

# Godunov Type Schemes for Low Froude Flows with Coriolis Term

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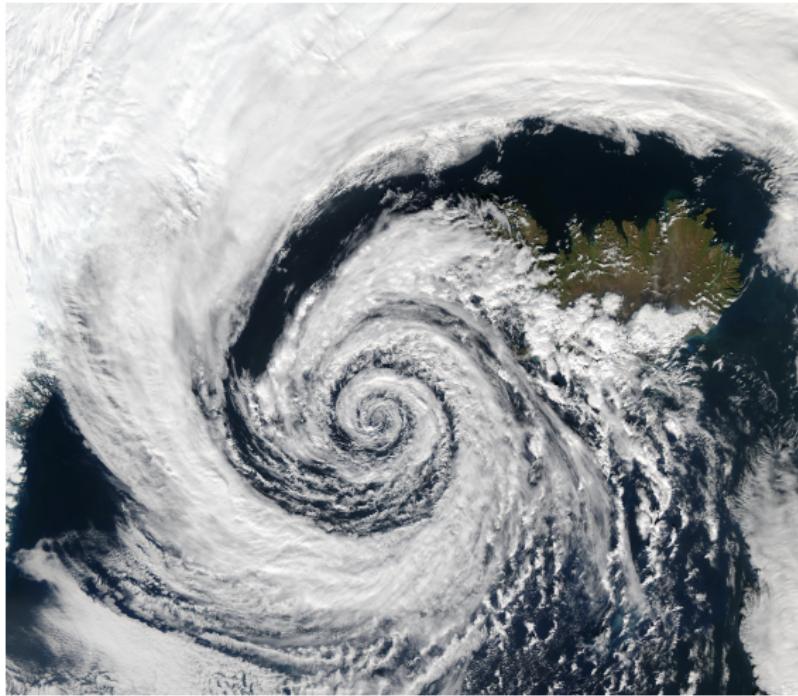


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November 6, 2015

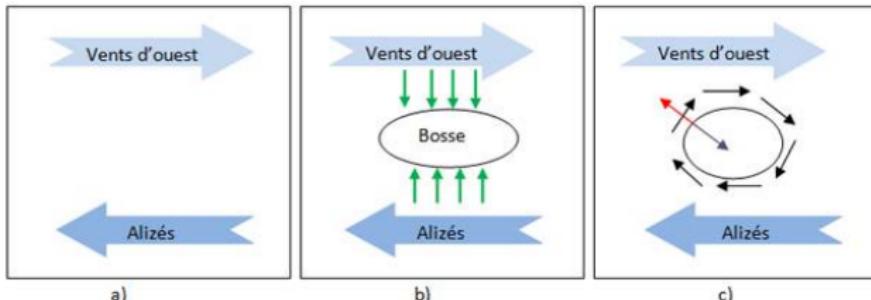
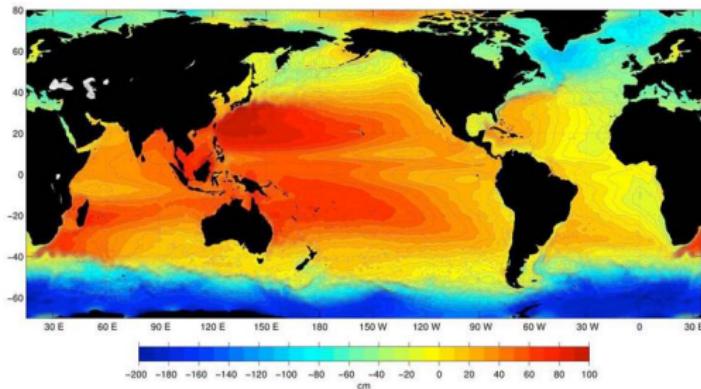
# Velocity field near a low-pressure zone



# Cyclones at different locations

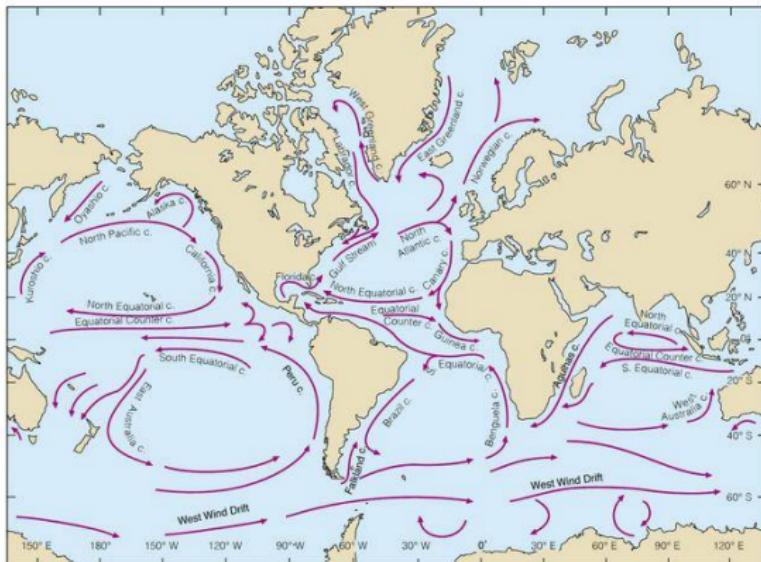


# In the ocean : Free surface elevation



- Transport d'Ekman
- Vitesse du courant
- Force de surpression
- Force de Coriolis

# In the ocean : Global circulation



<http://www.emse.fr/bouchardon/>

# Outline

- ▶ No more beautiful images
- ▶ Nor realistic numerical simulation
- ▶ Context : SWE with Coriolis term
  - ▶ Dimensionless Equations
  - ▶ Previous works
- ▶ Study of the 1d linear wave equation with Coriolis force
  - ▶ Kernel of the space operator
  - ▶ Stability of WB schemes
  - ▶ Accuracy at low Froude number
- ▶ Perspectives

# Shallow Water Equations with Coriolis Force

## ► Equations

$$\begin{aligned}\partial_t H + \nabla \cdot (H \mathbf{U}) &= 0 \\ \partial_t \mathbf{U} + \nabla \cdot (H \mathbf{U} \otimes \mathbf{u}) + \nabla \frac{gH^2}{2} &= -gH \nabla B - 2\Omega \times (H \mathbf{U})\end{aligned}$$

## ► Source terms

- Topography
- Coriolis Force

# Shallow Water Equations with Coriolis Force

## ► Dimensionless Equations

$$S_t \partial_t h + \nabla \cdot (h \mathbf{u}) = 0$$

$$\begin{aligned} S_t \partial_t \mathbf{u} + \nabla \cdot (h \mathbf{u} \otimes \mathbf{u}) + \frac{1}{F_r^2} \nabla \frac{h^2}{2} \\ = -\frac{1}{F_r^2} h \nabla b - \frac{1}{R_o} 2\omega \times (h \mathbf{u}) \end{aligned}$$

## ► Dimensionless Numbers

$$S_t = \frac{L}{UT}, \quad F_r = \frac{U}{\sqrt{gH}}, \quad R_o = \frac{U}{\|\Omega\| L}$$

- Strouhal : Advection vs. Non stationarity
- Froude : Advection vs. Pressure Gradient
- Rossby : Advection vs. Rotation

# Shallow Water Equations with Coriolis Force

- ▶ Dimensionless Equations

$$S_t \partial_t h + \nabla \cdot (h \mathbf{u}) = 0$$

$$\begin{aligned} S_t \partial_t \mathbf{u} + \nabla \cdot (h \mathbf{u} \otimes \mathbf{u}) + \frac{1}{F_r^2} \nabla \frac{h^2}{2} \\ = -\frac{1}{F_r^2} h \nabla b - \frac{1}{R_o} 2\omega \times (h \mathbf{u}) \end{aligned}$$

- ▶ Typical values in lakes or ocean

$$U = 1m/s, L = 10-10^3 km, H = 10-10^3 m, \|\Omega\| = 10^{-4} rad/s$$

- ▶ Lakes or Oceanic bay :  $F_r \approx 10^{-1}, R_o \approx 1$
- ▶ Deep Ocean :  $F_r \approx 10^{-2}, R_o \approx 10^{-2}$

# Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t \partial_t h + \nabla \cdot (h \mathbf{u}) = 0$$

$$\begin{aligned} S_t \partial_t \mathbf{u} + \nabla \cdot (h \mathbf{u} \otimes \mathbf{u}) + \frac{1}{F_r^2} \nabla \frac{h^2}{2} \\ = -\frac{1}{F_r^2} h \nabla b - \frac{1}{R_o} 2\omega \times (h \mathbf{u}) \end{aligned}$$

- Lake at rest

$$\nabla(h + b) = 0, \mathbf{u} = 0$$

- Geostrophic Equilibrium

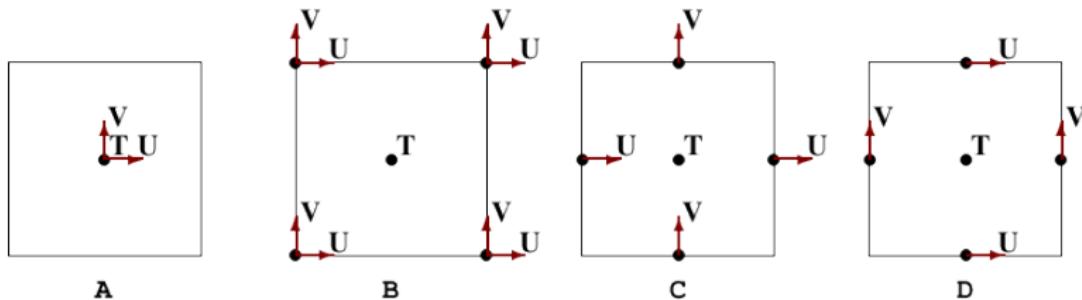
$$\nabla(h + b) + 2\omega \times \mathbf{u} = 0, \nabla \cdot \mathbf{u} = 0$$

# Numerical Simulations

- ▶ Stability of the scheme
  - ▶ OK for classical treatment of the homogeneous part
  - ▶ Modifications due to source terms
  - ▶ Linear and/or non linear studies
- ▶ Ability to preserve stationary states
  - ▶ Kernels of the continuous and discrete space operators
  - ▶ Impact on transient and long time results
- ▶ Accuracy at Low Froude
  - ▶ Stability of the kernel of the discrete space operator
  - ▶ Spurious numerical waves
- ▶ Dispersion laws
  - ▶ Linear case
  - ▶ More important for high order schemes

# "Industrial" codes

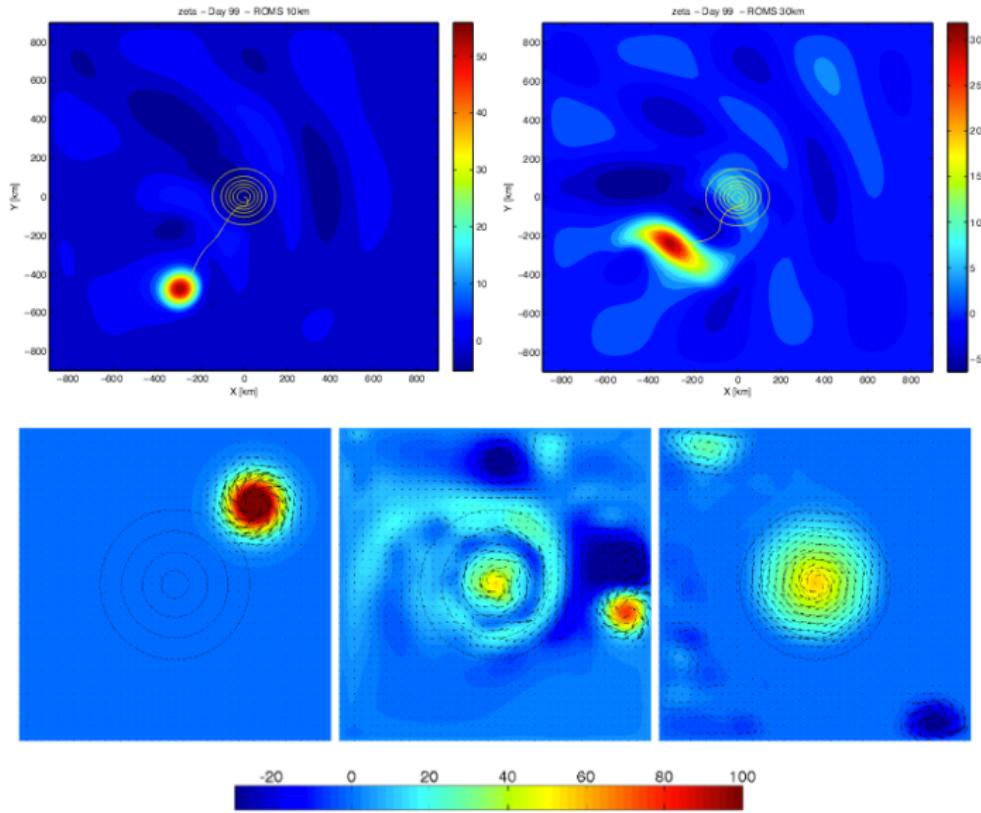
- ▶ ROMS, NEMO, HYCOM...
- ▶ Semi-implicit in time schemes
- ▶ Finite Difference Method on Arakawa grids (MCP, 1977)



On the Approx. of Coriolis Terms in C-Grid Model,  
Nechaev et al., AMS, 2004.

Num. Represent. of Geostrophic Modes on Arbitrarily Structured C-Grids,  
Thuburn et al., JCP, 2009.

# ANR COMODO



# Galerkin framework

- ▶ Semi-implicit in time schemes
- ▶ Many possible choices for the FE-DG element

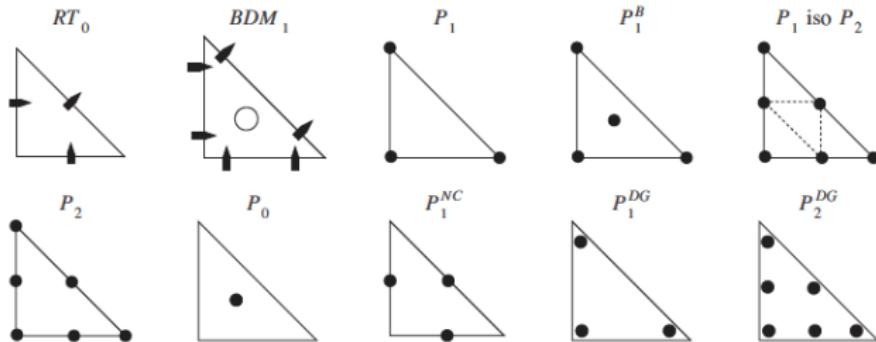


Fig. 1. Typical node locations represented by the symbols • for the  $RT_0$ ,  $BDM_1$ ,  $P_1$ ,  $P_1^B$ ,  $P_1$  iso  $P_2$ ,  $P_2$ ,  $P_0$ ,  $P_1^{NC}$ ,  $P_1^{DG}$  and  $P_2^{DG}$  finite elements.

Spurious Inertial Oscillations in SW Models,  
LeRoux, JCP, 2012.

# Galerkin framework

- ▶ Study of the kernel of space operator / Fourier analysis
  - ▶ Not the same number of velocity and pressure points
  - ▶ Spurious inertial oscillations

FE pair	$(p,q)$	nr	Geostrophic	Inertial ( $\nabla f$ , mult.)	Spurious $\eta$ modes	Inertia-gravity		
			0	$O(h^2)$		$O(1)$	$O(\frac{1}{h})$	
1	$P_1 - P_1$	(1,1)	3	<b>1</b>	Yes	2		
	$P_1^0 - P_1$	(3,1)	7	<b>1</b>		2		
	$P_1$ iso $P_2 - P_1$	(4,1)	9	<b>1</b>		2		
	$P_2 - P_0$	(4,2)	10	<b>2</b>		2	2	
	$P_2 - P_1$	(4,1)	9			2	2	
	$P_1^{NC} - P_0$	(3,2)	8	<b>2</b>		2	2	
	$P_1^{NC} - P_1$	(3,1)	7	<b>1</b>		2		
	$P_0 - P_1$	(2,1)	5	<b>1</b>		2		
	$P_1^{BG} - P_1$	(6,1)	13	<b>1</b>		2		
2	$P_1^{BG} - P_2$	(6,4)	16	<b>4</b>	No	2	6	
	$RT_0 - P_0$	(3,2)	5	<b>1</b>		2	2	
	$RT_0 - P_1$	(3,1)	4	<b>2</b>		2		
	$BDM_1 - P_0$	(6,2)	8	<b>4</b>		2	2	
	$BDM_1 - P_1$	(6,1)	7	<b>3</b>		2		

"We recommend to employ the same finite-element bases leading to  $p = q$  [...] to approximate surface-elevation and velocity."

# 1D Wave Equation with Coriolis Force

- ▶ Linearization of the SW model
  - ▶ Ocean study  $F_r = R_o = \epsilon$
  - ▶ Flat topography
  - ▶ Flow variables independent of  $y$  direction
  - ▶  $a, f = O(\epsilon^{-1})$

$$\begin{aligned}\partial_t h + a\partial_x u &= 0 \\ \partial_t u + a\partial_x h &= fv \\ \partial_t v &= -fu\end{aligned}$$

- ▶ Collocated Finite Volume Study
  - ▶ Ability to capture equilibrium states
  - ▶ Accuracy at Low Froude
  - ▶ Stability

## Works on the same subject

- ▶ Well balanced Schemes with Coriolis terms

Frontal Geostrophic Adjustment in 1D-Rotating SW  
Bouchut et al., JFM, 2004.

WB FV Evolution Galerkin Methods for the SWE  
Lukacova et al., JCP, 2007.

FV Simulation of the Geostrophic Adjustment in a Rotating SW System  
Castro et al., SIAM JSC, 2008

- ▶ Accuracy at low Mach number

Godunov type schemes for compressible Euler system at Low Mach Number,  
Dellacherie, JCP, 2010.

# 1D Wave Equation with Coriolis Force

- ▶ Kernel of the continuous space operator  
(1d Geostrophic equilibrium = Jet in rotating frame)

$$K_c = \{u = 0; a \partial_x h = fv\}$$

- ▶ Hodge type decomposition

$$K_c^\perp = \{a \partial_x v = fh\}$$

$$K_c \oplus K_c^\perp = L^2 \quad \text{and} \quad q(0) \in K_c^\perp \Rightarrow q(t) \in K_c^\perp$$

- ▶ Stability

$$\forall t > 0 \quad E(t) = E(0), \quad E = \|h\|^2 + \|u\|^2 + \|v\|^2$$

- ▶ Dispersion law

$$\omega = 0 \quad ; \quad \omega^2 = f^2 + (ak)^2$$

# Standard Explicit Godunov scheme

## ► Fully discrete scheme

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{a\kappa_h \Delta x}{2} \frac{h_{j+1}^n - 2h_j^n + h_{j-1}^n}{(\Delta x)^2} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} - \frac{a\kappa_u \Delta x}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} = fv_j^n$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{a\kappa_v \Delta x}{2} \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} = -fu_j^n$$

## ► Modified equation

$$\partial_t h + a \partial_x u + \mu_h \partial_{xx} h = 0$$

$$\partial_t u + a \partial_x h + \mu_u \partial_{xx} u = fv$$

$$\partial_t v + \mu_v \partial_{xx} v = -fu$$

# Properties of the modified equation

- ▶ Kernel of the continuous space operator  
 $(\neq 1d \text{ Geostrophic equilibrium})$

$$K_e = \{u = 0; v = 0; h = h_0\} \subsetneq K_c$$

- ▶ Hodge type decomposition

$$K_e^\perp = \left\{ \int h = 0 \right\}$$

$$K_e \oplus K_e^\perp = L^2 \quad \text{and} \quad q(0) \in K_e^\perp \Rightarrow q(t) \in K_e^\perp$$

- ▶ Stability

$$\forall t > 0 \quad \bar{E}(0) = \bar{E}(t) \leq E(t) \leq E(0), \quad \bar{E} = \|\bar{h}\|^2 + \|\bar{u}\|^2 + \|\bar{v}\|^2$$

- ▶ Dispersion law

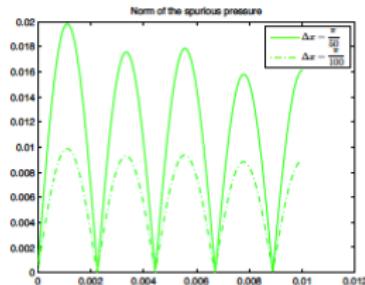
$$\omega = -i\mu \quad ; \quad \omega = -i\mu \pm \sqrt{f^2 + (ak)^2}$$

# Properties of the modified equation

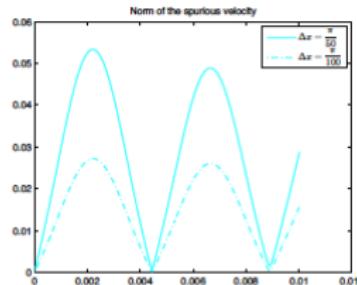
- ▶ Kernel of the continuous space operator  
 $\neq$  1d Geostrophic equilibrium)

$$K_e = \{u = 0; v = 0; h = h_0\} \subsetneq K_c$$

- ~~> For small time :  
Creation of inertia-gravity (= acoustic) spurious waves
- ~~> For long time :  
Non physical steady state



(a)  $\|p_h\|$



(b)  $\|u_h\|$

# Well-balanced schemes for Coriolis term - 1

- ▶ Apparent topography

$$\partial_x \tilde{b} = -fv$$

- ▶ Steady state

$$\partial_x(h + \tilde{b}) = 0$$

~~> Hydrostatic reconstruction

A fast and stable WB scheme with hydrostatic reconstr. for SW flows,  
AABKP, SIAM JSC, 2004.

~~> Path-conservative Schemes

On the WB prop. of Roe method for nonconservative hyperbolic systems,  
Pares et al., M2AN, 2004.

~~> Evolution Galerkin Methods

Finite volume evolution Galerkin methods for hyperbolic problems,  
Lukacova, SIAM JSC, 2004.

# Well-balanced schemes for Coriolis term - 1

## ► Fully discrete scheme

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{a\kappa_h \Delta x}{2} \frac{h_{j+1}^n - 2h_j^n + h_{j-1}^n}{(\Delta x)^2} + \frac{\kappa_h f}{2} \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} - \frac{a\kappa_u \Delta x}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} = f \frac{v_{j+1}^n + 2v_j^n + v_{j-1}^n}{4}$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = -f \frac{u_{j+1}^n + 2u_j^n + u_{j-1}^n}{4}$$

# Modified equation

- ▶ Modified equation

$$\begin{aligned}\partial_t h + a \partial_x u + \mu_h \partial_{xx} h + \nu_h \partial_x v &= 0 \\ \partial_t u + a \partial_x h + \mu_u \partial_{xx} u &= fv \\ \partial_t v &= -fu\end{aligned}$$

- ▶ Kernel of the continuous space operator  
(1d Geostrophic equilibrium = Jet in rotating frame)

$$\tilde{K}_e = \{u = 0; a \partial_x h = fv\} = K_c$$

- ▶ Stability

$$\forall t > 0 \quad \bar{E}(0) = \bar{E}(t) \leq E(t) \leq E(0),$$

## Well-balanced schemes for Coriolis term - 2

- ▶ No diffusion on the mass equation

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} - \frac{a\kappa_u \Delta x}{2} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} = fv_j^n$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = -fu_j^n$$

- ▶ Accuracy of Godunov scheme at low Mach number
- ↝ No diffusion on the momentum equ. (cartesian grid in 2D)

J. Jung talk yesterday

# Modified equation

- ▶ Modified equation

$$\begin{aligned}\partial_t h + a \partial_x u &= 0 \\ \partial_t u + a \partial_x h + \mu_u \partial_{xx} u &= fv \\ \partial_t v &= -fu\end{aligned}$$

- ▶ Kernel of the continuous space operator  
(1d Geostrophic equilibrium = Jet in rotating frame)

$$\hat{K}_e = \{u = 0; a \partial_x h = fv\} = K_c$$

- ▶ Stability

$$\forall t > 0 \quad \bar{E}(0) = \bar{E}(t) \leq E(t) \leq E(0),$$

# Kernels of the fully discrete scheme

- Scheme without diffusion on the mass equation

$$\hat{K}_e^{\Delta x} = \left\{ u_j = 0; a \frac{h_{j+1} - h_{j-1}}{2\Delta x} = f v_j \right\}$$

$\rightsquigarrow$  Spurious  $\eta$  modes

$$h_{2j} = C, \quad h_{2j+1} = -C, \quad v_j = 0$$

- Scheme with apparent topography

$$\tilde{K}_e^{\Delta x} = \left\{ \frac{u_j + u_{j+1}}{2} = 0; a \frac{h_{j+1} - h_j}{2\Delta x} = f \frac{v_j + v_{j+1}}{2} \right\}$$

$\rightsquigarrow$  Spurious velocity modes

$$h_j = h_0, \quad v_{2j} = v_0, \quad v_{2j+1} = -v_0$$

# Stability of the fully discrete scheme

- ▶  $L^2$  Stability of the homogeneous Godunov scheme

$$\sigma \leq \min \left( \kappa, \frac{1}{\kappa} \right), \quad \sigma = \frac{a\Delta t}{\Delta x}$$

- ▶  $L^2$  Stability of the EDO (Coriolis term)

- ▶ Explicit scheme : Unstable
- ▶  $\theta$ -scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \theta_u v_j^n + (1 - \theta_u) v_j^{n+1}$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \theta_v u_j^n + (1 - \theta_v) u_j^{n+1}$$

Stable if

$$\theta_u + \theta_v \leq 1$$

# Stability of the fully discrete scheme

- ▶ Fourier Analysis
  - ~~> Eigenvalues of  $3 \times 3$  matrices
- ▶ For both well balanced schemes

$$\lambda_1 = 1$$

- ▶ Scheme without diffusion on the mass equation

$$\sigma \leq \min \left( \frac{\kappa_u}{2}, \frac{1}{\kappa_u}, \frac{1}{\kappa_u(2 - (\theta_u + \theta_v))} \right),$$

- ▶ Scheme with apparent topography

- ▶ Explicit

$$\theta_u = 1, \quad \theta_v = 0$$

- ▶ Stability

$$\sigma \leq \min \left( \kappa, \frac{1}{\kappa} \right)$$

# All-Froude number modified scheme

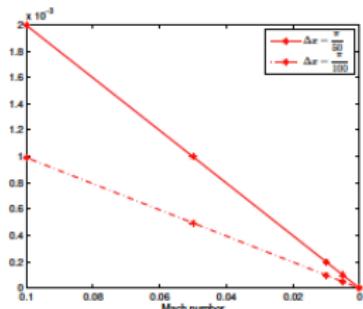
- ▶ Definition

$$\kappa_h = O(\epsilon)$$

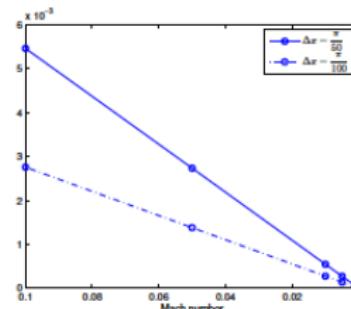
- ▶ Stability

$$\lambda'_1(\epsilon)|_{\epsilon=0} < 0$$

- ▶ Accuracy near geostrophic equilibrium



(a) The maximum of  $\|p_h\|$  as a function of M



(b) The maximum of  $\|u_h\|$  as a function of M

Figure 8: One step scheme with  $\kappa_r = M$ ,  $\kappa_M = 1$ ,  $\theta_1 = \theta_2 = 0$ .

# Perspectives

- ▶ Advection terms
  - ~~ Stability of the Low and All-Froude modified schemes ?
- ▶ 2d extension
  - ~~ Full incompressible limit

Preservation of the Discrete Geostrophic Equilibrium in SW Flows,  
AKNV, FVCA 6, 2011.

- ▶ Generic approach for linear WB schemes

~~ Work for 2d case

Structure of WB schemes for Friedrichs system with Linear Relaxation,  
Despres & Buet, JCP, 2015.

- ▶ Non linear study

~~ Stability criteria

~~ 2d non linear kernel  $\neq$  2d linear kernel

This is the end...

Thanks to the organizers  
for this wonderful conference !



But before, you may want to make comments...