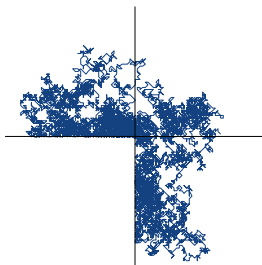


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Discrete harmonic functions in the three-quarter plane

Amélie Trotignon

- Institut für Algebra -
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Content

1. Introduction
2. Previous results in the quarter plane
3. Results in the three-quarter plane
4. Further objectives and perspectives

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Continuous harmonic functions

A continuous harmonic function annihilates the standard Laplacian Δf

if f is a harmonic function on an open set $\mathcal{U} \subset \mathbb{R}^2$, then f is twice differentiable and

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

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$f(x, y) = \ln(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 \setminus (0, 0)$.

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Applications and properties

- analysis – resolution of partial differential equations
- infinitely differentiable in open sets
- maximum principle; mean value properties; Harnack's inequalities (in the case of non-negative harmonic functions)

Discrete harmonic functions

Simplest discrete Laplacian in dimension 2

$$\Delta f(x, y) = f(x - 1, y) + f(x + 1, y) + f(x, y - 1) + f(x, y + 1) - 4f(x, y).$$

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Applications and properties

- discrete complex analysis
- probability of absorption at absorbing states of Markov chains or the Ising model
- satisfy multivariate linear recurrence relations (ubiquitous in combinatorics)

Gambler Ruin Problem and Harmonic Functions

S_n denotes the fortune at time n , with initial fortune of i gold coins.

At every step, the gambler bets 1 gold coin, and

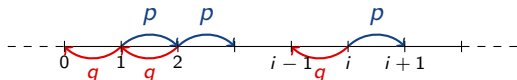
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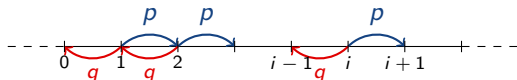
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Markov property

$$h_i = \mathbb{P}_i[\exists n \geq 0 : S_n = 0]$$

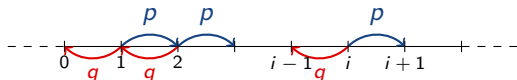
$$\begin{cases} h_0 &= 1 \\ h_i &= ph_{i+1} + qh_{i-1} \end{cases}$$

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Transition matrix and harmonicity

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ q & 0 & p & 0 & \cdots & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$h = (h_i)_{i \geq 0}$ satisfies $Ph = h$.

Application to random walks

Doob transform

Standard procedure in probability. From a Markov process and an associated harmonic function it defines a new random process.

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Example

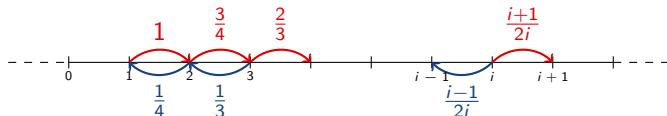
Let $(S_n)_{n \in \mathbb{N}}$ be a simple random walk over \mathbb{Z} .

The function $V(i) = i$ ($i \in \mathbb{N}$) is discrete harmonic for $(S_n)_{n \in \mathbb{N}}$.

Let $(T_n)_{n \in \mathbb{N}}$ be defined from $(S_n)_{n \in \mathbb{N}}$ by

$$\begin{cases} \mathbb{P}[T_{n+1} = i+1 \mid T_n = i] &= \frac{V(i+1)}{2V(i)} = \frac{i+1}{2i}, \\ \mathbb{P}[T_{n+1} = i-1 \mid T_n = i] &= \frac{V(i-1)}{2V(i)} = \frac{i-1}{2i}. \end{cases}$$

The process $(T_n)_{n \in \mathbb{N}}$ is a random walk over \mathbb{N}^* .



Applications to random walks

Asymptotic behavior of the number of excursions

Let $e_{(0,0) \rightarrow (i,j)}(n)$ be the number of n -excursions from the origin to (i, j) .

$$e_{(0,0) \rightarrow (i,j)}(n) \sim \kappa \cdot V(i, j) \cdot \rho^n \cdot n^{-\alpha}, \quad n \longrightarrow \infty$$

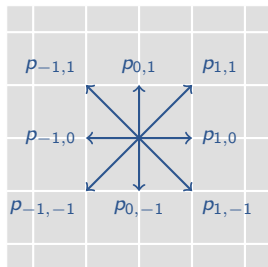
- V is a harmonic function
- ρ is the exponential growth
- α is the critical exponent

Furthermore, the growth of the harmonic function V is directly related to the critical exponent α

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Definition and properties



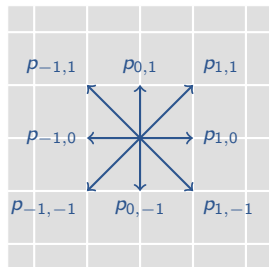
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discrete harmonic function

$$\tilde{f} = (\tilde{f}(i,j))_{(i,j) \in \mathcal{Q}}$$

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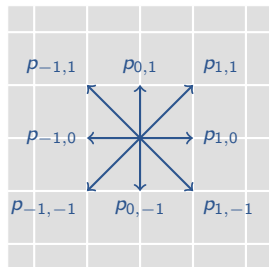
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Properties of the random walks

- homogeneous walk inside \mathcal{Q}
- zero drift

$$\sum_{-1 \leq i,j \leq 1} ip_{i,j} = \sum_{-1 \leq i,j \leq 1} jp_{i,j} = 0$$

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Properties of the harmonic functions

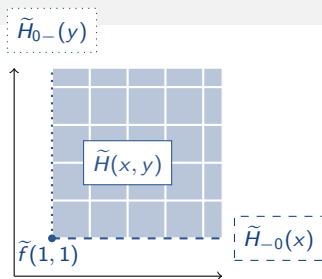
- For all $i \geq 1$ and $j \geq 1$,

$$\tilde{f}(i, j) = \sum_{-1 \leq i_0, j_0 \leq 1} p_{i_0, j_0} \tilde{f}(i + i_0, j + j_0)$$
- If $i \geq 0$, $\tilde{f}(i, 0) = 0$ and
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- If $i > 0$ and $j > 0$ then $\tilde{f}(i, j) > 0$

Strategy

Generating function

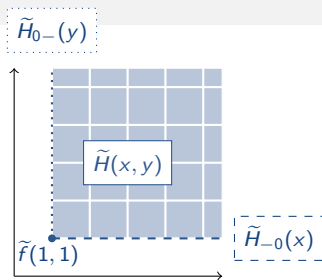
$$\tilde{H}(x, y) = \sum_{(i,j) \in \mathcal{Q}} \tilde{f}(i, j) x^{i-1} y^{j-1}.$$



Strategy

Generating function

$$\tilde{H}(x, y) = \sum_{(i,j) \in \mathcal{Q}} \tilde{f}(i, j) x^{i-1} y^{j-1}.$$



Functional equation

$$K(x, y)\tilde{H}(x, y) = K(x, 0)\tilde{H}_{-0}(x) + K(0, y)\tilde{H}_{-0}(y) - K(0, 0)\tilde{f}(1, 1)$$

where

$$\tilde{H}_{-0}(x) = \sum_{i \geq 1} \tilde{f}(i, 1) x^{i-1},$$

$$\tilde{H}_{-0}(y) = \sum_{j \geq 1} \tilde{f}(1, j) y^{j-1},$$

$$K(x, y) = xy \left[\sum_{-1 \leq i, j \leq 1} p_{i,j} x^{-i} y^{-j} - 1 \right].$$

Kernel of the random walks

$$K(x, y) = xy \left[\sum_{-1 \leq i, j \leq 1} p_{i,j} x^{-i} y^{-j} - 1 \right].$$

The Kernel: a polynomial of degree 2 in x and in y

$$K(x, y) = \tilde{\alpha}(y)x^2 + \tilde{\beta}(y)x + \tilde{\gamma}(y) = \alpha(x)y^2 + \beta(x)y + \gamma(x).$$

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Discriminant: $\tilde{\delta}(y) = \tilde{\beta}(y)^2 - 4\tilde{\alpha}(y)\tilde{\gamma}(y)$ and $\delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x)$.

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Zeros of the Kernel, $i = 0, 1$:

$$K(X_{\pm}(y), y) = 0$$

$$X_{\pm}(y) = \frac{-\tilde{\beta}(y) \pm \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}$$

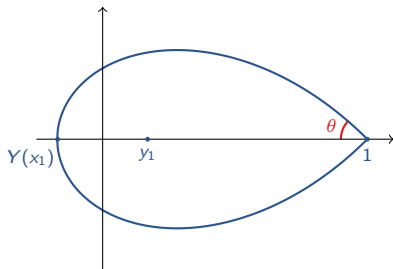
$$K(x, Y_{\pm}(x)) = 0$$

$$Y_{\pm}(x) = \frac{-\beta(x) \pm \sqrt{\delta(x)}}{2\alpha(x)}.$$

The roots of the kernel define analytic curves.

Branches of the Kernel

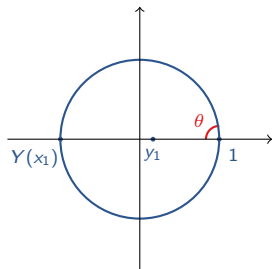
Curves $Y([x_1, 1]) = \{y \in \mathbb{C} : K(x, y) = 0 \text{ and } x \in [x_1, 1]\}$



Gouyou-Beauchamps model $(\rightarrow, \nearrow, \leftarrow, \searrow)$

$$(p_{1,0} = p_{-1,1} = p_{-1,0} = p_{1,-1} = 1/4)$$

$$\theta = \pi/4$$



Simple model $(\rightarrow, \uparrow, \leftarrow, \downarrow)$

$$(p_{1,0} = p_{0,1} = p_{-1,0} = p_{0,-1} = 1/4)$$

$$\theta = \pi/2$$

Generating function $H_{-0}(x)$ stated as a BVP

Functional Equation

$$K(x, y)\tilde{H}(x, y) = K(x, 0)\tilde{H}_{-0}(x) + K(0, y)\tilde{H}_{0-}(y) - K(0, 0)\tilde{f}(1, 1).$$

Boundary value problem

By evaluating the functional equation on the branch curves, we can transform the functional equation into the following **boundary value problem**:

For $x \in X([y_1, 1]) \setminus \{1\}$,

$$K(x, 0)\tilde{H}_{-0}(x) - K(\bar{x}, 0)\tilde{H}_{-0}(\bar{x}) = 0.$$

Explicit expression for the generating function

Theorem [Raschel, 2014]

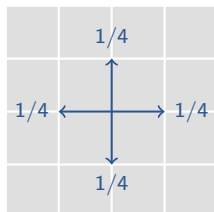
The function $\tilde{H}_{-0}(x)$ has the following explicit expression

$$\tilde{H}_{-0}(x) = \mu \frac{w(x) + \nu}{K(x, 0)},$$

where w is a conformal mapping vanishing at 0, and the constants ν and μ are defined by

$$\nu = -w(X_0(0)), \quad \mu = \tilde{f}(1, 1) \times \begin{cases} \frac{2p_{-1,1}}{w''(0)} & \text{if } p_{1,1} = 0 \text{ and } p_{0,1} = 0, \\ \frac{p_{0,1}}{w'(0)} & \text{if } p_{1,1} = 0 \text{ and } p_{0,1} \neq 0, \\ -\frac{p_{1,1}}{w(X_0(0))} & \text{if } p_{1,1} \neq 0. \end{cases}$$

Example of the simple random walk



$$K(x, 0) = \frac{x}{4}$$

$$w(x) = -\frac{2x}{(1-x)^2}$$

$$\nu = 0$$

$$\mu = -\frac{\tilde{f}(1, 1)}{8}$$

Expression for $H_{-0}(x)$

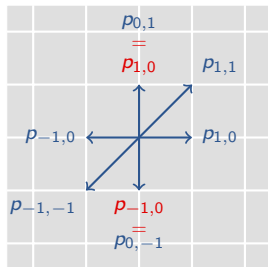
$$\tilde{H}_{-0}(x) = \frac{\tilde{f}(1, 1)}{(1-x)^2} = \tilde{f}(1, 1) \sum_{i \geq 1} ix^{i-1}$$

The function $f(i, j) = ij$ is discrete harmonic for the simple random walk

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Definitions and properties



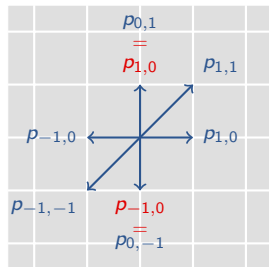
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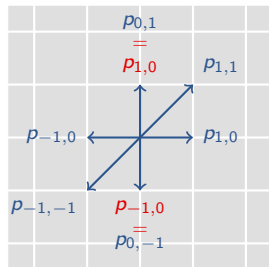
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Properties of the random walks

- homogeneous walk inside \mathcal{C}
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- zero drift

$$\sum_{-1 \leq i, j \leq 1} ip_{i,j} = \sum_{-1 \leq i, j \leq 1} jp_{i,j} = 0$$

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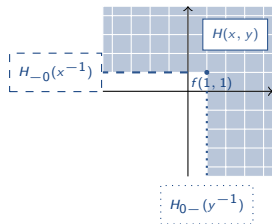
Properties of the harmonic functions

- For all $i \geq 1$ or $j \geq 1$,
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- If $i > 0$ or $j > 0$ then $f(i, j) > 0$
- For all $(i, j) \in \mathcal{C}$, $f(i, j) = f(j, i)$

A first functional equation

Generating function

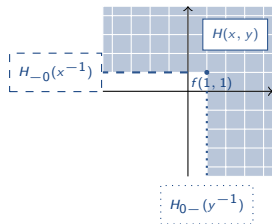
$$H(x, y) = \sum_{(i,j) \in \mathcal{C}} f(i, j) x^{i-1} y^{j-1}.$$



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A first functional equation

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where

$$H_{-0}(x^{-1}) = \sum_{i \leq 0} f(i, 1) x^{i-1},$$

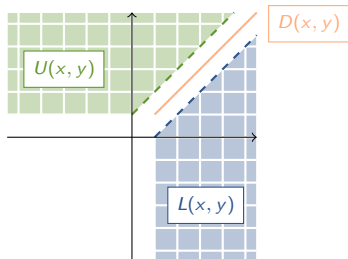
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Strategy

Generating function

$$H(x, y) = \sum_{(i,j) \in \mathcal{C}} f(i, j) x^{i-1} y^{j-1}.$$



Decomposition of the generating function

$$H(x, y) = L(x, y) + D(x, y) + U(x, y)$$

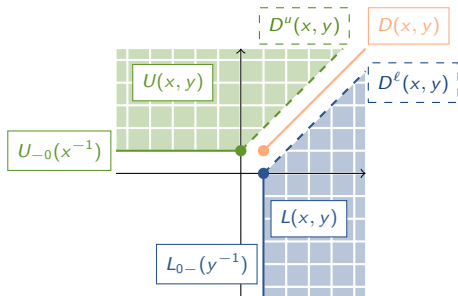
where

$$L(x, y) = \sum_{\substack{i \geq 1 \\ j \leq i-1}} f(i, j) x^{i-1} y^{j-1}, \quad D(x, y) = \sum_{i \geq 1} f(i, i) x^{i-1} y^{i-1}, \quad U(x, y) = \sum_{\substack{j \geq 1 \\ i \leq j-1}} f(i, j) x^{i-1} y^{j-1}.$$

Functional equations

Non-symmetric probability transitions & $p_{-1,1} = p_{1,-1} = 0$

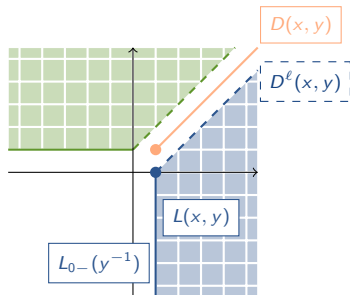
$$\left\{ \begin{array}{l} K(x, y)U(x, y) = -(p_{1,0}y + p_{0,-1}xy^2 + p_{1,1} + p_{-1,-1}x^2y^2 - xy) D(x, y) \\ \quad + (p_{0,1}x + p_{1,1}) U_{-0}(x^{-1}) - (p_{1,0}y + p_{0,-1}xy^2) D^\ell(x, y) \\ \quad + p_{1,1}f(1, 1) + p_{1,0}f(1, 0), \\ K(x, y)L(x, y) = -(p_{0,1}x + p_{-1,0}x^2y + p_{1,1} + p_{-1,-1}x^2y^2 - xy) D(x, y) \\ \quad + (p_{1,0}y + p_{1,1}) L_{0-}(y^{-1}) - (p_{0,1}x + p_{-1,0}x^2y) D^u(x, y) \\ \quad + p_{1,1}f(1, 1) + p_{0,1}f(0, 1). \end{array} \right.$$



Functional equations

Symmetric probability transitions & $p_{-1,1} \neq 0$; $p_{1,-1} \neq 0$

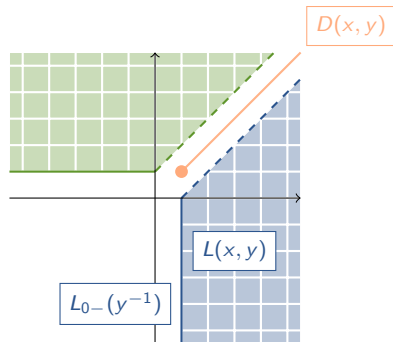
$$\begin{aligned}
 K(x, y)L(x, y) = & - \left(p_{0,1}x + p_{-1,1}x^2 + p_{-1,0}x^2y + \frac{1}{2} \left(p_{1,1} + p_{-1,-1}x^2y^2 - xy \right) \right) D(x, y) \\
 & + p_{1,-1} (y^2 - xy) D^\ell(x, y) + \left(p_{1,0}y + p_{1,-1}y^2 + p_{1,1} \right) L_{0-}(y^{-1}) \\
 & + p_{1,-1}yf(1, 0) + \frac{1}{2}p_{1,1}f(1, 1)
 \end{aligned}$$



Functional equations

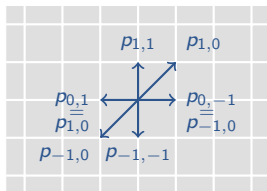
Symmetric probability transitions & $p_{-1,1} = p_{1,-1} = 0$

$$K(x, y)L(x, y) = - \left(p_{0,1}x + p_{-1,0}x^2y + \frac{1}{2} (p_{1,1} + p_{-1,-1}x^2y^2 - xy) \right) D(x, y) \\ + (p_{1,0}y + p_{1,1}) L_{0-}(y^{-1}) + \frac{1}{2} p_{1,1} f(1, 1)$$



Transforming the cones

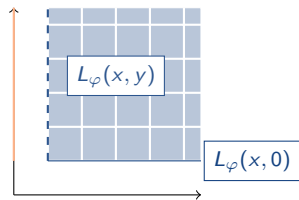
Change of variables $\varphi(x, y) = (xy, x^{-1})$



$$\begin{aligned} L(\varphi(x, y)) &= xL_{\varphi}(x, y) \\ &= x \sum_{i \geq 1} f(j, j-i)x^{i-1}y^{j-1} \end{aligned}$$

$$\begin{aligned} D(\varphi(x, y)) &= D_{\varphi}(y) \\ &= \sum_{i \geq 1} f(i, i)y^{i-1} \end{aligned}$$

$$D_{\varphi}^{\ell}(y)$$



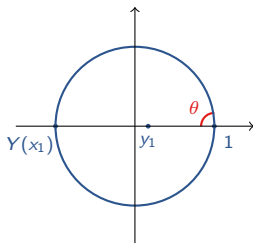
$$D_{\varphi}(y)$$

$$\begin{aligned} D^{\ell}(\varphi(x, y)) &= xD_{\varphi}^{\ell}(y) \\ &= x \sum_{i \geq 1} f(i, i-1)y^{i-1} \end{aligned}$$

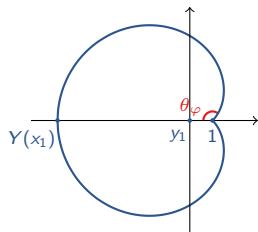
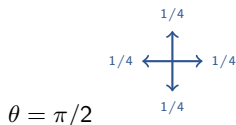
Transforming the probability transitions

$$K(\varphi(x, y)) = \frac{1}{x} K_\varphi(x, y)$$

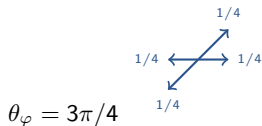
Curves $X([y_1, 1]) = \{x \in \mathbb{C} : K(x, y) = 0 \text{ and } y \in [y_1, 1]\}$



Simple model



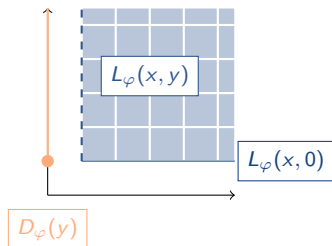
Gessel model



New functional equation

Functional Equation

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1).$$



Generating function $D_\varphi(y)$ stated as a BVP

Functional Equation

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1).$$

Boundary value problem

By evaluating the functional equation on the branch curves, we can transform the functional equation into the following **boundary value problem**:

For $y \in Y_\varphi([x_1, 1]) \setminus \{1\}$,

$$\sqrt{\tilde{\delta}_\varphi(y)}D_\varphi(y) - \sqrt{\tilde{\delta}_\varphi(\bar{y})}D_\varphi(\bar{y}) = 0.$$

Proof

For $x \in [x_1, 1]$,

$$(1) \quad X_-(y) = \frac{-\tilde{\beta}(y) - \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}.$$

$$(2) \quad K(x, Y_-(x)) = 0.$$

$$(3) \quad Y_+(x) = \overline{Y_-(x)}.$$

$$(4) \quad X_-(Y_-(x)) = x.$$

$$\text{Then } x = \frac{-\tilde{\beta}(Y_-(x)) - \sqrt{\tilde{\delta}(Y_-(x))}}{2\tilde{\alpha}(Y_-(x))}.$$

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1)$$

Proof

For $x \in [x_1, 1]$,

$$(1) X_-(y) = \frac{-\tilde{\beta}(y) - \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}.$$

$$(2) K(x, Y_-(x)) = 0.$$

$$(3) Y_+(x) = \overline{Y_-(x)}.$$

$$(4) X_-(Y_-(x)) = x.$$

$$\text{Then } x = \frac{-\tilde{\beta}(Y_-(x)) - \sqrt{\tilde{\delta}(Y_-(x))}}{2\tilde{\alpha}(Y_-(x))}.$$

$$0 = - \left[x\tilde{\alpha}_\varphi(Y_{\varphi,-}(x)) + \frac{1}{2}\tilde{\beta}_\varphi(Y_{\varphi,-}(x)) \right] D_\varphi(Y_{\varphi,-}(x)) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}\rho_{1,1}f(1, 1)$$

$$0 = - \left[x\tilde{\alpha}_\varphi(Y_{\varphi,+}(x)) + \frac{1}{2}\tilde{\beta}_\varphi(Y_{\varphi,+}(x)) \right] D_\varphi(Y_{\varphi,+}(x)) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}\rho_{1,1}f(1, 1)$$

Proof

For $x \in [x_1, 1]$,

$$(1) X_-(y) = \frac{-\tilde{\beta}(y) - \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}.$$

$$(2) K(x, Y_-(x)) = 0.$$

$$(3) Y_+(x) = \overline{Y_-(x)}.$$

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$$0 = - \left[x\tilde{\alpha}_\varphi(Y_{\varphi,+}(x)) + \frac{1}{2}\tilde{\beta}_\varphi(Y_{\varphi,+}(x)) \right] D_\varphi(Y_{\varphi,+}(x)) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1)$$

Proof

For $x \in [x_1, 1]$,

$$(1) X_-(y) = \frac{-\tilde{\beta}(y) - \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}.$$

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$$0 = \frac{\sqrt{\tilde{\delta}(Y_{\varphi,-}(x))}}{2} D_\varphi(Y_{\varphi,-}(x)) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1)$$

$$0 = \frac{\sqrt{\tilde{\delta}(Y_{\varphi,+}(x))}}{2} D_\varphi(Y_{\varphi,+}(x)) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1)$$

Proof

For $x \in [x_1, 1]$,

$$(1) X_-(y) = \frac{-\tilde{\beta}(y) - \sqrt{\tilde{\delta}(y)}}{2\tilde{\alpha}(y)}.$$

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$$\text{Then } x = \frac{-\tilde{\beta}(Y_-(x)) - \sqrt{\tilde{\delta}(Y_-(x))}}{2\tilde{\alpha}(Y_-(x))}.$$

$$\sqrt{\tilde{\delta}(Y_{\varphi,-}(x))}D_\varphi(Y_{\varphi,-}(x)) - \sqrt{\tilde{\delta}(Y_{\varphi,+}(x))}D_\varphi(Y_{\varphi,+}(x)) = 0$$

Then for $y \in Y_\varphi([x_\varphi, 1, 1])$,

$$\sqrt{\tilde{\delta}_\varphi(y)}D_\varphi(y) - \sqrt{\tilde{\delta}_\varphi(\bar{y})}D_\varphi(\bar{y}) = 0.$$

Expression for the Diagonal Section

The diagonal section is defined by:

$$D(x, y) = D_\varphi(xy) = \sum_{i \geq 1} f(i, i)x^{i-1}y^{i-1}.$$

Theorem [T., 2019]

The diagonal section of discrete harmonic functions *not necessarily positive* can be expressed as

$$D(x, y) = \frac{P(\tilde{w}_\varphi(xy))}{L_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

In particular, taking P of degree 1, we get the *unique positive* discrete harmonic function.

Expression for the Diagonal Section

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\tilde{w}'_\varphi(0)} \frac{\pi}{\theta_\varphi} \sqrt{-\frac{\tilde{\delta}''_\varphi(1)}{2\tilde{\delta}_\varphi(y)}} \sqrt{1 - \tilde{W}_\varphi(0)} \sqrt{\tilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, $\tilde{w}_\varphi(y)$ and \tilde{W}_φ are a conformal mappings, all depending on the step set.

Expression for the Diagonal Section

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with θ_φ an explicit angle, $\tilde{w}_\varphi(y)$ and \tilde{W}_φ are a conformal mappings, all depending on the step set.

Simple Walks

$$D_\varphi(y) = f(1,1) \frac{3}{8} \frac{\sqrt{\tilde{W}_\varphi(y)}}{\sqrt{\tilde{\delta}_\varphi(y)}} = f(1,1) \left(1 + \frac{44}{27}y + \frac{523}{243}y^2 + \frac{17168}{6561}y^3 + O(y^4) \right).$$

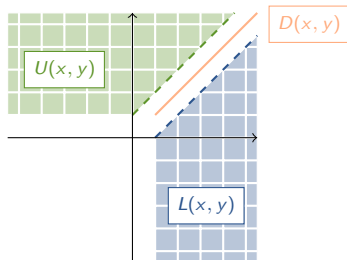
Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1).$$

Remember - Domain in three parts

$$H(x, y) = L(x, y) + D(x, y) + U(x, y).$$



Symmetry of the cut and the walk

$$\Rightarrow U(x, y) = L(y, x).$$

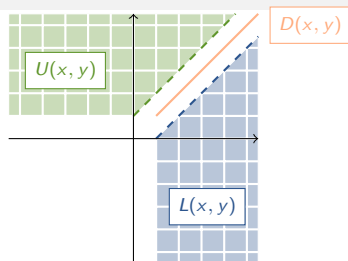
Expression for the generating functions

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- We have an expression of $D_\varphi(y)$;

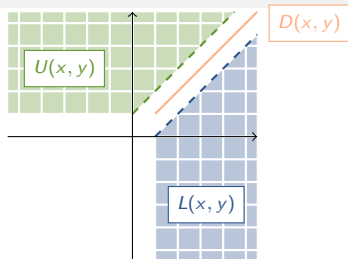
Expression for the generating functions

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Remember - Domain in three parts

$$H(x, y) = L(x, y) + D(x, y) + L(y, x).$$



- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x, y)$;

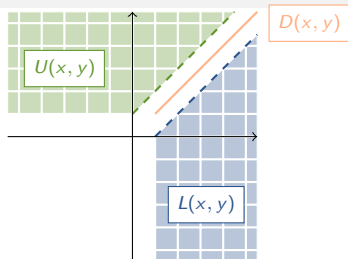
Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1).$$

Remember - Domain in three parts

$$H(x, y) = L(x, y) + D(x, y) + L(y, x).$$



- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x, y)$;
- With a change of variable we get an expression of $D(x, y)$ and $L(x, y)$;

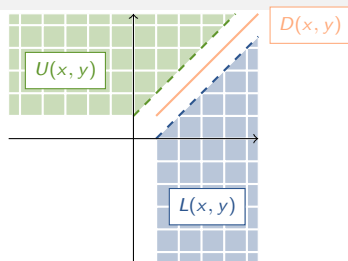
Expression for the generating functions

Remember - Functional Equation

$$K_\varphi(x, y)L_\varphi(x, y) = - \left[x\tilde{\alpha}_\varphi(y) + \frac{1}{2}\tilde{\beta}_\varphi(y) \right] D_\varphi(y) + K_\varphi(x, 0)L_\varphi(x, 0) + \frac{1}{2}p_{1,1}f(1, 1).$$

Remember - Domain in three parts

$$H(x, y) = L(x, y) + D(x, y) + L(y, x).$$



- We have an expression of $D_\varphi(y)$;
- With the functional equation we get an expression of $L_\varphi(x, y)$;
- With a change of variable we get an expression of $D(x, y)$ and $L(x, y)$;
- Then we have an expression of $H(x, y)$.

Asymptotics

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\tilde{w}'_\varphi(0)} \frac{\pi}{\theta_\varphi} \sqrt{-\frac{\tilde{\delta}''_\varphi(1)}{2\tilde{\delta}_\varphi(y)}} \sqrt{1 - \tilde{W}_\varphi(0)} \sqrt{\tilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, \tilde{w}_φ and \tilde{W}_φ conformal mappings, all depending on the step set.

Asymptotics

Expression for the Diagonal Section

$$D_\varphi(y) = -\frac{f(1,1)}{\tilde{w}'_\varphi(0)} \frac{\pi}{\theta_\varphi} \sqrt{-\frac{\tilde{\delta}''_\varphi(1)}{2\tilde{\delta}_\varphi(y)}} \sqrt{1 - \tilde{W}_\varphi(0)} \sqrt{\tilde{W}_\varphi(y)},$$

with θ_φ an explicit angle, \tilde{w}_φ and \tilde{W}_φ conformal mappings, all depending on the step set.

$$D_\varphi(y) = C \cdot \frac{\sqrt{\tilde{W}_\varphi(y)}}{\sqrt{\tilde{\delta}_\varphi(y)}}$$

Asymptotics

Angle of the step set

$$\theta = \arccos \left(- \frac{\sum_{-1 \leq i, j \leq 1} ij p_{i, j}}{\sqrt{\left(\sum_{-1 \leq i, j \leq 1} i^2 p_{i, j} \right) \cdot \left(\sum_{-1 \leq i, j \leq 1} j^2 p_{i, j} \right)}} \right); \quad \theta_\varphi = \pi - \frac{\theta}{2}$$

Example of the simple walks

$$\theta = \frac{\pi}{2}; \quad \theta_\varphi = \pi - \frac{\pi/2}{2} = \frac{3\pi}{4} \text{ (Gessel).}$$

Asymptotics

Angle of the step set

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Example of the simple walks

$$\theta = \frac{\pi}{2}; \quad \theta_\varphi = \pi - \frac{\pi/2}{2} = \frac{3\pi}{4} \text{ (Gessel).}$$

$$\begin{aligned} W_\varphi(y) = \frac{c + o(1)}{(1-y)^{\pi/\theta_\varphi}} &\Rightarrow D_\varphi(y) = - \frac{f(1,1)}{\tilde{w}'_\varphi(0)} \frac{\pi}{\theta_\varphi} \sqrt{-\frac{\tilde{\delta}''_\varphi(1)}{2\tilde{\delta}_\varphi(y)}} \sqrt{1 - \tilde{W}_\varphi(0)} \sqrt{\tilde{W}_\varphi(y)} \\ &= \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}} \text{ for } y \text{ close to } 1 \end{aligned}$$

Asymptotics

For y close to 1

In the three-quadrant

$$D_\varphi(y) = \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}}$$

In the quadrant

$$\tilde{D}_\varphi(y) = \frac{\tilde{c}_D + o(1)}{(1-y)^{\pi/\theta+1}}$$

Asymptotics

For y close to 1

In the three-quadrant

$$D_\varphi(y) = \frac{c_D + o(1)}{(1-y)^{\pi/(2\pi-\theta)+1}}$$

In the quadrant

$$\tilde{D}_\varphi(y) = \frac{\tilde{c}_D + o(1)}{(1-y)^{\pi/\theta+1}}$$

Theorem [Mustapha, 2019]

Let $\alpha_Q = \frac{\pi}{\theta}$ be the critical exponent of walks in the quadrant. Then the critical exponent α_C of walks in the three-quarter plane can be expressed as

$$\alpha_C = \frac{\alpha_Q}{2\alpha_Q - 1} = \frac{\pi}{2\pi - \theta}.$$

Content

1. Introduction
2. Previous results in the quarter plane
3. Results in the three-quarter plane
4. Further objectives and perspectives

Non-positive harmonic functions

Expression for the generating function

$$D(x, y) = \frac{P(\tilde{w}_\varphi(xy))}{G_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

More generally, for **any polynomial** P of degree n we get discrete harmonic functions (but not necessarily positive).

Non-positive harmonic functions

Expression for the generating function

$$D(x, y) = \frac{P(\tilde{w}_\varphi(xy))}{G_\varphi(xy)}, \quad P \in \mathbb{R}[y].$$

More generally, for **any polynomial** P of degree n we get discrete harmonic functions (but not necessarily positive).

Simple walks

$$P(y) = \frac{3}{4}y^2 - \frac{9}{16}$$

$$D_\varphi(y) = 1 + 4y + 9y^2 + 16y^3 + 25y^4 + O(y^5)$$

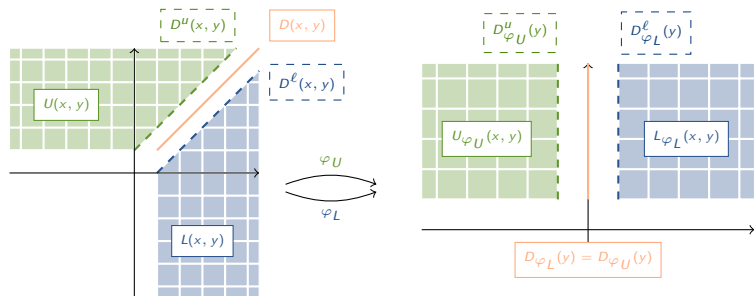
We find back the harmonic function $f(i, j) = ij$
(non-positive in the three-quadrant)

Non-positive harmonic functions

- ➔ Is every harmonic function completely determined by the polynomial P ?
- ➔ What is the structure of non-positive harmonic functions?
- ➔ How does the cone of restriction affect this structure?
- ➔ What are the properties of non-positive harmonic functions?

👁️ É. Fusy, K. Raschel, P. Tarrago and A. T.

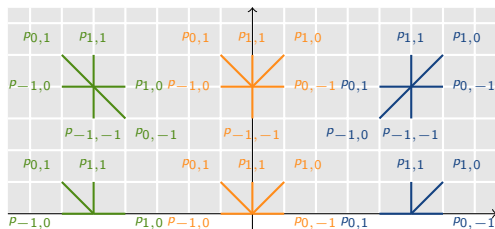
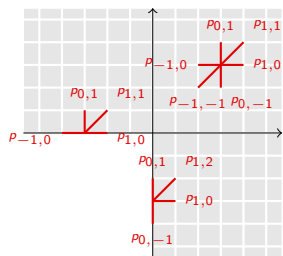
Non-symmetric case



$$\left\{ \begin{array}{l}
 K_{\varphi_U}(x, y) U_{\varphi_U}(x, y) = - (p_{1,0} x^{-1} y + p_{0,-1} x^{-1} y^2 + p_{1,1} + p_{-1,-1} y^2 - y) D_{\varphi_U}(y) \\
 \quad + (p_{0,1} x^2 + p_{1,1} x) U_{\varphi_U}(x, 0) - (p_{1,0} + p_{0,-1} y) D_{\varphi_L}^l(y) \\
 \quad + p_{1,1} f(1, 1) + p_{1,0} f(1, 0), \\
 K_{\varphi_L}(x, y) L_{\varphi_L}(x, y) = - (p_{0,1} x y + p_{-1,0} x y^2 + p_{1,1} + p_{-1,-1} y^2 - y) D_{\varphi_L}(y) \\
 \quad + (p_{1,0} + p_{1,1} x) L_{\varphi_L}(x, 0) - (p_{0,1} + p_{-1,0} y) D_{\varphi_U}^u(y) \\
 \quad + p_{1,1} f(1, 1) + p_{0,1} f(0, 1).
 \end{array} \right.$$

Non-symmetric case




Change of variables $\varphi_L(x, y) = (xy, x^{-1})$ & $\varphi_U(x, y) = (x, x^{-1}y)$



Random walks avoiding a quadrant can be seen as inhomogeneous walks in the half plane with probability transitions $\varphi_U((p_{i,j})_{-1 \leq i,j \leq 1})$ in the left quadrant and $\varphi_L((p_{i,j})_{-1 \leq i,j \leq 1})$ on the right quadrant.

References

References

-  S. Mustapha, *Non-D-Finite Walks in a Three-Quadrant Cone*, *Ann. Comb.* **23** (2019), no. 1, 143–158. MR 3921340
-  K. Raschel, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings*, *Stochastic Process. Appl.* **124** (2014), no. 10, 3147–3178, With an appendix by S. Franceschi. MR 3231615
-  A. Trotignon, *Discrete harmonic functions in the three-quarter plane*, *arXiv* **1906.08082** (2019), 1–26.

