On the Mach-Uniform Stability of Lagrange-Projection Scheme

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Objectives

We show that the Lagrange-projection scheme applied to the isentropic Euler equations is

- positivity-preserving
- L_∞-stable
- locally entropy dissipative
- asymptotic preserving (AP) w.r.t. the Mach number

under some Mach-uniform restrictions, and with well-prepared initial data.

Introduction: AP scheme

Isentropic Euler equations on domain $\Omega_T := \mathbb{T}(\Omega) \times \mathbb{R}_+$:

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + \frac{p}{\epsilon^2}) = 0, \tag{2}$$

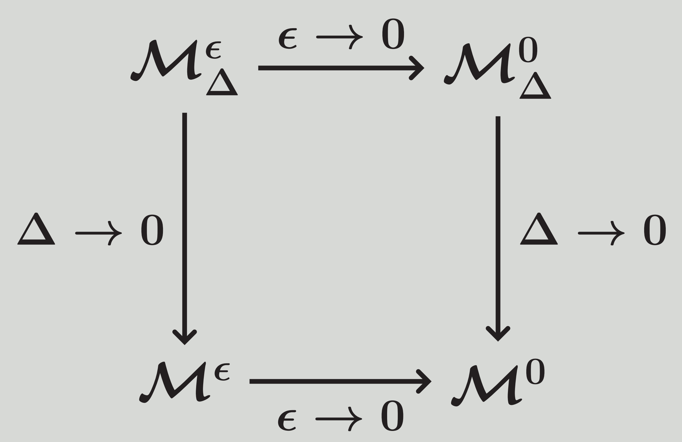
accompanied with energy inequality

$$\partial_t(\rho E) + \partial_x((\rho E + \frac{p}{\epsilon^2})u) \leqslant 0 \quad \text{in } \mathcal{D}'(\Omega_T). \tag{3}$$

ϵ	Mach number
$p(\rho) := \kappa\rho^\gamma$	Pressure function
$E(\rho) := \frac{\kappa/\epsilon^2}{\gamma-1}\rho^{\gamma-1} + \frac{u^2}{2}$	Total energy density

Asymptotic Preserving (AP) Scheme [Jin, 1995]:

- Consistency of the scheme with the underlying PDE as $\epsilon \to 0$
- Stability of the scheme uniformly in ϵ



Lagrange-projection approach

Solve Riemann problem in Lagrangian form

Project the solution on the fixed Eulerian grid

Equivalent to solve successively [Chalons et al., 2014]:

- **Acoustic system**

$$\tau_t - u_z = 0, \tag{4}$$

$$u_t + \frac{p_z}{\epsilon^2} = 0, \tag{5}$$

with $\tau := 1/\rho$ and $z := \rho \mathrm{d}x$.

- **Transport system** for $\mathbb{U} := (\rho, \rho u)^T$ with velocity u:

$$\partial_t \mathbb{U} + u \partial_x \mathbb{U} = 0. \tag{6}$$

Relaxation scheme for Lagrange step: Suliciu relaxation

Add an auxiliary equation to make the system linearly-degenerate [Chalons et al., 2014]:

$$\tau_t - u_z = 0,$$
$$u_t + \Pi_z = 0,$$
$$\Pi_t + \alpha^2 u_z = \Lambda(p - \pi),$$

where $\Pi := \frac{\pi}{\epsilon^2}, \alpha := \frac{a}{\epsilon}, \Lambda := \frac{\lambda}{\epsilon^2}$.

Liu's sub-characteristic condition

$$a^2 > \max(-p_\tau) \tag{7}$$

Using Riemann invariants [Coquel et al., 2010]:

$$\tau_t - u_z = 0,$$
$$\overrightarrow{w}_t + \alpha \overrightarrow{w}_z = 0, \qquad \overrightarrow{w} := \Pi + \alpha u,$$
$$\overleftarrow{w}_t - \alpha \overleftarrow{w}_z = 0, \qquad \overleftarrow{w} := \Pi - \alpha u.$$

Numerical scheme

- **Lagrange step** $(n \to n\dagger)$:

$$\tau_j^{n\dagger} = \tau_j^n + \frac{\Delta t}{\Delta z_j}(\tilde{u}^{n\dagger}_{j+1/2} - \tilde{u}^{n\dagger}_{j-1/2}), \tag{8}$$

$$\overrightarrow{w}_j^{n\dagger} = \overrightarrow{w}_j^n - \frac{a\Delta t}{\epsilon \Delta z_j}(\overrightarrow{w}_j^{n\dagger} - \overrightarrow{w}_{j-1}^{n\dagger}), \tag{9}$$

$$\overleftarrow{w}_j^{n\dagger} = \overleftarrow{w}_j^n + \frac{a\Delta t}{\epsilon \Delta z_j}(\overleftarrow{w}_{j+1}^{n\dagger} - \overleftarrow{w}_j^{n\dagger}), \tag{10}$$

where

$$\tilde{u}^{n\dagger}_{j+1/2} = \frac{u_j^{n\dagger} + u_{j+1}^{n\dagger}}{2} - \frac{1}{2a\epsilon}(\pi^{n\dagger}_{j+1} - \pi^{n\dagger}_j).$$

- **Projection step** $(n\dagger \to n+1)$:

$$\mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger} + \frac{\Delta t}{\Delta x}\Big[(\tilde{u}^{n\dagger}_{j-1/2})^+ \mathbb{U}^{n\dagger}_{j-1} - (\tilde{u}^{n\dagger}_{j+1/2})^- \mathbb{U}^{n\dagger}_{j+1} + \Big((\tilde{u}^{n\dagger}_{j+1/2})^- - (\tilde{u}^{n\dagger}_{j-1/2})^+\Big)\mathbb{U}^{n\dagger}_j\Big]. \tag{11}$$

Main theorem: Uniform stability

Definition (Well-prepared initial data)

For isentropic Euler equation, with constant p_0 and u_0:

$$p^0_{WP}(x) := p_0 + \mathcal{O}(\epsilon^2) p_2(x),$$
$$u^0_{WP}(x) := u_0 + \mathcal{O}(\epsilon) u_1(x).$$

Lemma (cf. [Coquel et al., 2010])

The time step restriction

$$\frac{\Delta t}{\Delta x} \leqslant \frac{2a/\epsilon}{(\overrightarrow{M}^n - \overleftarrow{m}^n)^+ - (\overrightarrow{m}^n - \overleftarrow{M}^n)^-}, \tag{12}$$

is uniform in ϵ, where $\overrightarrow{M}^n$ and $\overrightarrow{m}^n$ denote the maximum and minimum of $\overrightarrow{w}^n$, similarly for $\overleftarrow{w}$.

Theorem [Zakerzadeh, 2015] (cf. [Coquel et al., 2010])

The Lagrange-projection scheme for well-prepared initial data and under the ϵ-uniform CFL constraint (12)

- is in the locally conservative form
- is AP consistent, i.e.

$$\mathrm{div} u^n_{(0)} = 0, \quad \rho^n_{(0)j}, \rho^n_{(1)j} = const.$$

- is positivity-preserving
- is stable in L_∞-norm, uniformly in ϵ
- satisfies cell entropy inequality (consistent with (3)), under (7), i.e.

$$\frac{(\rho E)^{n+1}_j - (\rho E)^n_j}{\Delta t} + \frac{(\rho E \tilde{u} + \frac{\tilde{\pi}\tilde{u}}{\epsilon^2})^{n\dagger}_{j+1/2} - (\rho E \tilde{u} + \frac{\tilde{\pi}\tilde{u}}{\epsilon^2})^{n\dagger}_{j-1/2}}{\Delta x} \leqslant 0.$$

- Proof is similar to Coquel et al. [2010] with additional parameter ϵ
- Direct L_∞ estimate is a new result compared to [Coquel et al., 2010]

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