On the stability of IMEX schemes for singular hyperbolic PDE's

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joint with

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Key example

Isentropic gas dynamics

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) &= 0, \\ \partial_t (\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) &= 0. \end{aligned}$$

Key example

Isentropic gas dynamics

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) = 0.$$

Mach number:

$$\varepsilon = \frac{u_{\mathrm{ref}}}{c_{\mathrm{ref}}}$$

Challenges

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$\varepsilon \ll 1$: stiffness

$$\mathbf{u}\cdot\mathbf{n}, \quad \mathbf{u}\cdot\mathbf{n} \pm \frac{c}{\varepsilon}$$

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$\varepsilon \ll 1$: stiffness

$$\mathbf{u} \cdot \mathbf{n}, \quad \mathbf{u} \cdot \mathbf{n} \pm \frac{c}{\varepsilon}$$

$\varepsilon \to 0$: change of type

compressible to incompressible flow
Klainerman-Majda

Challenges

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Preserve the Asymptotics

Asymptotic Consistency

Asymptotic Stability

$\Rightarrow$ AP property (Shi Jin)

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Preserve the Asymptotics

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Efficiency:

implicit for stiff part
explicit for non-stiff part
$\Rightarrow$ IMEX

Todays Talk

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- a key stability structure

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- a new class of IMEX schemes

Todays Talk

- a key stability structure
- a new class of IMEX schemes
- examples, applications, stability

Prototype linear system

$$U_t + AU_x = 0.$$

with stiff eigenvalues

$$\lambda_{max} := \max|\lambda| = O\left(\frac{1}{\varepsilon}\right)$$

$$\lambda_{min} := \min|\lambda| = O(1)$$

Admissible Splittings

Definition

A splitting

$$A = \widetilde{A} + \widehat{A}.$$

is **admissible**, if

(i) both $\widetilde{A}$ and $\widehat{A}$ induce a hyperbolic system

(ii)

$$\widetilde{\lambda} := \rho(\widetilde{A}) = O\left(\frac{1}{\varepsilon}\right)$$

$$\widehat{\lambda} := \rho(\widehat{A}) = O(1)$$

CFL Conditions

$$\nu := \lambda_{max} \frac{\Delta t}{\Delta x} \qquad \text{full} \quad \text{CFL number}$$

$$\widehat{\nu} := \widehat{\lambda} \frac{\Delta t}{\Delta x} \qquad \text{nonstiff CFL number}$$

$$\nu = O(1) \quad \Rightarrow \quad \widehat{\nu} = O(\varepsilon) \qquad \text{stable} \quad \text{inefficient}$$

$$\nu = O\left(\frac{1}{\varepsilon}\right) \quad \Leftarrow \quad \widehat{\nu} = O(1) \qquad \text{unstable} \quad \text{efficient}$$

Flux-Splitting & IMEX Time-Discretization

Implicit-explicit discretization

Klein 1996
Degond, Tang 2011
Haack, Jin, Liu 2011

$$U^{n+1} = U^n + \widetilde{A}U_x^{n+1} + \widehat{A}U_x^n$$

Examples of stability/instability

Numerical experiments:

IMEX schemes which are

- based on admissible splittings
- asymptotic consistent

can be stable and unstable

Noelle, Bispen, Arun, Lukacova, Munz SISC 2014
Euler, Low Mach, IMEX, weakly AP

Bispen, Arun, Lukacova, Noelle CiCP 2014
Shallow water, Low Froude, IMEX, AP

Modified equation, cf. Warming/Hyett 1974

Theorem (Schütz, Noelle JSC 2014)

The modified equation of the IMEX scheme is

$$w_t + Aw_x = \frac{\Delta t}{2} C \, w_{xx}$$

with diffusion matrix

$$C := (\widehat{\alpha} + \widetilde{\alpha}) \frac{\Delta x}{\Delta t} \mathbf{I} - (\widehat{A} - \widetilde{A})(\widehat{A} + \widetilde{A})$$

and numerical upwind viscosities $\widehat{\alpha}$, $\widetilde{\alpha}$.

the crucial commutator

Is C positive definite?

$$\begin{aligned} C &= \left((\widehat{\alpha} + \widetilde{\alpha}) \frac{\Delta x}{\Delta t} \mathbf{I} - \widehat{A}^2 \right) &+& \left(\widetilde{A}\widehat{A} - \widehat{A}\widetilde{A} \right) &+& \widetilde{A}^2 \\ &= O(1) &+& O\left(\frac{1}{\varepsilon}\right) &+& O\left(\frac{1}{\varepsilon^2}\right) \end{aligned}$$

Certainly yes, if commutator $[\widetilde{A}, \widehat{A}] = 0$

Example (Schütz, Noelle 2014)

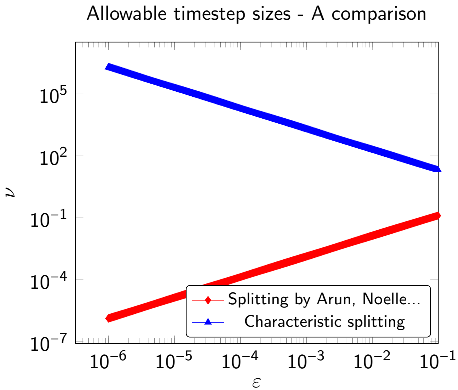
Fourier stability analysis for prototype system

$$A = \begin{pmatrix} a & 1 & 0 \\ \frac{1}{\varepsilon^2} & a & \frac{1}{\varepsilon^2} \\ 0 & 1 & a \end{pmatrix}$$

$a > 0$, eigenvalues

$$\lambda = a, a \pm \frac{\sqrt{2}}{\varepsilon}$$

Euler: classical versus characteristic splitting



Comparison of classical versus characteristic splitting

How to recover stability

Need e.g.

$$\widehat{A} \text{ and } \widetilde{A} \text{ symmetric}$$

or

$$\widetilde{A}\widehat{A} - \widehat{A}\widetilde{A} = O(1)$$

or

$$\widehat{A} = O(\varepsilon)$$

Theorem (Stability for Haack-Jin-Liu (Zakerzadeh 2015))

For the isentropic Euler equations, the Haack-Jin-Liu scheme with Mach-uniform CFL condition, has (strictly) stable modified equation in the sense of Majda-Pego, i.e. it is AP stable.

Reference-Solution IMEX

Nonlinear hyperbolic system of balance laws

$$\partial_t U(x,t;\varepsilon) + \nabla \cdot F(U,x,t;\varepsilon) = S(U,x,t;\varepsilon)$$

with

$$U : \mathbb{R}^d \times \mathbb{R}_+ \times (0,1] \to \mathbb{R}^m, \quad (x,t;\varepsilon) \mapsto U(x,t;\varepsilon)$$

- Challenge: Stiffness as $\varepsilon \to 0$
- Goal: Asymptotic stability

Reference solution and scaled perturbation: $U = \overline{U} + D\,V$

$$\begin{array}{llllll} \overline{U}: & \mathbb{R}^d \times \mathbb{R}_+ & \to \mathbb{R}^m, & (x,t) & \mapsto & \overline{U}(x,t) \\ V: & \mathbb{R}^d \times \mathbb{R}_+ \times (0,1] & \to \mathbb{R}^m, & (x,t;\varepsilon) & \mapsto & U(x,t;\varepsilon) \end{array}$$

and

$$D = \operatorname{diag}(\varepsilon^{k_1}, \ldots, \varepsilon^{k_m})$$

Taylor expansion with remainder of F and S around $\overline{U}$:

$$F = F(\overline{U}) + A(\overline{U})\ DV + \widehat{F}(\overline{U}, V) = D\big(\overline{G} + \widetilde{G} + \widehat{G}\big)$$

$$S = S(\overline{U}) + \overline{S}'V\,DV + \widehat{S}(\overline{U}, V) = \underbrace{D\big(\overline{Z} + \widetilde{Z} + \widehat{Z}\big)}_{RS+IM+EX}$$

Theorem (Modified equation for RS-IMEX (Noelle 2014))

$$B_0 W_t = -\nabla \cdot B_1 + B_2 + \nabla \cdot (B_3 \cdot \nabla W)$$

with

$$B_0 := I - \frac{\Delta t}{2}(\widetilde{Z}' - \widehat{Z}'),$$

$$B_1 := \widetilde{G} + \widehat{G} + \frac{\Delta t}{2}\Big((\widetilde{G}' - \widehat{G}')(\widetilde{Z}' + \widehat{Z}' - \widetilde{G}_x - \widehat{G}_x)\Big),$$

$$B_2 := \widetilde{Z} + \widehat{Z} + \frac{\Delta t}{2}(\widetilde{Z}_t - \widehat{Z}_t),$$

$$B_3 := \frac{(\widehat{\alpha} + \bar{\alpha})\Delta x}{2} I + \frac{\Delta t}{2}(\widetilde{G}' - \widehat{G}')(\widetilde{G}' + \widehat{G}').$$

Study this for each application!

RS-IMEX is AP for isentropic Euler

Theorem (Consistency for RS-IMEX (Zakerzadeh 2015))

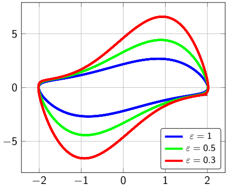
For isentropic Euler equations, the RS-IMEX scheme is consistent with the asymptotic limit in the fully-discrete settings, i.e. it is AP consistent.

- *(Schöbel 2015) AP consistency for semi-discrete scheme*

Theorem (Stability for RS-IMEX (Zakerzadeh 2015))

For isentropic Euler equations, the RS-IMEX scheme with Mach-uniform CFL condition, has (strictly) stable modified equation in the sense of Majda-Pego, i.e. it is AP stable.

van der Pol and IMEX (Schütz, Kaiser 2015)



- Prototype example

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix}.$$

- 'Traditional' splitting: $\begin{pmatrix} 0 \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix}$

van der Pol and IMEX

- 'Reference solution' (RS) $\varepsilon \to 0$:

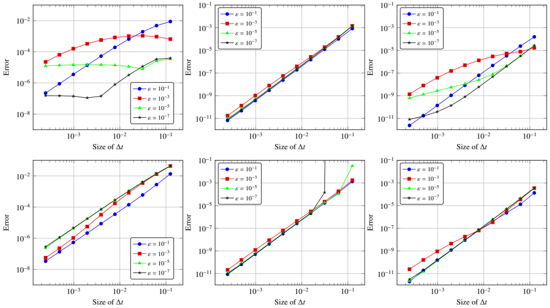
$$\begin{pmatrix} y'_{(0)} \\ 0 \end{pmatrix} = \begin{pmatrix} z_{(0)} \\ g(y_{(0)}, z_{(0)}) \end{pmatrix}.$$

- *RS-IMEX* splitting based on $w_{(0)}$:

$$f(w) = f(w_{(0)}) + f'(w_{(0)})(w - w_{(0)}) + \text{Rest}$$

- Motivation: $w - w_{(0)} = O(\varepsilon)$.

RS-IMEX + Runge-Kutta



(Left to right) DPA-242, BHR-553, BPR-353. (Top to bottom) Standard / RS-IMEX

- IMEX Runge-Kutta (Pareschi, Russo, Boscarino ...)
- standard splitting looses convergence order
- RS-IMEX gives full order of accuracy

Numerical Comparison for Isentropic Euler Equations

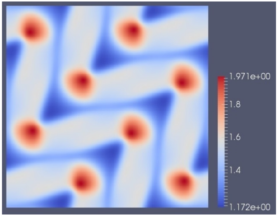
Haack-Jin-Liu scheme

RS-IMEX scheme

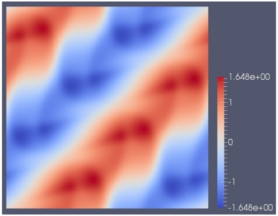
initial data from Degond-Tang (CiCP 2011) / Haak-Jin-Liu (JSC 2012)

$$
\begin{aligned}
\rho(x,y,0) &= 1 + \varepsilon^2 \sin^2(2\pi(x+y)) \\
\rho u(x,y,0) &= \sin(2\pi(x-y)) + \varepsilon^2 \sin(2\pi(x+y)) \\
\rho v(x,y,0) &= \sin(2\pi(x-y)) + \varepsilon^2 \cos(2\pi(x+y))
\end{aligned}
$$

Results

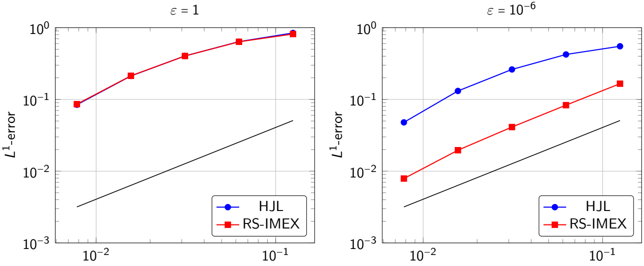


(a) Density



(b) Momentum

Results



Numerical comparison of HJL vs. RS-IMEX