Low Mach Number Modeling of Stratified Astrophysical Flows

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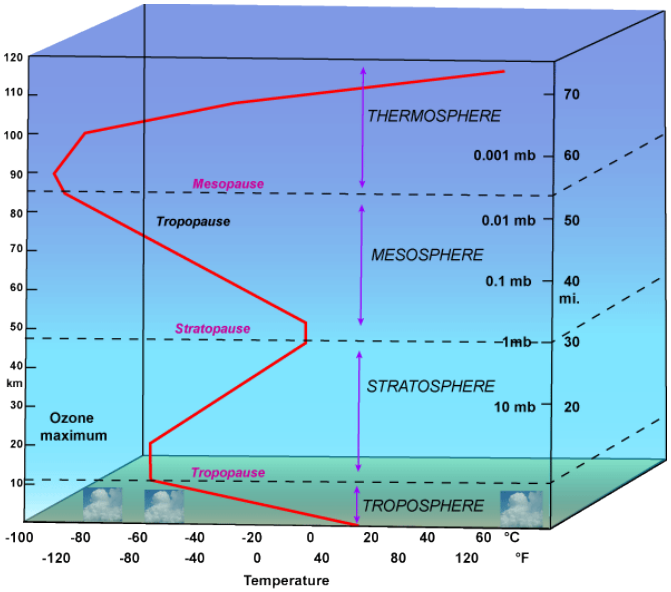
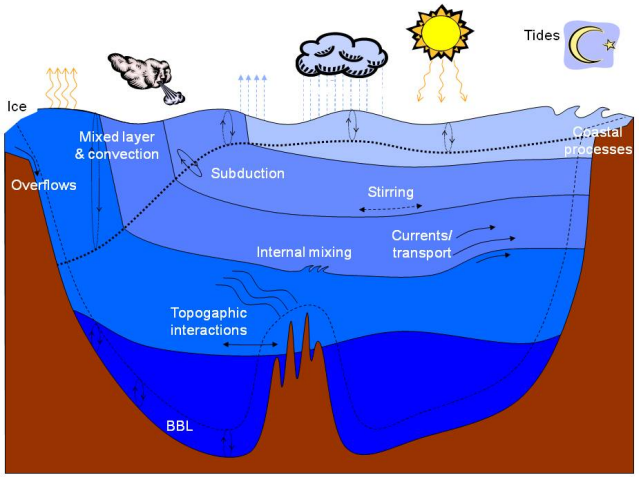
Workshop on Low Velocity Flows
November 5-6, 2015

Examples of Stratified Flows

When we think of stratified flows, we often think of the atmospheric or ocean.



Atmospheric Stratification



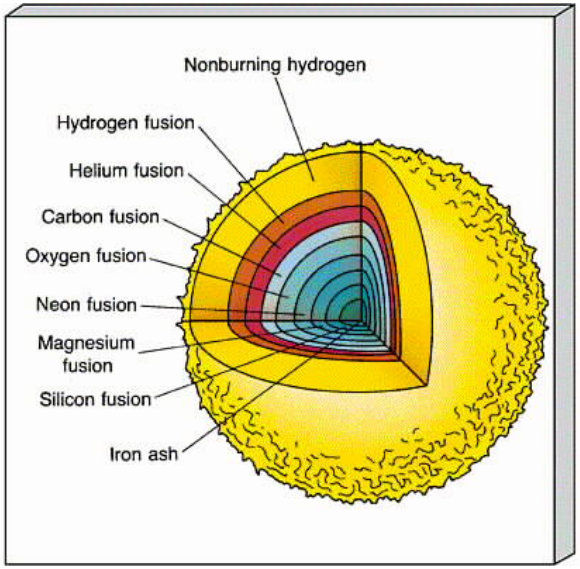
Oceanic Stratification

Note that these pictures distort the length scales.

Atmospheres and oceans are really thin layers on a sphere.

More Stratified Flows

We might also think of stars ...



Interior of a highly evolved star

Unlike in the atmosphere and ocean, stellar convection can occur either within thin layers or throughout the whole star.

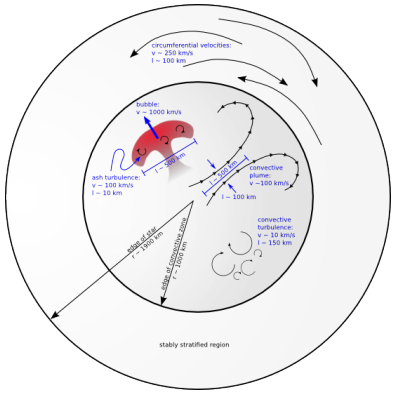
Low Mach Number Modeling

Atmospheric and oceanic convection tends to be slow relative to the sound speed.

- air : $c \approx 340$ m/s, $v \approx 3 - 4$ m/s in Paris ($v > 33$ m/s to be defined as a hurricane)
- ocean: $c \approx 1500$ m/s, $v \approx 6$ m/s in Gulf Stream

Stellar convection tends to be slow relative to the sound speed as well.

- star : $c \approx 5e6$ m/s, $v \approx 1e5$ m/s



Astrophysical Flows

But many of the most interesting astrophysical flows are explosive.

- supernovae (explosion of whole star)
- gamma-ray bursts (brightest electromagnetic events in universe; result from collapse of star)

- classical novae (burst from accreted H/He layer on white dwarf)
- X-ray bursts (burst from accreted He layer on neutron star)

SN 1994d

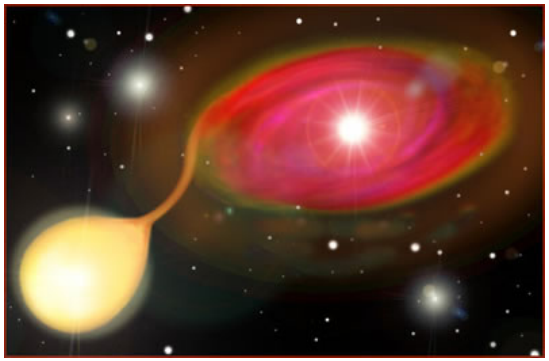
Type Ia Supernovae

- Largest thermonuclear explosions in the universe
- Brightness rivals that of host galaxy, L 10^{43} erg / s
- Definition: no H line in the spectrum, Si II line at 6150A.

SNe Ia: Theory

Suppose we want to study Type Ia supernovae ...

One of the models of a SN Ia progenitor is a carbon/oxygen white dwarf in a binary pair.

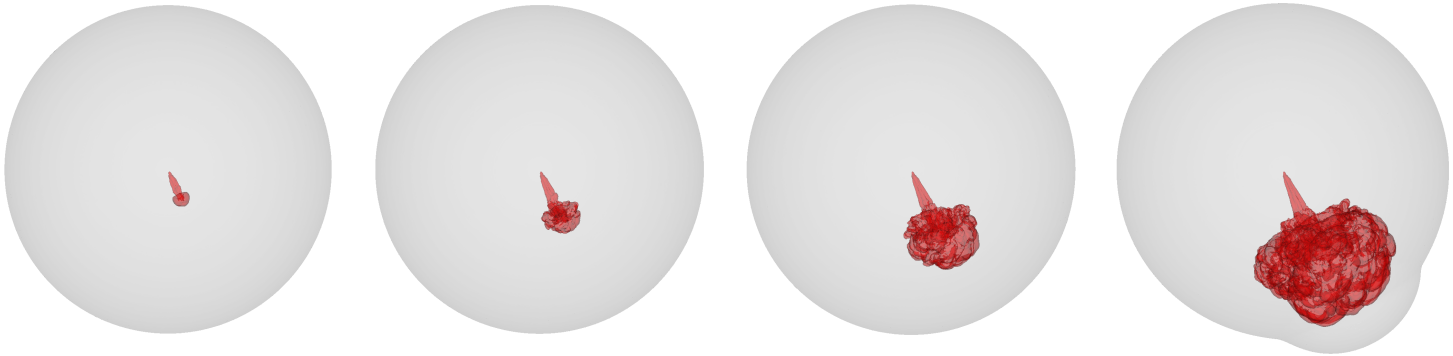


A carbon-oxygen white dwarf accretes mass from a binary companion ($\approx$ 10 million years to reach Chandrasekhar limit)

- Over a period of centuries, carbon burning near the core drives convection and temperature slowly increases.
- Over the last few hours, (low Mach number!) convection becomes more vigorous as the heat release intensifies and convection can no longer carry away the heat.
- Eventually, the star ignites, and finally explodes within seconds.

SNe Ia: Modeling

Traditional modeling approaches focus on the last few seconds.



Initial conditions:

- Radial profile from 1d stellar evolution code
- Assumptions about when & where of ignition ”hot spots”

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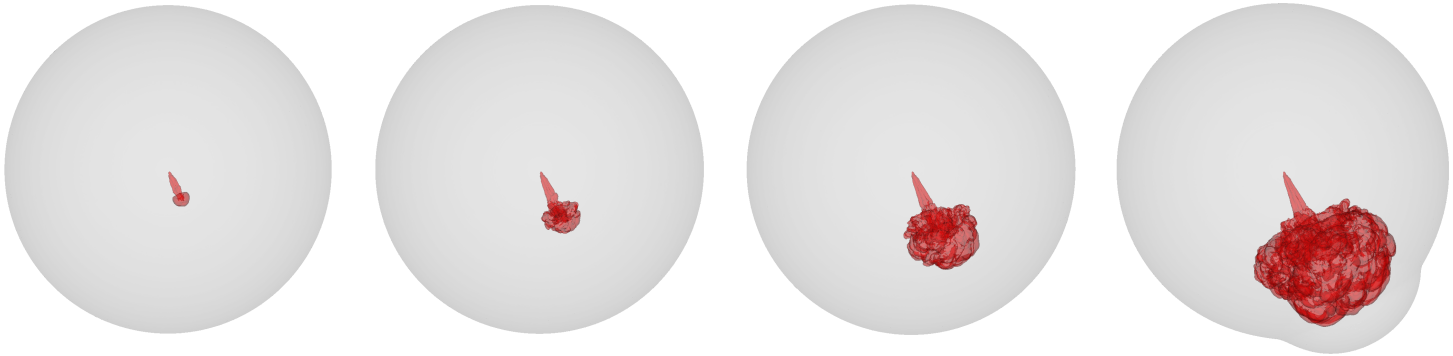

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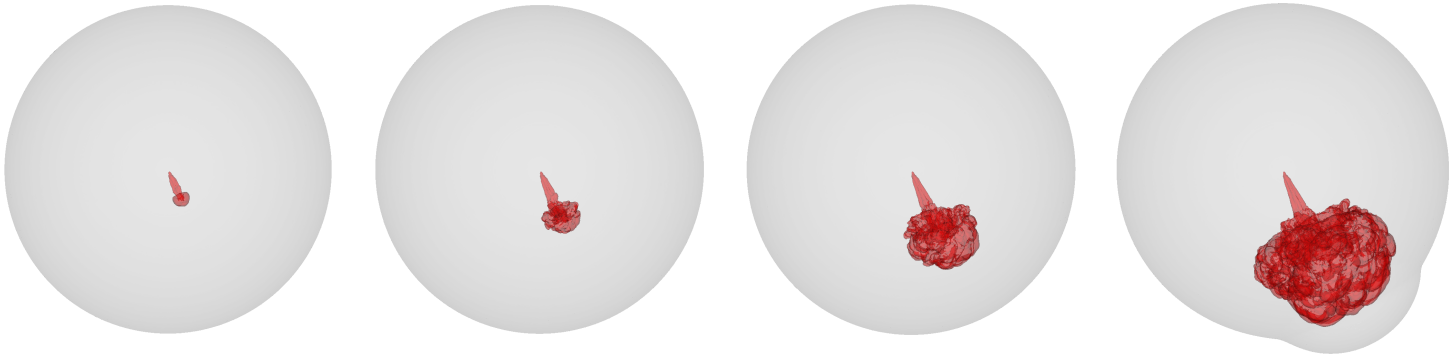

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- Radial profile from 1d stellar evolution code
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But... the simulated explosions are very sensitive to the initial conditions.

⇒ We need to know more about how SNe Ia ignite.

Modeling of Type Ia Supernovae

Typically, numerical simulations of SNe Ia have used the compressible Navier-Stokes equations with reactions:

$$
\begin{aligned}
\frac{\partial(\rho X_k)}{\partial t} + \nabla \cdot (\rho U X_k) &= \rho \dot{\omega}_k \\
\frac{\partial(\rho U)}{\partial t} + \nabla \cdot (\rho U U + p) &= -\rho g \mathbf{e}_r \\
\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho U E + U p) &= -\rho g (U \cdot \mathbf{e}_r) + \rho \sum_k q_k \dot{\omega}_k
\end{aligned}
$$

ρ	density	e	internal energy
U	velocity	X_k	mass fractions
p	pressure	$\dot{\omega}_k$	X_k production rate
$E = e + U^2/2$	total energy	$\vec{g}$	gravity

with Timmes equation of state:

$$p(\rho, T, X_k) = p_{ele} + p_{rad} + p_{ion}$$

where

$$p_{ele} = \text{fermi}, \; p_{rad} = aT^4/3, \; p_{ion} = \frac{\rho k T}{m_p} \sum_m X_k / A_m$$

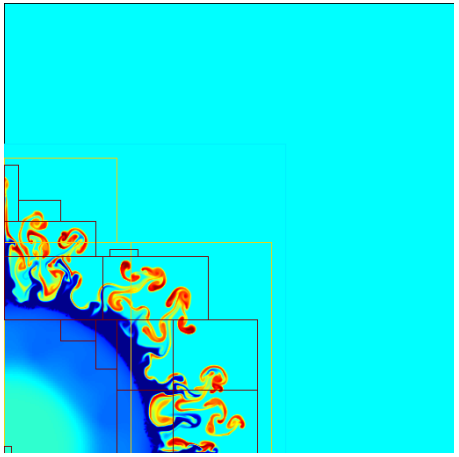
Compressible Formulation

Time-explicit methods for hyperbolic conservation laws with source terms:

$$\mathbf{U}_t + \nabla \cdot \mathbf{F} = \mathbf{S}$$

have the advantages that they are

- easy to program
- easy to parallelize – great weak scaling to 200K cores
- straightforward with AMR (synchronization is explicit as well)



But the time step is the problem – to capture ignition we need to simulate 2 hours, not 2 seconds.

Low Mach Number Approach

We want to develop a model based on separation of scales between fluid motion and acoustic wave propagation.

One approach is based on asymptotic expansion in the Mach number, $M = |U|/c$, which leads to a decomposition of the pressure into thermodynamic and dynamic components:

$$p(\mathbf{x}, t) = p_0(r, t) + p'(\mathbf{x}, t)$$

where $p'/p_0 = O(M^2)$.

- p_0 replaces p in the thermodynamics; p' appears only in the momentum equation,
- Physically: acoustic equilibration is instantaneous; sound waves are “filtered” out
- Mathematically: resulting equation set is no longer strictly hyperbolic; a constraint equation is added to the evolution equations
- Computationally: time step is dictated by fluid velocity, not sound speed.

Criteria for a New Model

We want to eliminate acoustic waves (so they don't limit the time step) but make as few additional limiting assumptions as possible.

New model, in addition to allowing a larger Δt, needs to incorporate

- Buoyancy
- Large variation from background state (or the star will never ignite!)
- Background stratification
- Nonideal equation of state (i.e. not constant γ)
- Reactions and heat release
- Overall expansion of the star

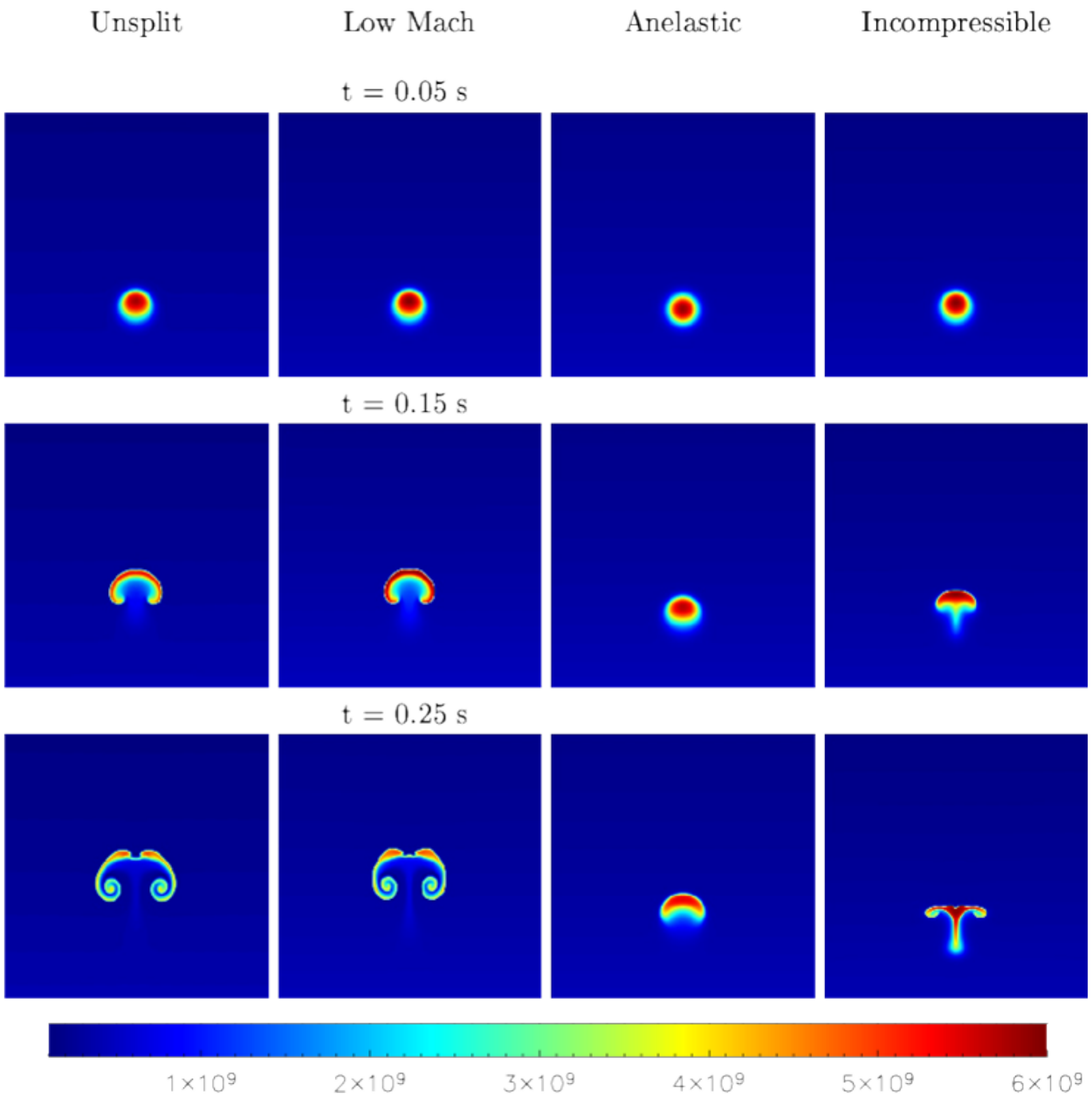
and, in the end, must have lower time-to-solution.

A hierarchy of possible models

- Incompressible $\nabla \cdot U = 0$
  - No compressibility effects
- Anelastic $\nabla \cdot (\rho_0 U) = 0$
  - Compressibility due to static stratified atmosphere
  - Only valid for small thermodynamic perturbations from a static hydrostatic (usually isentropic) background
- Low Mach number combustion $\nabla \cdot U = S$
  - Local compressibility due to heat release and diffusion
  - Large variation in density and temperature allowed
  - No stratification
- Pseudo-incompressible $\nabla \cdot (p_0^{1/\gamma} U) = S$
  - Compressibility due to both background stratification and heat release
  - Static background
  - Ideal gas EOS

None of these quite works.

Buoyant bubble rise



Low Mach Number Model

$$
\begin{aligned}
\frac{\partial(\rho X_k)}{\partial t} &= -\nabla\cdot(U\rho X_k) + \rho\dot{\omega}_k \ , \\
\frac{\partial(\rho h)}{\partial t} &= -\nabla\cdot(U\rho h) + \frac{Dp_0}{Dt} - \sum_k \rho q_k \dot{\omega}_k \ , \\
\frac{\partial U}{\partial t} &= -U\cdot\nabla U - \frac{\beta_0}{\rho}\nabla\left(\frac{p'}{\beta_0}\right) - \frac{(\rho-\rho_0)}{\rho} g\mathbf{e}_r \ , \\
\nabla\cdot(\beta_0 U) &= \beta_0\left(S - \frac{1}{\bar{\Gamma} p_0}\frac{\partial p_0}{\partial t}\right)
\end{aligned}
$$

where, by differentiating the EOS, we can define

$$
S = -\sigma\sum_k \xi_k\dot{\omega}_k + \frac{1}{\rho p_\rho}\sum_k p_{X_k}\dot{\omega}_k
$$

Use average heating to evolve base state.

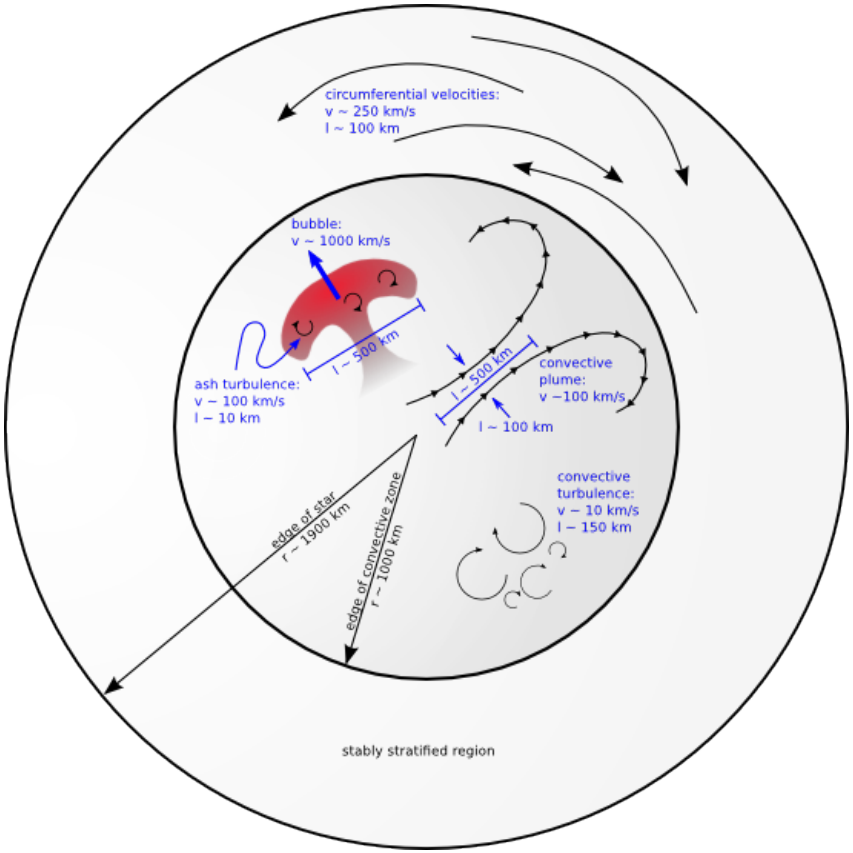
$$
\frac{\partial p_0}{\partial t} = -w_0\frac{\partial p_0}{\partial r} \quad \text{where} \quad \mathrm{w_0(r,t)} = \int_{\mathrm{r_0}}^{\mathrm{r}} \overline{\mathrm{S}}(\mathrm{r}',\mathrm{t})\ \mathrm{dr}'
$$

MAESTRO: Low Mach number method

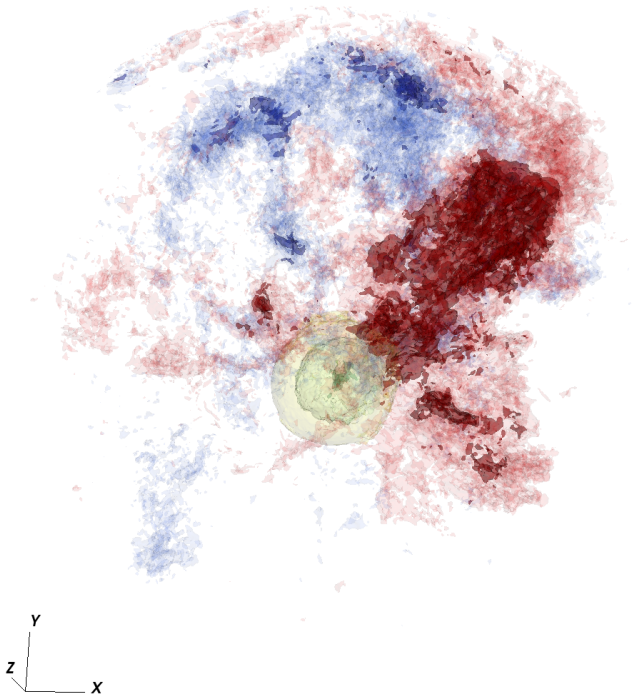
- Numerical approach based on generalized projection method
- 2nd-order accurate fractional step scheme
  - Advance velocity and thermodynamic variables – unsplit Godunov method
  - Project solution back onto constraint – involves solving an elliptic equation for the pressure perturbation (using multigrid)
- Strang splitting (or better coupling) for reaction terms – local implicit ODE integration
- Also need to advance background state
- Built in BoxLib, a reusable software framework for block-structured AMR application codes:
  - supports block-structured AMR
  - scales to 100000's of processors
  - linear solvers for solving elliptic and parabolic equations
  - hybrid MPI / OpenMP
  - modular EOS and reaction networks – "plug 'n play"
- BoxLib, MAESTRO and CASTRO are freely available to all via: https://github.com/BoxLib-Codes/

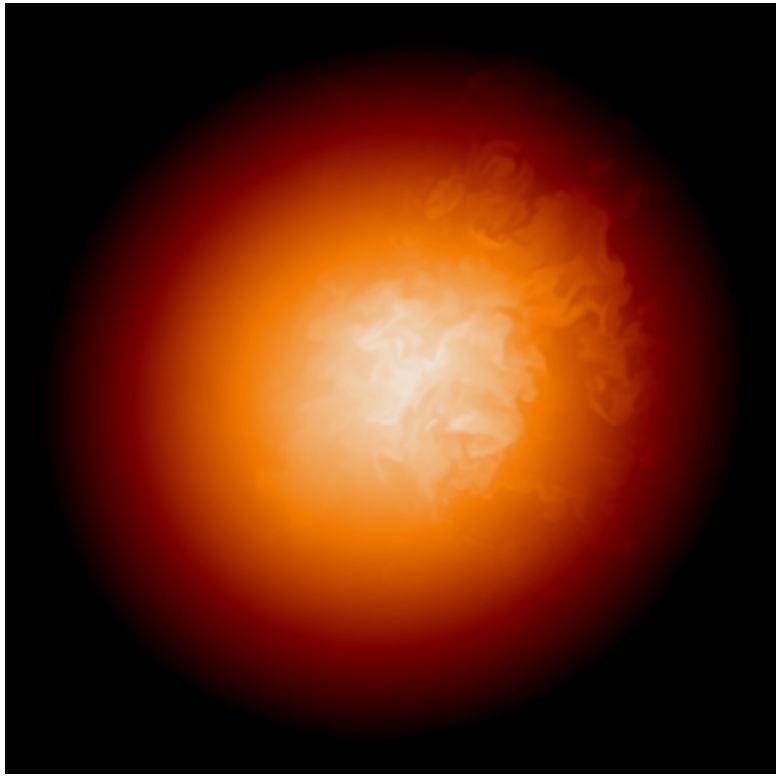
White Dwarf Convection

Using MAESTRO, we can simulate the flow before the star ignites.



White dwarf convection





Convective flow pattern on inner 1000 km of star

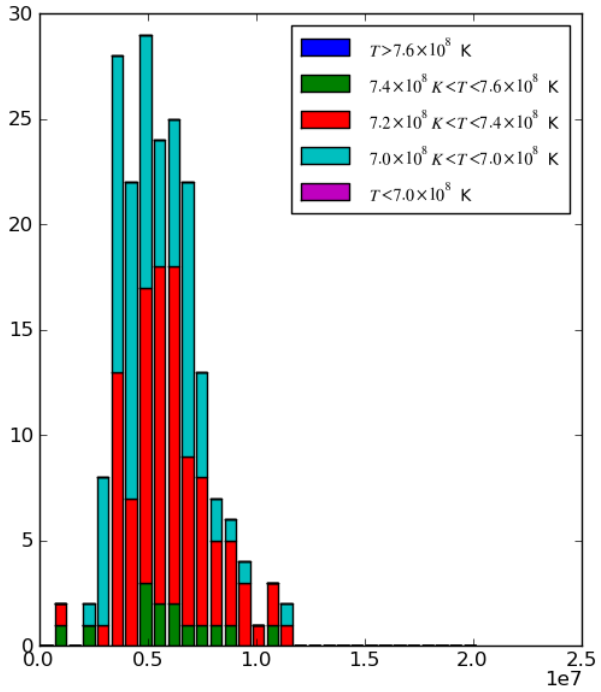
- Red / blue is outward / inward radial velocity
- Yellow / green shows burning rate

Two dimensional slices of temperature a few minutes before ignition

Where does the star ignite?

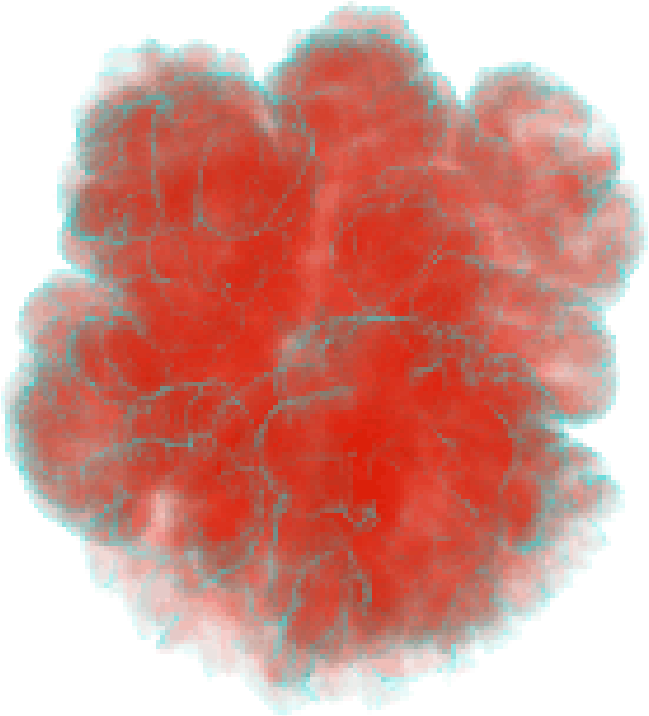
We would like to know the the distribution of ignition sites (note there is not a single ”answer”)

- Monitor peak temperature and radius during simulation
- Filter data
- Bin data to form histogram
- Assume that hot spot locations are “almost” ignitions



Back to the Supernova

Using the output from MAESTRO to define the initial velocity field and possible ignition points, we can initialize a fully compressible simulation to model the explosion itself.



This movie is courtesy of Haitao Ma and Stan Woosley of UC Santa Cruz.

Where Do We Go From Here?

More physics:

- More astrophysical applications – other SN Ia progenitor scenarios, Type 1 X-ray bursts, convection in main sequence stars...
- More combustion applications – new fuels $\Rightarrow$ new flame behavior

More algorithmic features done / in progress (due to larger Δt):

- Low Mach number method for moist atmospheric flows – larger Δt has implications for how you handle the phase change of water – see Duarte et al
- Improved coupling of different processes – e.g. spectral deferred corrections (SDC) instead of Strang splitting for reactions

More algorithmic features in future:

- Add stellar rotation – this makes the base state no longer spherically symmetric
- Long wavelength acoustics – how to add these to low Mach number flows, both in stratified media and combustion
- Are there better ways to use perturbational pressure from low M simulation to feed into compressible solution?