Constructive Matrix Theory for Hermitian Higher Order Interaction

Vincent Rivasseau

Laboratoire de Physique des 2 Infinis Irène Joliot-Curie
Université Paris-Saclay

Seminar "Tensor Journal Club", June 24, 2020

- Joint work with Thomas Krajewski and Vasily Sazonov, arXiv:1910.13261
- Constructive *loop vertex expansion* for stable matrix models with (single trace) interactions of arbitrarily high even order in the Hermitian case.
- We prove analyticity in the coupling constant of the free energy for such models in a domain *uniform in the size N of the matrix*.
- It relies on a new and simpler method which can also be applied in the general case of non-Hermitian matrices (which was earlier treated by the same authors, arXiv:1712.05670).

- $d\mu(H)$ GUE measure with iid covariance $1/N$

$$d\mu(H) = \frac{1}{\pi^{N^2}} e^{-N \mathrm{Tr} H^2} dH,$$

$$dH = \prod_i dH_{ii} \prod_{i<j} dH_{ij} d\bar{H}_{ij}.$$

- The H_{2p} model is defined with an action

$$S(\lambda, H) := \lambda \mathrm{Tr} H^{2p}, \quad p \geq 3.$$

The partition function and free energy of the model

$$Z(\lambda, N) := \int d\mu(H)\, e^{-NS(\lambda,H)}, \tag{1}$$

$$F(\lambda, N) := \frac{1}{N^2} \log Z(\lambda, N). \tag{2}$$

Write

$$K := H\sqrt{1+\lambda H^{2p-2}}, \quad K^2 = H^2 + \lambda H^{2p}, \tag{3}$$

and put $T := \frac{H^2}{K^2}$. The Fuss-Catalan equation is:

$$zT^p(z) - T(z) + 1 = 0\,, \tag{4}$$

with $z := -\lambda K^{2p-2}$. The change of variables inverts to $H(K) := K\sqrt{T(z)}$.

Let us define

$$f_\lambda(u) := \sqrt{T(-\lambda u^{2p-2})}, \quad h_\lambda(u) := u f_\lambda(u), \quad k_\lambda(v) := v\sqrt{1+\lambda v^{2p-2}} \tag{5}$$

$$h_\lambda \circ k_\lambda(z) = k_\lambda \circ h_\lambda(z) = z \tag{6}$$

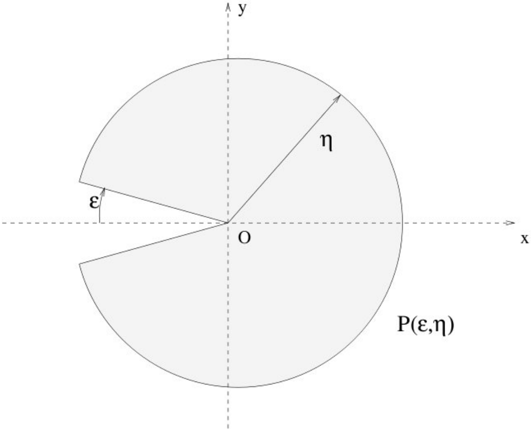
The *Jacobian* of the change of variables produces a new non-polynomial interaction

$$\left|\frac{\delta H}{\delta K}\right| = \left|\det \frac{H\otimes \mathbf{1} - \mathbf{1}\otimes H}{K\otimes \mathbf{1} - \mathbf{1}\otimes K}\right| . \tag{7}$$

Theorem

For any $\varepsilon > 0$ *there exists* η *small enough such that the expansion is absolutely convergent and defines an analytic function of* λ*, uniformly bounded in* N*, in the uniform in* N *"pacman domain"*

$$P(\varepsilon, \eta) := \{0 < |\lambda| < \eta, |\arg \lambda| < \pi - \varepsilon\}. \tag{8}$$



A pacman domain with radius $\leq \eta$ and angular size $\leq 2\pi - 2\varepsilon$.

We first expand the partition function as

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\mu(K) \prod_{a=1}^{n} \mathcal{S}(\lambda, K_a) . \quad (9)$$

Then we apply the BKAR formula. The BKAR formula is a Taylor expansion formula with integral remainder in several variables. The result is a sum over the set $\mathfrak{F}_n$ of forests $\mathcal{F}$ on n labeled vertices

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F} \in \mathfrak{F}_n} \int dw_{\mathcal{F}} \, \partial_{\mathcal{F}} \mathcal{Z}_n \Big|_{x_{ab} = x_{ab}^{\mathcal{F}}(w)}, \quad (10)$$

$$\int dw_{\mathcal{F}} := \prod_{(a,b) \in \mathcal{F}} \int_0^1 dw_{ab} \, , \quad \partial_{\mathcal{F}} := \prod_{(a,b) \in \mathcal{F}} \frac{\partial}{\partial x_{ab}} \, , \quad (11)$$

$$\mathcal{Z}_n := \int d\mu_{C(x)}(\{K\}) \prod_{a=1}^{n} \mathcal{S}(\lambda, K_a), \quad (12)$$

$$x_{ab}^{\mathcal{F}} := \begin{cases} \min_{(k,l) \in P^{\mathcal{F}}_{a \leftrightarrow b}} w_{kl} & \text{if } P^{\mathcal{F}}_{a \leftrightarrow b} \text{ exists} \, , \\ 0 & \text{if } P^{\mathcal{F}}_{a \leftrightarrow b} \text{ does not exist} \, . \end{cases} \quad (13)$$

An example of the BKAR forest formula

The formidable-looking formula is in fact quite simple.

There is *a BKAR formula for each complete graph*:

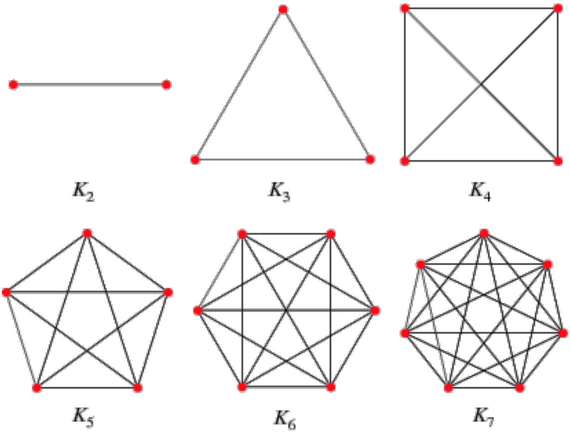


Figure: The list of complete graphs up to order 7

For $n = 2$, there are *two forests* (including the empty one), and the BKAR forest formula is simply

$$F(1) = F(0) + \int_0^1 dw \; F'(w). \tag{14}$$

We recognize the Taylor formula at order one with integral remainder!

An example of the BKAR forest formula

For $n = 3$ there are seven forests. The formula decomposes into

- the empty forest, which is always associated to $0, 0, ..., 0$,
- three singleton forests, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ with a single parameter,
- three doubleton forests $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ with two parameters. This is where the *min* formula appears for the first time.

$$
\begin{aligned}
F(1,1,1) &= F(0,0,0) + \int_0^1 dw_1 \; \partial_1 F(w_1,0,0) + \int_0^1 dw_2 \; \partial_2 F(0,w_2,0) \\
&+ \int_0^1 dw_3 \; \partial_3 F(0,0,w_3) + \int_0^1 \int_0^1 dw_1 dw_2 \; \partial^2_{12} F(w_1,w_2,\min(w_1,w_2)) \\
&+ \int_0^1 \int_0^1 dw_1 dw_3 \; \partial^2_{13} F(w_1,\min(w_1,w_3),w_3) \\
&+ \int_0^1 \int_0^1 dw_2 dw_3 \; \partial^2_{23} F(\min(w_2,w_3),w_2,w_3). \qquad (15)
\end{aligned}
$$

Introducing the condensed notations

$$d\mu = d\mu_{C(x)}(\{K\}), \quad \partial_{\mathcal{F}}^{K} = \prod_{(a,b)\in\mathcal{F}} \mathrm{Tr}\Big[\frac{\partial}{\partial K_a^\dagger}\frac{\partial}{\partial K_b}\Big], \quad \mathcal{S}_n = \prod_{a=1}^{n} \mathcal{S}(\lambda, K_a), \quad (16)$$

we obtain

$$Z(\lambda, N) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}\in\mathfrak{F}_n} A_{\mathfrak{F}} \quad (17)$$

$$A_{\mathfrak{F}} = N^{-|\mathcal{F}|} \int dw_{\mathcal{F}} \int d\mu \, \partial_{\mathcal{F}}^{K} \mathcal{S}_n \Big|_{x_{ab}=x_{ab}^{\mathcal{F}}(w)}. \quad (18)$$

The good thing is that the free energy $F(\lambda, N)$ is computed by *the same sum of the same amplitude* but made of *spanning trees* $\mathfrak{T}_n$

$$F(\lambda, N) = \frac{1}{N^2}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\in\mathfrak{T}_n} A_{\mathcal{T}}, \qquad (19)$$

The statement "if the partition function is made of some discrete objects, the logarithm is made *of same objects but restricted to connected case*" is true for a much wider class than the *class of graphs* or the *class of forests*. It pertains to the class of *combinatoric species* defined by André Joyal and developed by Canadians mathematicians François Bergeron, Gilbert Labelle and Pierre Leroux.

In essence it is an abstract, systematic method for counting discrete structures, made of graphs, of permutations, of matroids ...

- The constructive theory of matrices is the initial application of the LVE (R, arXiv 2007...).
- For ϕ^4 models, any Schwinger function S is expressed in a constructive way as simply a rearrangement of the perturbative series *in the intermediate expansion* $\sqrt{\lambda}\phi^2\sigma$

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G,T) A_G = \sum_T A_T, \quad A_T = \sum_{G \supset T} w(G,T) A_G. \quad (20)$$

with

$$\sum_T |A_T| < +\infty \ , \qquad (21)$$

where the tree weights $w(G,T)$ are defined by the *percentage of the Hepp's sectors from which the Kruskal tree is leading* (R, Tanasa, Zhituo...).

The constructive theory of *tensor models* bears on LVE still further (Delepouve, Herbin, Gurau, Lahoche, Lionni, Magnen, Noui, R, Smerlak, Tamaazousti, Vignes-Tourneret...).

We *fix* the tree (with at least $n \geq 2$ nodes, the case $n = 1$ requiring a special treatment).

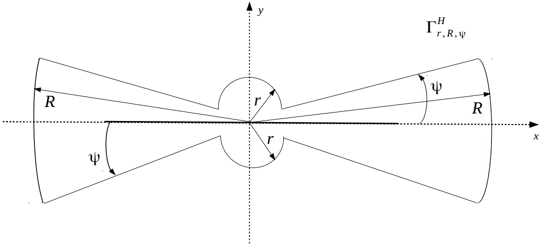
We put $g(u) = h(u) - u$. Notice that $g_\lambda(u)$ vanishes at $\lambda = 0$, so that:

$$g_\lambda(u) = \int_0^\lambda dt \partial_t g_t(u) = -\frac{1}{2} \int_0^\lambda dt \ u^{2p-1} \frac{T'(u)}{T(u)} f_t(u). \tag{22}$$

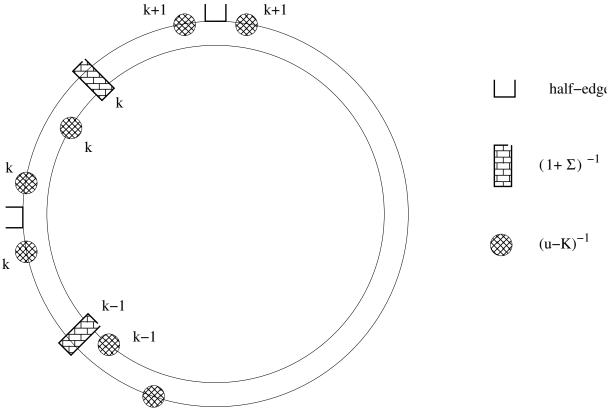
We use factorization through holomorphic calculus

$$f(H) = \oint_{\Gamma} dv \frac{f(v)}{v - H}$$

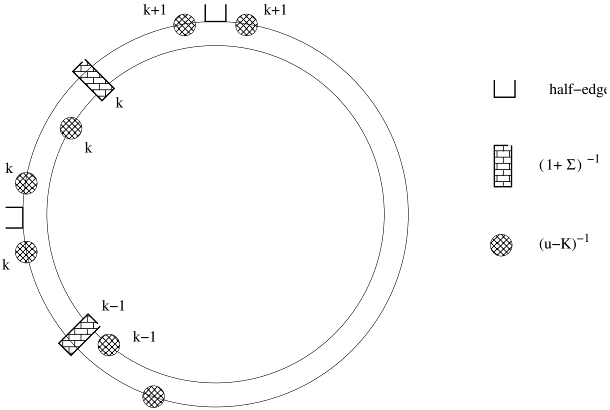
provided the contour Γ encloses the full spectrum of H.



Here are shown a keyhole contour Γ encircling the spectrum of H, which, for H Hermitian, lies on a real axis segment like the one shown in boldface.



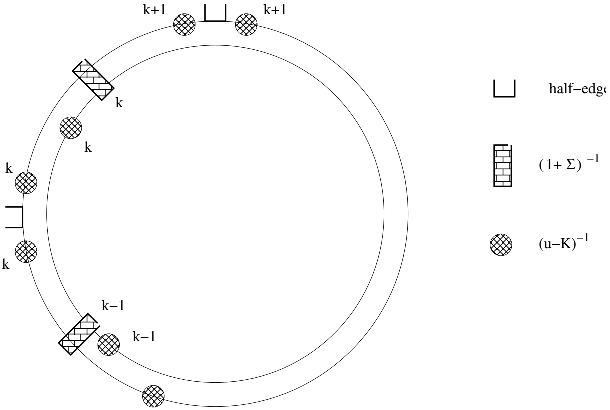
Here is a picture of a vertex with some of its corners operators, noted $O^{c_k}(u_k, u_{k+1})$. There is only one operator K per vertex (this is due to the fact that the formula BKAR depends only on a vertex). Therefore there is *only one eigenvector base*, $e_a \otimes e_b$, since the two borders factorize independently.



The operator $O^{c_k}(u_k, u_{k+1})$ is *diagonal on the basis* $e_a \otimes e_b$, with eigenvalues

$$O^{c_k}_{ab}(u_k, u_{k+1}) = \left[\mathbf{1}_\otimes + \Sigma\right]^{-1}_{ab} \Big[\frac{1}{u_k - \mu_a} \frac{1}{u_{k+1} - \mu_a} \otimes \frac{1}{u_k - \mu_b} + \frac{1}{u_k - \mu_a} \otimes \frac{1}{u_k - \mu_b} \frac{1}{u_{k+1} - \mu_b} \Big]. \qquad (23)$$

with the two tensors terms being symmetric of each other.



Calling $\nu_a = h_\lambda(\mu_a)$, Σ is defined by

$$(\mathbf{1}_\otimes + \Sigma)^{-1}_{ab} = \frac{k_\lambda(\nu_a) - k_\lambda(\nu_b)}{\nu_a - \nu_b}. \qquad (24)$$

The label k indicates the corresponding contour variable. The upper left corner between the two half-edges indicating by the $\sqcup$ symbols contains three $\frac{1}{u-K}$ operators with indices k, k and $k+1$.

The following lemmas are a bit technical; I could give some explanations during the questions.

The theorem depends on a key lemma:

Lemma (1)

On the contour Γ we have the bound

$$|g_\lambda(u)| \le O(1)|\lambda|^{\frac{1}{4p^2}}|u|^{1+\frac{1}{2p}-\frac{1}{2p^2}}. \tag{25}$$

The next lemma bounds $(\mathbf{1}_\otimes + \Sigma)^{-1}_{ab}$

Lemma (2)

For complex λ such that $|arg(\lambda)| \le \pi - \varepsilon$ there exists some constant $O(1)$ such that

$$|(\mathbf{1}_\otimes + \Sigma)^{-1}_{ab}| \quad \le \quad O(1)\sup\{1, \Lambda_{ab}\} \tag{26}$$

$$\Lambda_{ab} \quad := \quad |\lambda|^{\frac{1}{2p}} \sup\{|\mu_a|, |\mu_b|\}^{1-\frac{1}{p}}. \tag{27}$$

Lemma (3)

For complex λ such that $|arg(\lambda)| \leq \pi - \varepsilon$

$$\|O^{c_k}(u_k, u_{k+1})\| \leq O(1) \frac{1}{(1+|u_k|)^{1+\frac{1}{p}}} \frac{1}{1+|u_{k+1}|}. \tag{28}$$

Lemma (4)

There exists some constant $O(1)$ such that

$$|A_{\mathcal{T}}| \leq [O(1)]^n \prod_v \prod_{k=1}^{m(v)} \oint_\Gamma |g_\lambda(u_k)| \left[\frac{1}{1+|u_k|}\right]^{2+\frac{1}{p}} du_k. \tag{29}$$

Lemma (5)

There exists some constant $O(1)$ such that

$$\oint_\Gamma |g_\lambda(u)| \left[\frac{1}{1+|u|}\right]^{2+\frac{1}{p}} du \leq O(1)|\lambda|^{\frac{1}{4p^2}}. \tag{30}$$

With the five preceding lemmas the proof is now complete.

- The LVE expresses the free energy *constructively*: it has the great advantage of being a *convergent sum*, while perturbative techniques in Feyman graphs *fail in this respect*.
- The trick of involving a parameter for *each corner of each vertex* is the key of this theorem. It simplifies also the non-Hermitian complex case or the case of symmetric or symplectic matrices.
- As was said, the case $n = 1$ required a special treatment. It is actually quite subtile; it involves five terms each with a different structure. I could again give some explanations during the questions.

Thank you for your attention !

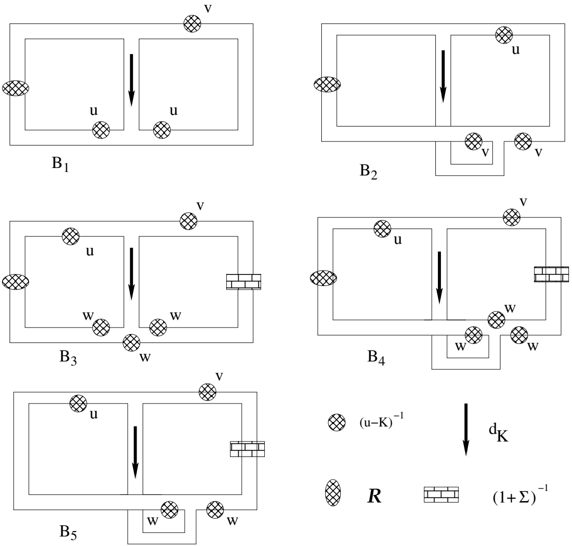


Figure: The five terms B_1, B_2, B_3, B_4 and B_5. The arrow indicates the action of the ∂_K matrix derivative.

We define $\mathbf{R}_{diag}$ and $(\mathbf{1}_\otimes + \Sigma)^{-1}_{diag}$ as the diagonal "single thread" N by N matrix with eigenvalue $\mathbf{R}^a_{diag} := \mathbf{R}_{aa}$ or $(\mathbf{1}_\otimes + \Sigma)^{-1}_{aa}$ on e_a, and perform a careful analysis of the tensor threads involved, hopefully helped by Figure 2. It gives

$$A_2 = N^{-3} \int d\mu \int_0^\lambda dt \oint_\Gamma du \oint_{\Gamma'} dv \, \partial_t \phi(t, u, v) \big[B_1 + B_2 + B_3 + B_4 + B_5 \big], \quad (31)$$

where the first two terms are obtained when ∂_K hits $\left[\frac{1}{u-K} \otimes \frac{1}{v-K}\right]$, giving

$$B_1 = \mathrm{Tr}_{\otimes^3} \Big[[\mathbf{R} \otimes 1] \big[\frac{1}{u-K} \otimes \frac{1}{v-K} \otimes \frac{1}{u-K} \big] \Big], \quad (32)$$

$$B_2 = \mathrm{Tr} \, \mathbf{R}_{diag} \frac{1}{(u-K)(v-K)^2}, \quad (33)$$

The last three terms B_3, B_4 and B_5 come from ∂_K hitting $\mathbf{R}$

$$\partial_K \mathbf{R} = \partial_K (1 \otimes f)(1 + \Sigma)^{-1} \quad (34)$$

$$= -\mathbf{R} \, [\partial_K \Sigma] \, (1 + \Sigma)^{-1} + [1 \otimes \partial_K f](1 + \Sigma)^{-1}. \quad (35)$$

We can use the rather standard estimates on T_p and $E_p(z) = \frac{T_p'}{T_p}(z)$. In particular it is proven there that in a domain avoiding a small angular opening ε around the cut of T_p we have

$$T_p(z) \le (1+|z|)^{-\frac{1}{p}}, \quad E_p(z) \le \frac{O(1)}{(1+|z|)}. \tag{36}$$

In our case this means that on our contour Γ, for any $0 < \delta < 1$ there is a constant C_δ such that

$$|e_t f_t(u)| \le \frac{C_\delta}{[|t||u|^{2p-2}]^{(1+\frac{1}{2p})(1-\delta)}}, \tag{37}$$

Choosing $\delta = \frac{1}{2p}$ gives (25).

Calling $\nu_a = h_\lambda(\mu_a)$, (23) means that

$$(\mathbf{1}_\otimes + \Sigma)^{-1}_{ab} = \frac{k_\lambda(\nu_a) - k_\lambda(\nu_b)}{\nu_a - \nu_b}. \tag{38}$$

hence it is bounded by $\sup_{\nu \in [\nu_a, \nu_b]} |k'_\lambda(\nu)|$ where the sup is taken along the $[\nu_a, \nu_b]$ segment. k'_λ can be explicitly computed and from the large z behavior of the function $T(z) \sim z^{-1/p}$ derived from its functional equation (4). The bound follows easily on the pacman domain.

Suppose eg $\Lambda_{ab} = |\lambda^{\frac{1}{2p}}||\mu_a|^{1-\frac{1}{p}}$. We bound the $\frac{1}{u_k - \mu_a}$ factor in (23) as

$$\left|\frac{1}{u_k - \mu_a}\right| \leq \left[\frac{1}{1+|u_k|}\right]^{1/p} \left[\frac{1}{1+|\mu_a|}\right]^{1-1/p}. \tag{39}$$

Combining with (26) leads to

$$|O_{ab}^{c_k}(u_k, u_{k+1})| \leq O(1) \left[\frac{1}{1+|u_k|}\right]^{1+1/p} \frac{1}{1+|u_{k+1}|}. \tag{40}$$

The other cases $\Lambda_{ab} = |\lambda^{\frac{1}{2p}}||\mu_b|^{1-\frac{1}{p}}$ or $\Lambda_{ab} = 1$ are obviously similar. Since the bound (40) is independent of a and b, it implies (28), with $C_{\varepsilon,2} = 2C_\varepsilon [C_{\varepsilon,1}]^3$.

We bound recursively all tree traces. The simplest way to understand how it works is to start from a leaf f, which has $r = m = 1$. The associated operator is therefore a single contour-corner operator O^c whose norm, by (28), is bounded by $O(1)[\frac{1}{1+|u_k|}]^{2+\frac{1}{p}}$. The amplitude for $A_{\mathcal{T}}$ contains a partial trace on one $\mathcal{H}$ factor of the tensor product $\mathcal{H} \otimes \mathcal{H}$ of the leaf vertex, leading to a simpler operator on $\mathcal{H}$ only, with norm bounded by $NO(1)[\frac{1}{1+|u_k|}]^{2+\frac{1}{p}}$. After gluing this factor between the two appropriate corners in the parent vertex $v(f)$ we can find a new leaf and iterate. This leads to the bound. Indeed this induction collects exactly $n+1$ factors N (since the last vertex of the tree brings two such factors). This exactly compensates with the N^{-n-1} factor in (19). Finally the $\int dw_{\mathcal{T}} \int d\mu_{C(x)}(\{K\})$ integrals are *normalized* so do not add anything to the bounds.

Inserting (25) proves (30) since the integral $\oint_\Gamma \frac{|u|^{1+\frac{1}{2p}-\frac{1}{2p^2}}}{(1+|u|)^{2+\frac{1}{p}}} du$ is absolutely convergent and bounded by a constant at fixed p.