Mathematical analysis of low Mach number limit of some two phase flow models

B. Desjardins

Fondation Mathématique Jacques Hadamard FMJH
CMLA ENS Cachan
61, avenue du Président Wilson
94235 Cachan cedex
France
Email: benoit.desjardins@fondation-hadamard.fr

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Low Mach number flows workshop
Coworkers

A series of joint works, and ongoing work... with:

- Didier Bresch (CNRS, Université de Savoie at Chambéry)
- Jean-Michel Ghidaglia (CMLA ENS Cachan)
- Emmanuel Grenier (UMPA, ENS Lyon)

ANR DYFICOLTI managed by D. LANNES
Outline

1. Two phase one pressure models: some mathematical properties
2. Low Mach number limit for single phase models
3. Generalization to two phase flows.
Complex free surface flows

Interfaces subject to Variable accelerations $g(t)$, shocks, shear
Mixing induced by instabilities

1. Rayleigh-Taylor
2. Richtmyer-Meshkov
3. Kelvin-Helmholtz

…in the fully turbulent regime

See I.W. Kokkinakis, Dimitris Drikakis, David Youngs, R.J.R. Williams,
Two-equation and multi-fluid turbulence models for Rayleigh-Taylor mixing
Rayleigh Taylor Instabilities (2/3)

3 to 1 density ratio \( \text{Atwood} = 0.5 \), \( 3072^3 \) simulation

grid spacing $\Delta \sim \eta$, Kolmogorov scale
Two phase flow models are widely used in industrial framework (turbulent mixing in nuclear industry, reactive flows, propulsion, sprays . . .)
We consider here only barotropic flows (no temperature dependance of the e.o.s. $P^\pm = P^\pm(\rho, \theta)$)

1. Each phase $\pm$ is characterized by its velocity $u^\pm$, density $\rho^\pm$, and equation of state $P^\pm = P(\rho^\pm)$.
2. The volume fraction of phase $\pm$ is denoted $\alpha^\pm$: $\alpha^+ + \alpha^- = 1$.
3. Pressure equilibrium: pressure relaxation effects are assumed to be more rapid than any other physical phenomena: $P^+ = P^-$.
4. Turbulent diffusion of velocity: stress tensor $S^\pm = 2\alpha^\pm \rho^\pm \nu^\pm D(u^\pm)$, where $\nu^\pm > 0$ are constant (zero bulk viscosity).
5. Surface tension expressed as diffuse interface models: $\text{div} K^\pm = \sigma^\pm \alpha^\pm \rho^\pm \nabla \Delta \alpha^\pm \rho^\pm$ where $\sigma^\pm \geq 0$ are constants.
The four equations - one pressure two phase model writes as:

\[
\begin{align*}
\alpha_+ + \alpha_- &= 1, \\
\partial_t (\alpha^+ \rho^+) + \text{div} (\alpha^+ \rho^+ u^+) &= 0, \\
\partial_t (\alpha^- \rho^-) + \text{div} (\alpha^- \rho^- u^-) &= 0, \\
\partial_t (\alpha^+ \rho^+ u^+) + \text{div} (\alpha^+ \rho^+ u^+ \otimes u^+) + \alpha^+ \nabla P &= -D + \alpha^+ \rho^+ g + \text{div} S^+ + \text{div} K^+, \\
\partial_t (\alpha^- \rho^- u^-) + \text{div} (\alpha^- \rho^- u^- \otimes u^-) + \alpha^- \nabla P &= +D + \alpha^- \rho^- g + \text{div} S^- + \text{div} K^-, \\
P &= P_-(\rho_-) = P_+(\rho_+),
\end{align*}
\]

with

\[\mp D = \text{momentum exchange} \sim \mp (u^+ - u^-)|u^+ - u^-|^\beta \text{ or interfacial pressure } \pi \nabla \alpha^+, \]

\[0 \leq \alpha_\pm \leq 1, \quad S^\pm \text{ denotes the stress tensor of phase } \pm.\]
As a consequence of the pressure equilibrium assumption, the system (with right hand side = 0) is non-conservative, and non-hyperbolic if

$$0 \leq |u^+ - u^-| < c_m, \quad \text{(when } \nu^\pm = 0 \text{ and } \sigma^\pm = 0 \text{)}$$

with

$$c_m^2 = c_-^2 c_+^2 \frac{((\alpha^+ \rho^+)^{1/3} + (\alpha^- \rho^-)^{1/3})^3}{\alpha^+ \rho^- c_-^2 + \alpha^- \rho^+ c_+^2}.$$

In general, $c_m$ is large compared to $u^+$ and $u^-$ and therefore flow belongs to non-hyperbolic region.

where $c_{\pm}^2 = P_{\pm}'(\rho_{\pm})$ denotes the sound speed of phase $\pm$.

Properties of the model with zero diffusion, capillary and momentum exchange terms

- Non linear and non conservative equations:
  Existence of global in time weak solutions is open

- Regularizations
  - Hyperbolization with additional physical effects such as interfacial pressure
  - Viscosity and capillarity at high frequency

- What happens when one phase or the two are only slightly compressible?
  \[ \rho^+ \rightarrow \text{constant}, \text{ or } \rho^\pm \rightarrow \text{constant} \]
Physical Models for Two Phase Flows

Hyperbolization: interfacial pressure

Interfacial pressure models $\pi$ are often used as a low frequency stabilization effect

\[ \alpha_+ + \alpha_- = 1, \]
\[ \partial_t (\alpha^+ \rho^+) + \text{div} (\alpha^+ \rho^+ u^+) = 0, \]
\[ \partial_t (\alpha^- \rho^-) + \text{div} (\alpha^- \rho^- u^-) = 0, \]
\[ \partial_t (\alpha^+ \rho^+ u^+) + \text{div} (\alpha^+ \rho^+ u^+ \otimes u^+) + \alpha^+ \nabla P + \pi \nabla \alpha^+ = 0, \]
\[ \partial_t (\alpha^- \rho^- u^-) + \text{div} (\alpha^- \rho^- u^- \otimes u^-) + \alpha^- \nabla P + \pi \nabla \alpha^- = 0, \]
\[ P = P_-(\rho_-) = P_+(\rho_+), \]

with

\[ 0 \leq \alpha_\pm \leq 1. \]
In Cathare Thermohydraulic code, D. Bestion introduced the interfacial pressure

$$\pi = \delta \frac{\alpha^+ \alpha^- \rho^+ \rho^-}{\alpha^+ \rho^- + \alpha^- \rho^+} (u^+ - u^-)^2.$$ 

with $\delta > 1$ to get hyperbolicity for small relative velocity.

If $\rho^+ >> \rho^-$ and $\alpha^- << 1$, hyperbolicity if $|u^+ - u^-|^2 < c^2$. 


With viscosity and capillarity effects, the two phase system rewrites:

\[
\begin{align*}
\alpha^+ + \alpha^- &= 1, \\
\partial_t (\alpha^\pm \rho^\pm) + \text{div}(\alpha^\pm \rho^\pm u^\pm) &= 0, \\
\partial_t (\alpha^\pm \rho^\pm u^\pm) + \text{div}(\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) \\
&\quad + \alpha^\pm \nabla p = \text{div}(\alpha^\pm S^\pm) + \sigma^\pm \alpha^\pm \rho^\pm \nabla \Delta (\alpha^\pm \rho^\pm). \\
\end{align*}
\]

with

\[
S^\pm = 2\mu^\pm D(u^\pm) + \lambda^\pm \text{div} u^\pm \text{Id}
\]

\[
p = p^\pm(\rho^\pm) = A^\pm(\rho^\pm)\gamma^\pm \quad \text{where } \gamma^\pm \text{ are given constants greater than 1}
\]

Assume \( \mu^\pm(\rho^\pm) = \nu^\pm \rho^\pm, \quad \lambda^\pm(\rho^\pm) = 0. \)
Then, using the mathematical structure associated with the additional entropy and additional regularization due to capillary stress tensor,

**Theorem (D. Bresch, B.D., J.M. Ghidaglia, E. Grenier, 2009)**

*Stability of weak solutions in $\mathbb{T}^d$ in dimension $d \geq 1$.*


In dimension $d = 1$, no capillarity effects are required

**Theorem (D. Bresch, X. Huang, J., 2011)**

*Stability of weak solutions in $\mathbb{T}^1$ without capillarity.*

Formally, total energy is conserved:

\[
\sum_{\pm} \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \alpha^\pm \rho^\pm |u^\pm|^2 + \alpha^\pm \rho^\pm e^\pm (\rho^\pm) + \frac{\sigma^\pm}{2} |\nabla (\alpha^\pm \rho^\pm)|^2 \right)
\]

\[+ \sum_{\pm} \int_{\Omega} 2\nu^\pm \alpha^\pm \rho^\pm |D(u^\pm)|^2 = - \sum_{\pm} \int_{\Omega} p \partial_t \alpha^\pm = 0\]

where the internal energy is given by

\[e^\pm (\rho^\pm) = A^\pm \frac{(\rho^\pm)^{\gamma^\pm-1}}{\gamma^\pm - 1}\]
Global weak solutions

**Definition of weak solutions.** We shall say that \((\rho^\pm, \alpha^\pm, u^\pm)\) is a weak solution on \((0, T)\) if the following three conditions are fulfilled:

- The following regularity properties hold
  \[
  \alpha^\pm \rho^\pm |u^\pm|^2 \in L^\infty(0, T; L^1(\Omega)),
  \]
  \[
  \nabla \sqrt{\alpha^\pm \rho^\pm} \in L^\infty(0, T; L^2(\Omega)^3),
  \]
  \[
  \sqrt{\alpha^\pm \rho^\pm} \nabla u^\pm \in L^2((0, T) \times \Omega)^{3 \times 3},
  \]
  \[
  \sqrt{\sigma^\pm \nabla (\alpha^\pm \rho^\pm)} \in L^\infty(0, T; L^2(\Omega)^3),
  \]
  \[
  \sqrt{\sigma^\pm \Delta (\alpha^\pm \rho^\pm)} \in L^2(0, T; L^2(\Omega)),
  \]

with following time continuity properties

\[
\alpha^\pm \rho^\pm \in C([0, T]; H^s(\Omega)), \text{ for all } s < 1/2,
\]
\[
\alpha^\pm \rho^\pm u^\pm \in C([0, T]; H^{-s}(\Omega)^3) \text{ for some positive } s,
\]
Global weak solutions

- the initial conditions $\alpha^\pm \rho^\pm |_{t=0} = R_0^\pm$ and $\alpha^\pm \rho^\pm u^\pm |_{t=0} = m_0^\pm$ hold in $\mathcal{D}'(\Omega)$.

- "Mass" equations hold in $\mathcal{D}'((0,T) \times \Omega)$ and momentum equations multiplied by $\alpha^\pm \rho^\pm$ hold in $\mathcal{D}'((0,T) \times \Omega)^3$: for all $\psi^\pm \in C^\infty([0,T] \times \Omega)^d$ and denoting $R^\pm = \alpha^\pm \rho^\pm$, one has

$$
\int_\Omega R^\pm u^\pm(t,\cdot) \cdot \psi^\pm(t,\cdot) - \int_\Omega R_0^\pm m_0^\pm \cdot \psi(0,\cdot)
$$

$$
= \int_0^T \int_\Omega \left[ D(\psi^\pm) : \left( R^\pm u^\pm \otimes R^\pm u^\pm - 2\nu^\pm R^2 u^\pm D(u^\pm) 
+ \sigma^\pm R^\pm \nabla R^\pm \otimes \nabla R^\pm \right) + \sigma^\pm |\nabla R^\pm|^2 \psi^\pm \cdot \nabla R^\pm - \alpha^\pm R^\pm \psi^\pm \cdot \nabla p 
- R^\pm^2 \text{div } u^\pm (u^\pm \cdot \psi^\pm) - 2\nu^\pm R^\pm \psi^\pm \cdot (D(u^\pm) \cdot \nabla R^\pm) + R^\pm^2 u^\pm \cdot \partial_t \psi^\pm 
- \sigma^\pm \text{div } \psi^\pm \left( \Delta R^\pm^3 / 3 - R^\pm |\nabla R^\pm|^2 \right) \right]
$$
Why a degenerate viscosity may help?

In dimension \( d \geq 1 \), the auxiliary velocity

\[
w^\pm = u^\pm - g^\pm, \text{ where } g^\pm = \nu^\pm \frac{\nabla R^\pm}{R^\pm}
\]

satisfies

\[
\partial_t (R^\pm w_i^\pm) + \text{div}(R^\pm u^\pm w^\pm) + \frac{R^\pm}{\rho^\pm} \partial_i p
\]

\[
= \text{div}(\nu^\pm R^\pm \nabla u_i^\pm) \mp D_i + \sigma^\pm R^\pm \partial_i \Delta R^\pm
\]

(1)

so that

\[
\frac{d}{dt} \int_\Omega \left( R^\pm \frac{|w^\pm|^2}{2} + R^\pm \frac{|g^\pm|^2}{2} + \frac{\sigma^\pm}{2} |\nabla R^\pm|^2 \right)
\]

\[
+ \int_\Omega \left( \nu^\pm R^\pm |\nabla u^\pm|^2 + \sigma^\pm \nu^\pm |\Delta R^\pm|^2 \right)
\]

\[
+ \int_\Omega \frac{R^\pm}{\rho^\pm} \nabla p \cdot w^\pm = 0
\]
Formal energy conservation

Summing up the two phases $\pm$, one gets

$$\sum_{\pm} \nu^\pm \frac{d}{dt} \int_\Omega \left( R^\pm \frac{|w^\pm|^2}{2} + R^\pm \frac{|g^\pm|^2}{2} + R^\pm e^\pm (\rho^\pm) + \frac{\sigma^\pm}{2} |\nabla R^\pm|^2 \right)$$

$$+ \nu^+ \nu^- \sum_{\pm} \int_\Omega \left( R^\pm |\nabla u^\pm|^2 + R^\pm c^\pm 2 \frac{\nabla |\rho^\pm|^2}{\rho^\pm 2} + \sigma^\pm |\Delta R^\pm|^2 \right)$$

$$= - \sum_{\pm} \int_\Omega \nu^\pm p \partial_t \alpha^\pm$$
When one of the two phases becomes incompressible $\rho^+ \to \text{constant}$ and phase $-$ remain compressible
  
  - Regular limit
  - No acoustic waves

\[ p = p^-(\rho^-), \quad \rho^+ = \rho_0^+ \text{ constant} \]

Invariant regions

**Definition**

A closed subset $S \subset \mathbb{R}^n$ is invariant for the local solution $w$ over $[0, T]$ if $w(t, x) \in S$ for all $(t, x) \in [0, T] \times \Omega$ as soon as $w(0, x) \in S$.

**Theorem (J.M. Ghidaglia)**

*For smooth solutions to inviscid equation, the region $\alpha^- \geq 0$ is invariant under the evolution if and only if phase $+$ is compressible.*

Numerical schemes sometimes lead to negative values for $\alpha^-$ and therefore require ”clipping techniques” which consists in modifying (increasing) variable $\alpha^-$ when it gets too small.

However, this procedure destroys mass conservation and very often leads to nonphysical results.
The two phase inviscid model may be rewritten in $w = (u^+, u^-, \alpha, p)$, where $\alpha = \alpha^+ - \alpha^-:$

$$\partial_t u^+ + u^+ \cdot \nabla u^+ + \nabla p/\rho^+ = 0,$$
$$\partial_t u^- + u^- \cdot \nabla u^- + \nabla p/\rho^- = 0,$$
$$\partial_t \alpha + u_{\alpha} \cdot \nabla \alpha + (1 - \alpha^2)F(w, \nabla w) = 0,$$
$$\partial_t p + u_{p} \cdot \nabla p + A \text{div}(\alpha^+ u^+ + \alpha^- u^-) = 0,$$

where $(a, b) \mapsto F(a, b)$ is a smooth function of $a$ and $b$, so that

$$\partial_t w + \sum_{i=1}^{3} M_i(w) \partial_i w = 0,$$

where $M_i$ are smooth functions of $w$. 
Invariant regions: proof

Boundary $\alpha^- = 0$ corresponds to $w_3 = \alpha = 1$ and matrices $w \mapsto M_i(w)$ are smooth functions of $w$.

Let $[0, T]$ be a time interval for which local smooth solutions exist.

From CHUEH, CONWAY, SMOLLER (Springer-Verlag (1983)), $w_3 = 1$ is invariant if and only if $dw_3 = (0, 0, 0, 0, 0, 0, 1, 0)$ is a left eigenvector of $M_i(v^+, v^-, 1, q)$ where $v^\pm$ and $q > 0$ are arbitrary.

- In the case when both phases are compressible

  $$(0, 0, 0, 0, 0, 0, 1, 0) \cdot M_i(v^+, v^-, 1, q) = v_i^- (0, 0, 0, 0, 0, 0, 1, 0).$$

- When phase $+$ is incompressible,

  $$(0, 0, 0, 0, 0, 0, 1, 0) \cdot M_i(v^+, v^-, 1, q) = (2, 2, 2, 0, 0, 0, v_i^+, 0).$$
In the case when the two phases are incompressible (constant densities $\rho^\pm$)

\[
\alpha^- + \alpha^+ = 1,
\]

\[
\partial_t(\alpha^+) + \text{div}(\alpha^+ u^+) = 0,
\]

\[
\partial_t(\alpha^-) + \text{div}(\alpha^- u^-) = 0,
\]

\[
\rho^+(\partial_t(\alpha^+ u^+) + \text{div}(\alpha^+ u^+ \otimes u^+)) + \alpha^+ \nabla P + \pi \nabla \alpha^+ = 0,
\]

\[
\rho^-(\partial_t(\alpha^- u^-) + \text{div}(\alpha^- u^- \otimes u^-)) + \alpha^- \nabla P + \pi \nabla \alpha^- = 0,
\]

with $\rho^-$ and $\rho^+$ constants and $P$ the Lagrangian multiplier associated with the constraint $\alpha^+ + \alpha^- = 1$. 
Incompressible - Incompressible Model

Hyperbolic with Bestion interfacial pressure:

\[ \pi = \delta \frac{\alpha^+ \alpha^- \rho^+ \rho^-}{\alpha^+ \rho^- + \alpha^- \rho^+} (u^+ - u^-)^2 \]

with \( \delta > 1 \).

Remark: The two-layers shallow-water system between rigid lids shares the same form.

In this model, \( \pi = 0 \) and a term \( \text{cst} \nabla \alpha^+ \) appears in the + momentum component.

Model and Theorem

The model (SW) in $\Omega = T^2$ or $R^2$

$$h_t + \text{div} (h v_1) = 0,$$

$$-h_t + \text{div} ((1 - h)v_2) = 0,$$

$$(v_1)_t + (v_1 \cdot \nabla)v_1 + \frac{\rho - 1}{\rho} \nabla h + \frac{1}{\rho} \nabla p = 0,$$

$$(v_2)_t + (v_2 \cdot \nabla)v_2 + \nabla p = 0.$$

**Remark.** Indices 1 and 2 refer to the bottom and top layer respectively. Density of bottom layer $\rho = \rho_1/\rho_2 > 1$, the top one equals 1. The depth of the bottom layer is $h_1 = h$ and top $h_2 = 1 - h$. Gravity $g$ is taken equal to 1.

**Theorem (D. Bresch, M. Renardy, 2011)**

Let $\rho > 1$ and $s > 2$. Assuming that $(h_0, v^0_1, v^0_2) \in (H^s)^5$ with $0 < h_0 < 1$ are such that

$$|v^0_1 - v^0_2|^2 < (\rho - 1)(h_0 + \rho(1 - h_0))/\rho. \quad (2)$$

is satisfied and, moreover, $\text{div} (h_0 v^0_1 + (1 - h_0)v^0_2) = 0$. Then, there exists $T_{\text{max}} > 0$, and a unique maximal solution $(h, v_1, v_2) \in C([0, T_{\text{max}}); (H^s)^5)$ (and a corresponding pressure $p$) to the system (SW), which satisfies the initial condition

$$(h, v_1, v_2)|_{t=0} = (h_0, v^0_1, v^0_2).$$
First result in the non-irrotational case

**Main result:** Local well-posedness under optimal restrictions on the data by rewriting the system in an appropriate form which fits into the abstract theory of T.J.R. HUGHES, T. KATO and J.E. MARSDEN related to second order quasi-linear hyperbolic systems.

**Idea:** Isolate the “essential” part, using the total derivative $\partial_t + \mathbf{V} \cdot \nabla$ operator with $\mathbf{V}$ the weighted average velocity $\mathbf{V} = (\rho(1 - h)\mathbf{v}_1 + h\mathbf{v}_2)/(\rho(1 - h) + h)$. 
Single phase low Mach number limit
Incompressible limit of single phase barotropic flows can be expressed as:

\[
\begin{align*}
\partial_t \rho^\varepsilon + \text{div}(\rho^\varepsilon u^\varepsilon) &= 0 \\
\partial_t (\rho^\varepsilon u^\varepsilon) + \text{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{\nabla P^\varepsilon}{\gamma \varepsilon^2} &= \text{div} S^\varepsilon = \mu \Delta u^\varepsilon + (\lambda + \mu) \nabla \text{div} u^\varepsilon \\
\rho^\varepsilon &= P(\rho) = A \rho^{\gamma}, \\
\varepsilon &= \frac{U}{c} \quad \text{denotes the Mach number, where} \quad c^2 = P'(\rho).
\end{align*}
\]

Formally when \( \varepsilon \to 0 \), \( P^\varepsilon \) (and \( \rho^\varepsilon \)) tends to a constant value \( \overbar{P} \) (and \( \overbar{\rho} = 1 \)), and \( \text{div} \overbar{u} = 0 \). Introducing density fluctuations \( \Psi^\varepsilon = (\rho^\varepsilon - 1)/\varepsilon \), and momentum \( m^\varepsilon = \rho^\varepsilon u^\varepsilon \)

\[
\begin{align*}
\partial_t \Psi^\varepsilon + \frac{\text{div} m^\varepsilon}{\varepsilon} &= 0 \\
\partial_t m^\varepsilon + \frac{\nabla \Psi^\varepsilon}{\varepsilon} &= -\text{div}(u^\varepsilon \otimes m^\varepsilon) + \text{div} S^\varepsilon - \nabla \frac{P(\rho^\varepsilon) - P(1) - P'(1)(\rho^\varepsilon - 1)}{\varepsilon^2}
\end{align*}
\]
In the well prepared case: \( u^\varepsilon_0 \to \bar{u}_0 \) where \( \text{div} \bar{u}_0 = 0 \), \( u^\varepsilon \) converges strongly to \( \bar{u} \) solution of the incompressible Navier Stokes equations

\[
\partial_t \bar{u} + \text{div}(\bar{u} \otimes \bar{u}) + \nabla \pi = \nu \Delta \bar{u}, \quad \text{div} \bar{u} = 0, \quad \bar{u}_{|t=0} = \bar{u}_0.
\]


**Periodic Boundary Conditions**

In the ill prepared case \( u^\varepsilon - \bar{u} - \mathcal{L}(-t/\varepsilon) \bar{v} \to 0 \) in \( L^2((0, T) \times \mathbb{T}^d) \), where \( \bar{v} \) solves the nonlinear acoustic equation where \( t \mapsto \mathcal{L}(t) \) denotes the linear wave operator where \( (\psi, m) = \mathcal{L}(t)(\psi_0, m_0) \) solves

\[
\partial_t \Psi + \text{div} m = 0 \quad \partial_t m + \nabla \Psi = 0 \quad (\Psi, m)_{|t=0} = (\Psi_0, m_0).
\]

i.e. \( (\partial_t^2 - \Delta) \Psi = 0 \).

**Rk:** \( t \mapsto \mathcal{L}(t) \) is an isometry over \( H^s \ \forall s \in \mathbb{R} \).
Whole space
Acoustic waves disperse and tend to zero locally in \((t, x)\): \(u^\varepsilon \to \bar{u}\) in \(L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^d)\),
due to Strichartz estimates:

\[
\|L(t/\varepsilon)\phi\|_{L^q(\mathbb{R}^+; L^p(\mathbb{R}^d))} \leq C_{p,q} \varepsilon^{1/q} \|\phi_0\|_{\dot{H}^s(\mathbb{R}^d)}
\]

where \((p, q) \in (2, \infty) \times (2, \infty)\) and \(\sigma \in (0, \infty)\) such that

\[
\frac{2}{q} = (d - 1) \left(\frac{1}{2} - \frac{1}{p}\right), \quad \sigma q = \frac{d + 1}{d - 1}.
\]

Bounded domain
Most of acoustic waves are damped near the boundary \(u^\varepsilon \to \bar{u}\) in \(L^2((0, T) \times \mathbb{R}^d)\),
unless there are acoustic modes \(\psi \neq 0\) such that

\[
-\Delta \psi = \lambda \psi \quad \text{in} \quad \Omega,
\]

\[
\psi = Cst \quad \text{and} \quad \partial_n \psi = 0 \quad \text{on} \quad \partial\Omega
\]

\[(3) \quad \text{(Schiffer conjecture: } \exists \psi \neq 0 \text{ satisfying (3) } \implies \Omega \text{ is a ball}).
\]
Ongoing work: the incompressible limit in the two phase framework

When one of the two phases becomes incompressible $\rho^+ \to \text{constant}$ and phase $-$ remain compressible

- Regular limit
- No acoustic waves

Double incompressible limit: sound speeds $C^+$ and $C^-$ are of same order of magnitude $\to \infty$.

- No relevant sound speed for two phase mixtures.
- Asymptotic behavior of the spectrum of linearized operator.
The scaled two phase model for the double incompressible limit can be written as

\[ \alpha_+ + \alpha_- = 1, \]
\[ \partial_t (\alpha^+ \rho^+) + \text{div} (\alpha^+ \rho^+ u^+) = 0, \]
\[ \partial_t (\alpha^- \rho^-) + \text{div} (\alpha^- \rho^- u^-) = 0, \]
\[ \partial_t (\alpha^+ \rho^+ u^+) + \text{div} (\alpha^+ \rho^+ u^+ \otimes u^+) + \frac{\alpha^+}{\epsilon^2} \nabla P = -D + \alpha^+ \rho^+ g + \text{div} S^+, \]
\[ \partial_t (\alpha^- \rho^- u^-) + \text{div} (\alpha^- \rho^- u^- \otimes u^-) + \frac{\alpha^-}{\epsilon^2} \nabla P = +D + \alpha^- \rho^- g + \text{div} S^-, \]
\[ P = P_-(\rho^-) = P_+(\rho^+), \]
Two phase low Mach limit: reformulation of the system

The two phase system can be rewritten in terms of \((\alpha^+, P, v_r, U)\) variables

\[
\partial_t \alpha^+ + V_\rho \cdot \nabla \alpha^+ + B_\rho \text{ div } v_r + \frac{B_\rho}{P} \left( \frac{\rho^+}{\gamma^+} - \frac{\rho^-}{\gamma^-} \right) \left( \hat{V}_\rho \cdot \nabla P + A_\gamma P \text{ div } U \right) = 0 \tag{4}
\]

\[
\partial_t P + \hat{V}_\gamma \cdot \nabla P + PA_\gamma \text{ div } U = 0 \tag{5}
\]

where

\[
V_\rho = \frac{\rho^+ u^+}{\alpha^+} + \frac{\rho^- u^-}{\alpha^-}, \quad \hat{V}_\rho = \frac{\alpha^+ u^+}{\rho^+} + \frac{\alpha^- u^-}{\rho^-}, \quad A_\rho = \frac{1}{\frac{\alpha^+}{\rho^+} + \frac{\alpha^-}{\rho^-}}, \quad B_\rho = \frac{1}{\frac{\rho^+}{\alpha^+} + \frac{\rho^-}{\alpha^-}},
\]

\[
V_\gamma = \frac{\gamma^+ u^+}{\alpha^+} + \frac{\gamma^- u^-}{\alpha^-}, \quad \hat{V}_\gamma = \frac{\alpha^+ u^+}{\gamma^+} + \frac{\alpha^- u^-}{\gamma^-}, \quad A_\gamma = \frac{1}{\frac{\alpha^+}{\gamma^+} + \frac{\alpha^-}{\gamma^-}}, \quad B_\gamma = \frac{1}{\frac{\gamma^+}{\alpha^+} + \frac{\gamma^-}{\alpha^-}},
\]

\[
U = \alpha^+ u^+ + \alpha^- u^-, \quad v_r = \rho^+ u^+ - \rho^- u^-, \quad u_r = u^+ - u^-, \quad \text{and} \quad K = \alpha^+ \rho^- + \alpha^- \rho^+.
\]
Reformulation of the two phase system

\[ \partial_t v_r + V_\rho \cdot \nabla v_r + A_\gamma \left( \frac{\rho^+ u^+}{\gamma^+} - \frac{\rho^- u^-}{\gamma^-} \right) \text{div} U + A_\rho u_r \cdot \nabla U - A_\rho u_r (u_r \cdot \nabla \alpha^+) \\
+ \left( \frac{\rho^+ u^+}{\gamma^+} - \frac{\rho^- u^-}{\gamma^-} \right) (\hat{V}_\gamma - V_\rho) \cdot \frac{\nabla P}{P} - \pi \left( \frac{1}{\alpha^+} + \frac{1}{\alpha^-} \right) \nabla \alpha^+ = 0. \tag{6} \]

and

\[ \partial_t U + \hat{V}_\rho \cdot \nabla U + \frac{1}{A_\rho \varepsilon^2} \nabla P + B_\rho u_r \text{div} v_r + u_r B_\rho \left( \frac{\rho^+}{\gamma^+} - \frac{\rho^-}{\gamma^-} \right) A_\gamma \text{div} U \\
+ B_\rho u_r \cdot \nabla v_r + u_r (V_\rho - \hat{V}_\rho) \cdot \nabla \alpha^+ - B_\rho^2 u_r \left( u_r \cdot \frac{\nabla P}{P} \right) \left( \frac{\rho^+}{\alpha^+ \gamma^+} + \frac{\rho^-}{\alpha^- \gamma^-} \right) \\
+ \pi \left( \frac{1}{\rho^-} - \frac{1}{\rho^+} \right) \nabla \alpha^+ + U \left( \frac{\alpha^+ u^+ \rho^-}{\gamma^-} + \frac{\alpha^- u^- \rho^+}{\gamma^+} \right) \cdot \frac{\nabla P}{PK} \\
- \left( \frac{\alpha^+ u^+ \rho^-}{\gamma^-} + \frac{\alpha^- u^- \rho^+}{\gamma^+} \right) U \cdot \frac{\nabla P}{PK} = 0. \tag{7} \]
First step: uniform local existence

Methodology "à la Métivier Schochet" to prove local existence over an interval \([0, T]\) independent of \(\varepsilon\)

\[
\begin{align*}
    a(\partial_t q + v \cdot \nabla q) + \frac{1}{\varepsilon} \text{div}_x u &= 0, \\
    b(\partial_t m + v \cdot \nabla m) + \frac{1}{\varepsilon} \nabla \psi &= 0
\end{align*}
\]

with \(a(t, x)\) and \(b(t, x)\) known and positive, and

\[
q = \frac{1}{\varepsilon} Q(t, x, \varepsilon \psi), \quad m = \mu(t, x) u, \quad v = V(t, x, u, q),
\]

where \(Q, \mu\) and \(V\) are smooth functions of their arguments with \(Q(t, x, 0) = 0\), \(\partial_\theta Q > 0\) and \(\mu > 0\). Note that \(q\) is not singular in \(\varepsilon\) and satisfies

\[
q = Q_1(t, x, \varepsilon \psi) \psi, \quad \text{with} \quad Q_1 > 0.
\]

\(a\) and \(b\) depend on functions satisfying a non singular equation (like \(\alpha^+\) and \(\nu_r\)).

**Theorem** \(\exists T > 0, \varepsilon_0 > 0\) such that

\[
\sup_{t \in [0, T], \varepsilon < \varepsilon_0} \| (u^\varepsilon(t), \psi^\varepsilon(t)) \|_{H^s} \leq C_T \| (u_0^\varepsilon, \psi_0^\varepsilon) \|_{H^s}
\]
Asymptotic behavior of solutions depend on averaging phenomena in variable $t/\varepsilon$, close to what happens in systems like

$$
\begin{align*}
\varepsilon \dot{\varphi} &= \omega(I) + \varepsilon g(I, \varphi), & \varphi(0) = \varphi^* \\
\dot{I} &= f(I, \varphi), & I(0) = I^*
\end{align*}
$$

Under generic non resonance assumptions, for a.e. initial data $(\varphi^*, I^*)$, $I^\varepsilon$ converges to $I$ such that

$$
\dot{I} = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} f(I, \varphi) d\varphi, & I(0) = I^*.
$$

Reduction of (4)(5)(6)(7) to (11) requires constant multiplicity of the spectrum of the linearized operator.
Second step: analysis of the spectrum

Detailed description of the spectrum of system (4)(5)(6)(7) is needed to complete the analysis.

Eigenvalues are the roots of the polynomial in $X$

$$P(X) = (X - \lambda)^2(X + \lambda)^2 - K_1(X - \lambda)^2 - K_2(X + \lambda)^2 + K_3$$

with

$$K_1 = \alpha^- \rho^+ - \alpha^+ \pi/C^2,$$
$$K_2 = \alpha^+ \rho^- - \alpha^- \pi/C^2,$$
$$K_3 = -\pi/\gamma^2,$$
$$\lambda = u_r/2\gamma$$
$$\gamma^2 = C^2/(\alpha^- \rho^+ + \alpha^+ \rho^-),$$

Questions currently being investigated
- Resonances ? Eigenvalue crossings ?
- Local existence in $[0, T]$ uniformly in $\varepsilon$, and convergence to the incompressible - incompressible system.
- Wave dispersion / damping depending on domain geometry ?