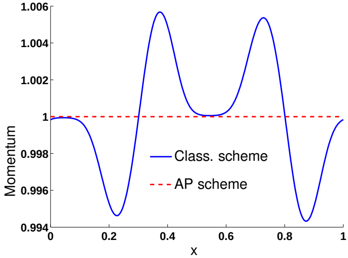
$\varepsilon = 10^{-4}$

Class : 4036 loops
CPU time 0.82

AP : 57 loops
CPU time 0.14





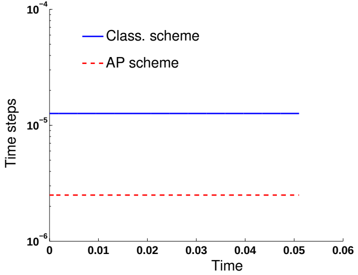


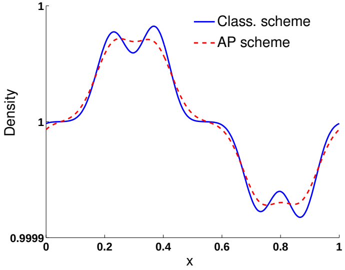
Non linear case $\gamma = 2$ and $\varepsilon \ll 1$

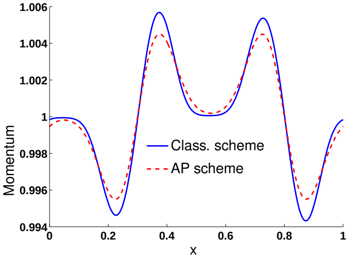
$\varepsilon = 10^{-4}$

Underlying of

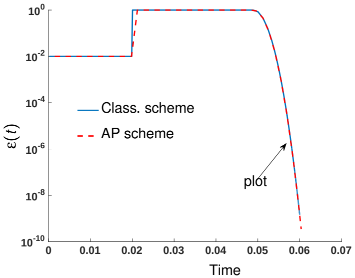
the viscosity





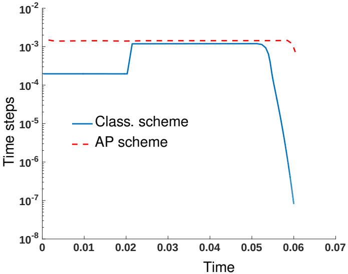


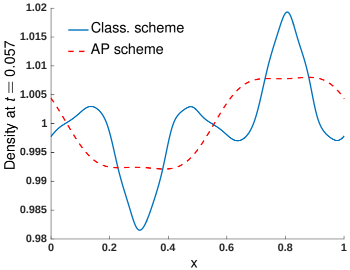
Same test case $\gamma = 2$, $\varepsilon = \varepsilon(t)$, $\Delta x = 1/300$



Class : 6906 loops
CPU time 361.62

AP : 43 loops –> $\div 160$
CPU time 2.33 –> $\div 155$





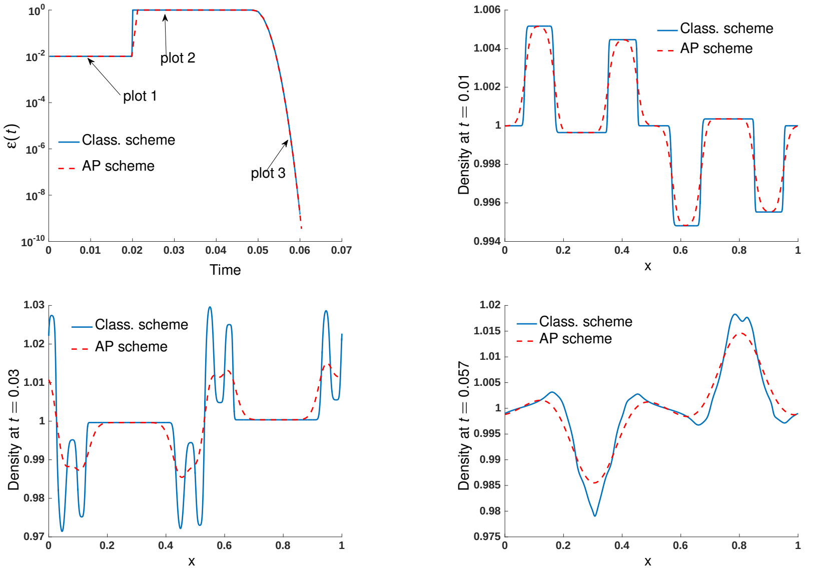
If $T_{final} = 0.058$:

Class : 400 loops, CPU time 22.19

AP : 41 loops, CPU time 2.47

–> $\div 10$

$\varepsilon = \varepsilon(t)$, $\Delta x = 1/3000$



$T_{final} = 0.058$: Class : 4087 loops, CPU time 195.18

AP : 448 loops, CPU time 25.95

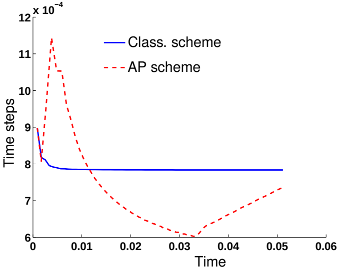
–> $\div 9.1$

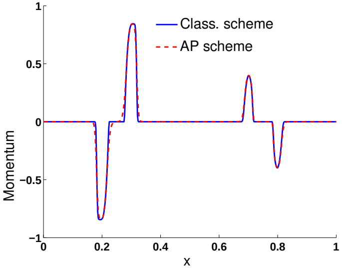
Non linear case $\gamma = 2$, zero velocity

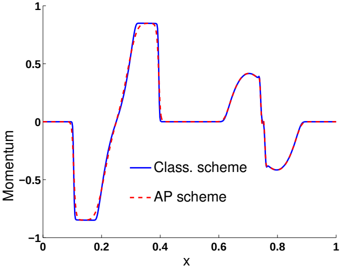
$\varepsilon = 0.99$

Same initial density

But zero initial momentum



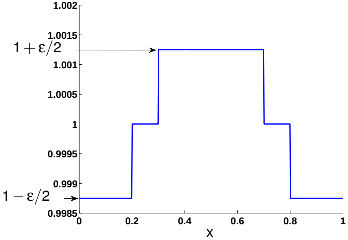




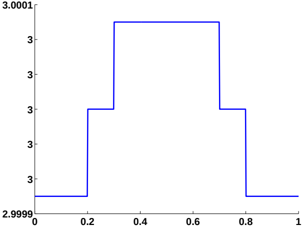
Outline

Full Euler system

- Domain $[0,1]$, $\Delta x = 1/500$
- Final time : $T = 0.05$,
- $\gamma = 1.4$, $\rho(t=0) = 1$, $p(t=0) = 1$



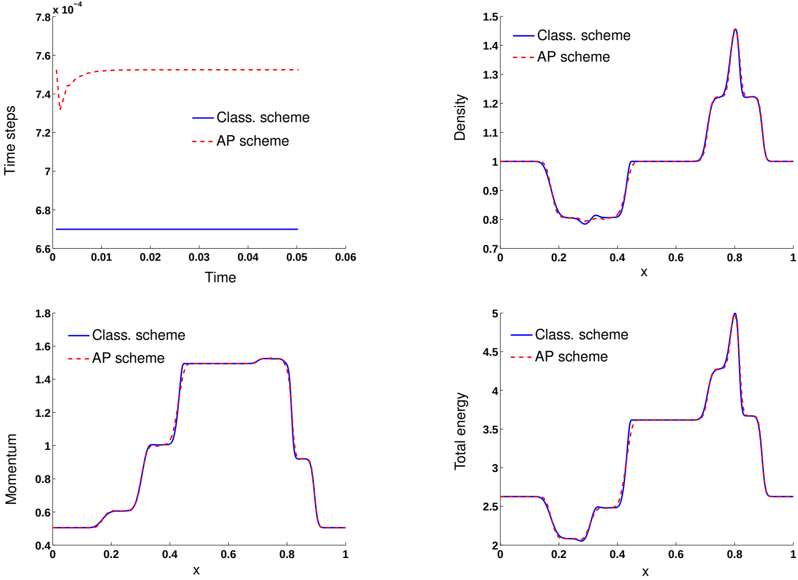
Initial momentum



Initial total energy

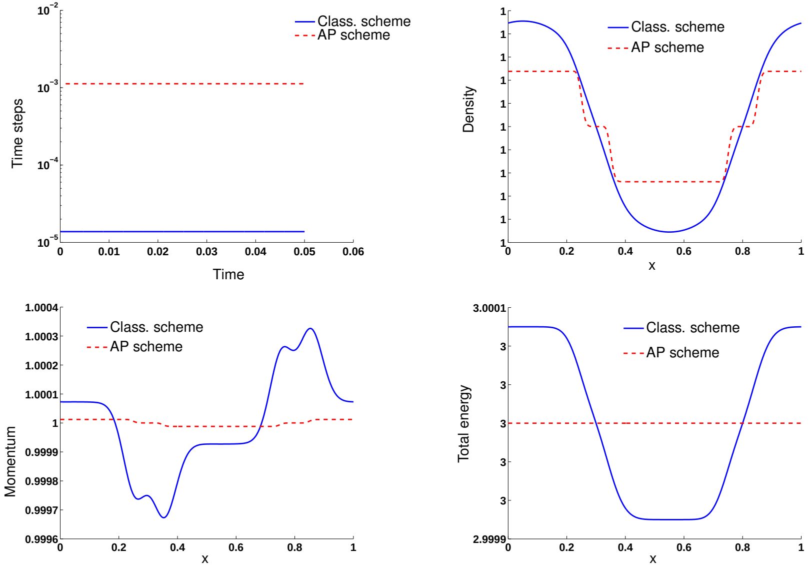
Full Euler system, $\varepsilon = 0.99$

Class : 75 loops, CPU 0.02 **AP** : 67 loops, CPU 0.08



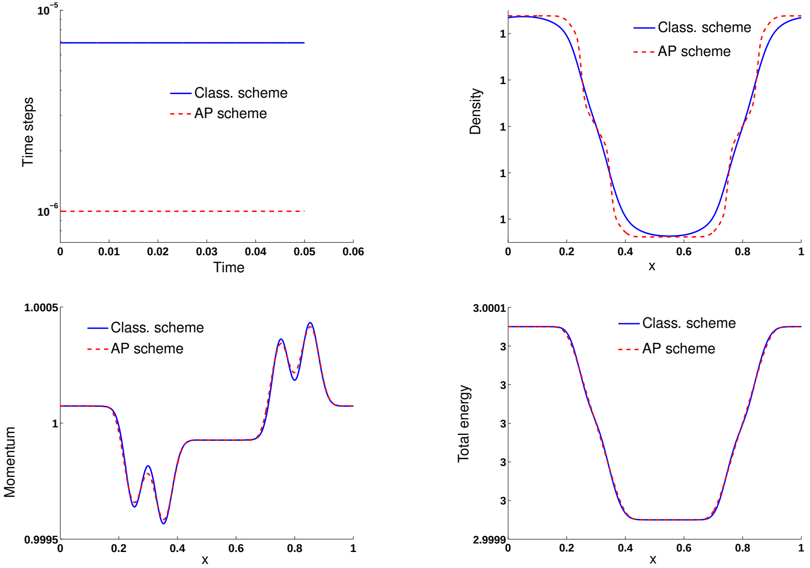
Full Euler system, $\varepsilon = 10^{-4}$

Class : 3629 loops, CPU 0.86 **AP** : 45 loops, CPU 0.06

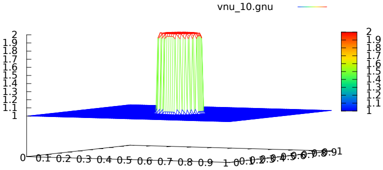


Full Euler system, $\varepsilon = 10^{-4}$

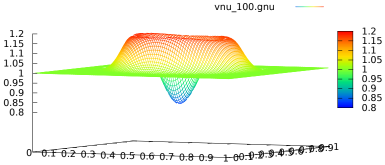
Underlying of the viscosity, 1000 cells



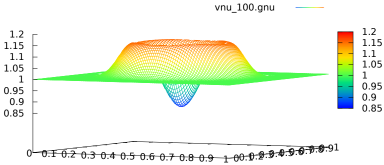
2D results in progress, $\gamma = 1$



Initial density and zero velocity



Classical density $\varepsilon = 0.99$



AP density $\varepsilon = 0.99$

- For $\varepsilon = 0.01$,

Class. 801 loops, CPU 102.22

AP 89 loops, CPU 29.13

–> $\div 3.5$

Works in progress

- Analysis of the AP scheme, convergence in the low-Mach number regime, staggered grids [Herbin, Latché, Saleh, V, in preparation]

- High order space and time algorithms preserving the AP properties, multi-dimensional test cases in physical variables ($\varepsilon = \varepsilon(x)$)

- Initial and boundary conditions not well prepared to the low Mach number regime, [Métivier, Schochet, 2001], [Alazard, 2005]

- Use the AP scheme only in the intermediate zones

$$\Rightarrow \quad \text{Domain decomposition}$$