Automorphisms of p-completed classifying spaces of groups of Lie type

Bob Oliver

For any space X, let $\mathrm{Out}(X)$ be the group of homotopy classes of self homotopy equivalences of X. If G is a finite group, then the natural homomorphism $\mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG)$ is an isomorphism: this is an easy consequence of elementary obstruction theory. However, if p is a prime and $BG^\wedge_p$ denotes the Bousfield-Kan p-completion of BG, then $\mathrm{Out}(BG^\wedge_p)$ can be very different from $\mathrm{Out}(BG) \cong \mathrm{Out}(G)$.

In joint work with Carles Broto and Jesper Møller, we were surprised to find that when G is a finite simple group of Lie type, there is still a close relation between these two groups. If p is the defining characteristic of G, then $\mathrm{Out}(BG^\wedge_p) \cong \mathrm{Out}(G)$ (the natural homomorphism is an isomorphism) with just one exception: $G = SL_3(2)$. If p is different from the defining characteristic, then the natural homomorphism is in general neither surjective nor injective, but there is always another finite group G^* such that $BG^{*\wedge}_p \simeq BG^\wedge_p$, and the natural map $\mathrm{Out}(G^*) \longrightarrow \mathrm{Out}(BG^{*\wedge}_p)$ is split surjective.

The methods used to prove this are partly geometric and partly algebraic.