Tensor Model Holography

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Holographic duality

Considerable evidence that quantum gravity in asymptotically AdS_{d+1} spacetime is equivalent to CFT_d.

Smooth classical geometry in gravity is strong coupling and large N in the CFT. Single trace primaries of CFT give single particle states in gravity. In spite of all constructed evidence, the duality remains mysterious.

Famous example: duality between $U(N)$ $\mathcal{N}=4$ SYM and IIB string theory on asymptotically $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime with N units of 5-form flux.

A "much easier" example is provided by the duality between the free $O(N)$ vector model and higher spin gravity.

What we are shooting for

To demonstrate a holographic duality you should demonstrate the following:

1. The dual theory should exhibit an extra non-compact dimension. Physics is local in this extra dimension.

2. The loop expansion parameter is $1/N$.

A complete understanding should also demonstrate that:

3. The dual theory is a theory of gravity.

Holography Report Card

Single matrix model	A
Multi matrix model	F
Vector model	B
Tensor model	F

Comments: Listens and follows directions well. Does not complete activities in a timely manner. Too easily distracted. Does not express ideas clearly.

Single Matrix Holography

Consider the matrix quantum mechanics described by the action

$$S = \int dt \mathrm{Tr} \left(\frac{1}{2}\dot{M}^2 - \frac{1}{2}M^2 - gM^4 \right) \tag{1}$$

There is a global $U(N)$ symmetry acting as

$$M \to U^\dagger M U \tag{2}$$

Focus on the singlet sector. Introduce the invariant variable

$$\phi(k) = \mathrm{Tr}\left(e^{ikM}\right) \tag{3}$$

$$\phi(x) = \int dk e^{-ikx}\phi(k) = \sum_{i=1}^{N} \delta(x - \lambda_i) \tag{4}$$

Single Matrix Holography

Rewrite the Hamiltonian

$$H = \mathrm{Tr}\left(\frac{1}{2}P^2 + \frac{1}{2}M^2 + gM^4\right) \qquad P_{ij} = -i\frac{\partial}{\partial M_{ji}} \tag{5}$$

In terms of the collective variables. The potential terms are easily rewritten using

$$\begin{aligned} \mathrm{Tr}(X^n) &= \sum_{i=1}^{N}\lambda_i^n = \int dx\,\sum_{i=1}^{N}\delta(x-\lambda_i)\,x^n \\ &= \int dx\,x^n\phi(x) \end{aligned} \tag{6}$$

so that

$$\mathrm{Tr}\left(\frac{1}{2}M^2 + gM^4\right) = \int dx\left(\frac{1}{2}x^2 + gx^4\right)\phi(x)$$

Single Matrix Holography

Rewrite the Hamiltonian

$$H = \mathrm{Tr}\left(\frac{1}{2}P^2 + \frac{1}{2}M^2 + gM^4\right) \qquad P_{ij} = -i\frac{\partial}{\partial M_{ji}} \tag{7}$$

In terms of the collective variables. The kinetic term becomes

$$\begin{aligned} \mathrm{Tr}(P^2) \to \quad & - \int dk_1 \int dk_2 \frac{\partial\phi(k_1)}{\partial M_{ji}} \frac{\partial\phi(k_2)}{\partial M_{ij}} \frac{\partial}{\partial\phi(k_1)} \frac{\partial}{\partial\phi(k_2)} \\ & - \int dk \frac{\partial^2\phi(k)}{\partial M_{ji}\partial M_{ij}} \frac{\partial}{\partial\phi(k)} \end{aligned} \tag{8}$$

This is not manifestly Hermittian! Why not? How should we interpret this?

Single matrix holography

In changing from M to ϕ there is a non-trivial Jacobian

$$\int [dM]\Phi(M)^*\Psi(M) \to \int [d\phi] J[\phi]\Phi[\phi]^*\Psi[\phi] \tag{9}$$

"Jacobian" is an abuse of language. We are changing from the N eigenvalues of $M(t)$ to a function $\phi(t,x)$ of a continuous variable x. At large N we will assume $\phi(t,x)$ at different x are independant variables. To remove the non-trivial measure

$$\Psi[\phi] \to J^{\frac{1}{2}}\Psi[\phi] \qquad H \to H_{\mathrm{coll}} = J^{\frac{1}{2}} H J^{-\frac{1}{2}} \tag{10}$$

One obtains a differential equation for the Jacobian from the requirement

$$H_{\mathrm{coll}} - H_{\mathrm{coll}}^{\dagger} = 0 \tag{11}$$

Single matrix holography

$$H = \int dx \left(\frac{1}{2}\partial_x \pi \phi \partial_x \pi + x^2 \phi(x) + g x^4 \phi(x) - \mu(\phi(x) - \frac{N}{V}) + \frac{\pi^2}{6}\phi^3(x) \right)$$

Set

$$\phi(x) = \phi_0(x) + \frac{1}{N}\partial_x \eta(x) \tag{12}$$

where $\phi_0(x)$ extremizes the effective potential. The theory of the η field is a two dimensional theory, with interactions local in the extra dimension (with coordinate x) and it has $1/N$ as loop expansion parameter.

Why A on the report card? Why not A^+? The 2d black hole has not yet been described. Work of Kazakov, Kostov, Kutasov suggest we need to go beyond singlet sector. How do we do that?

References for single matrix holography

A. Jevicki and B. Sakita, "The Quantum Collective Field Method and Its Application to the Planar Limit," Nucl.Phys.B **165**, 511 (1980).

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Summary: Constructive Holography

1. Identify a suitable set of invariant variables that are independent at large N. (for the one matrix problem this was $\phi(k) = \mathrm{Tr}(e^{ikM})$.)

2. Rewrite the theory in terms of the invariant variables. This entails the construction of a Jacobian which supplies highly non-trivial and non-linear interactions.

3. Construct the large N configuration and expand about it. The field theory of fluctuations about the large N configuration is the holographic dual: it has $1/N$ as loop counting parameter and is higher dimensional.

Constructive Holography: Two Matrices

1. Identify a suitable set of invariant variables that are independent at large N. (for the one matrix problem this was $\phi(k) = \mathrm{Tr}(e^{ikM})$.)

For one matrix we could also have taken $\mathrm{Tr}(M^n)$, $n = 1, 2, 3...N$. At large N the power n is not restricted.

For two matrices: $\mathrm{Tr}(A)$,$\mathrm{Tr}(B)$ and then $\mathrm{Tr}(A^2)$, $\mathrm{Tr}(AB)$, $\mathrm{Tr}(B^2)$ and then $\mathrm{Tr}(A^3)$, $\mathrm{Tr}(A^2B)$, $\mathrm{Tr}(AB^2)$, $\mathrm{Tr}(B^3)$ and then ....

The number of invariants grows rapidly as the number of matrices in the trace increases...

Constructive Holography: Two Matrices

For two matrices: $\mathrm{Tr}(A)$,$\mathrm{Tr}(B)$ and then $\mathrm{Tr}(A^2)$, $\mathrm{Tr}(AB)$, $\mathrm{Tr}(B^2)$ and then $\mathrm{Tr}(A^3)$, $\mathrm{Tr}(A^2B)$, $\mathrm{Tr}(AB^2)$, $\mathrm{Tr}(B^3)$ and then ....

$$\begin{aligned} F(x,y) &= (x+y)+\left(x^2+xy+y^2\right)+\left(x^3+x^2y+xy^2+y^3\right) \\ &+ \left(x^4+x^3y+2x^2y^2+xy^3+y^4\right) \\ &+ \left(x^5+x^4y+2x^3y^2+2x^2y^3+xy^4+y^5\right) \\ &+ \left(x^6+x^5y+3x^4y^2+4x^3y^3+3x^2y^4+xy^5+y^6\right)+\cdots \end{aligned}$$

$$\begin{aligned} F(\alpha,\alpha) &= 2\alpha+3\alpha^2+4\alpha^3+6\alpha^4+8\alpha^5+14\alpha^6+20\alpha^7+36\alpha^8 \\ &+ 60\alpha^9+108\alpha^{10}+188\alpha^{11}+352\alpha^{12}+632\alpha^{13}+1182\alpha^{14} \\ &+ 2192\alpha^{15}+4116\alpha^{16}+7712\alpha^{17}+14602\alpha^{18}+27596\alpha^{19} \\ &+ 52488\alpha^{20}+99880\alpha^{21}+190746\alpha^{22}+364724\alpha^{23}+699252\alpha^{24} \\ &+ 1342184\alpha^{25}+2581428\alpha^{26}+4971068\alpha^{27}+\cdots \end{aligned}$$

$$F_1(\alpha)=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\cdots$$

Constructive Holography: Two Matrices

Truncating to a smaller space? Go just a little beyond one matrix:

$$\phi_n = \mathrm{Tr}(A^n) \qquad \psi_n = \mathrm{Tr}(A^n B)$$

$$\begin{aligned} \frac{\partial}{\partial A^i_j}\frac{\partial}{\partial A^j_i} &= \frac{\partial \phi_n}{\partial A^i_j}\frac{\partial \phi_m}{\partial A^j_i}\frac{\partial}{\partial \phi_n}\frac{\partial}{\partial \phi_m} + \frac{\partial^2 \phi_n}{\partial A^i_j \partial A^j_i}\frac{\partial}{\partial \phi_n} \\ &+ \frac{\partial \psi_n}{\partial A^i_j}\frac{\partial \psi_m}{\partial A^j_i}\frac{\partial}{\partial \psi_n}\frac{\partial}{\partial \psi_m} + \frac{\partial^2 \psi_n}{\partial A^i_j \partial A^j_i}\frac{\partial}{\partial \psi_n} \end{aligned} \tag{13}$$

Loop splitting and joining:

$$\frac{\partial \phi_n}{\partial A^i_j}\frac{\partial \phi_m}{\partial A^j_i} = nm\phi_{n+m-2} \qquad \frac{\partial^2 \phi_n}{\partial A^i_j \partial A^j_i} = \sum_{r=0}^{n-2} \phi_r \phi_{n-r-2}$$

$$\frac{\partial \psi_n}{\partial A^i_j}\frac{\partial \psi_m}{\partial A^j_i} = \sum_k \mathrm{Tr}(BA^{n+m-2-k}BA^k)$$

Constructive Holography: Two Matrices

1. Identify a suitable set of invariant variables that are independent at large N. (for the one matrix problem this was $\phi(k) = \mathrm{Tr}(e^{ikM})$.)

We fail on the first step - its hard to find a nice way to package all the invariant variables into one (or many) collective fields whose dynamics we could study. The F on the report card was generous!

We now consider a problem in which more than one collective field plays a role: the $O(N)$ vector model. The holographic dual to the vector model is Vasiliev's higher spin gravity.

Higher spin theories in AdS

In any dimension Vasiliev wrote a set of consistent, fully non-linear, gauge invariant equations of motion for higher spin gravity. They admit a vacuum solution which is AdS spacetime.

In the simplest bosonic 4d theory, the linearized spectrum around the AdS vacuum is an infinite tower of massless higher spin fields with spin $s = 2, 4, 6, \ldots$ plus a scalar with mass $m^2 = -2/l_{AdS}^2$.

Full quantum theory is not yet known. We don't know of any action that reproduces Vasiliev's equations.

Linearized HS Spectrum

Linearizing the Vasiliev equations around AdS vacuum solution, one finds the usual equations for a scalar (with mass $m^2 = -2$), linearized graviton and free massless Higher Spin fields (Fronsdal).

$$(\nabla^2 - \kappa^2)\varphi_{\mu_1\cdots\mu_s} = 0 \qquad \kappa^2 = (s-2)(s+d-3) - 2 \tag{14}$$

$$\nabla^\mu \varphi_{\mu\mu_2\cdots\mu_s} = 0 \qquad g^{\mu\nu}\varphi_{\mu\nu\nu_3\cdots\mu_s} = 0 \tag{15}$$

At higher orders in perturbation theory, one reads off cubic, quartic, etc. couplings. Could in principle reconstruct a Lagrangian in terms of the physical Higher Spin fields but its very hard in practice.

Free equations are ordinary wave equations. Interactions involve higher derivatives of arbitrarily high order so the Vasiliev theory is a higher derivative theory of gravity.

Higher Spins and AdS/CFT

The AdS/CFT correspondence predicts that such theories exist.

Vasiliev higher spin theories have the correct spectrum to be dual to simple CFT's with matter transforming in the vector rep of the symmetry group.

Consider a free theory of N free real scalar fields

$$S = \int d^3x \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i \qquad i = 1, 2, \cdots, N \tag{16}$$

Has $O(N)$ global symmetry under which the scalar transforms as a vector. Vasiliev theory is dual to the singlet sector of this $O(N)$ vector model.

This is different from familiar examples of AdS/CFT, where the CFT side is usually a gauge theory with matrix (adjoint) fields.

Single Trace CFT Primaries

The free theory has an infinite tower of conserved Higher Spin currents which are primary operators of the form

$$j_{\mu_1\mu_2\cdots\mu_s} \sim \sum_k c_k \; \partial_{\mu_1}\cdots\partial_{\mu_k}\phi^i \; \partial_{\mu_{k+1}}\cdots\partial_{\mu_s}\phi^i \tag{17}$$

$$\partial_\mu j^{\mu\mu_2\cdots\mu_s} = 0 \tag{18}$$

$$\Delta(j^{\mu_1\cdots\mu_s}) = s+1 = s+d-2 \tag{19}$$

In the singlet sector, these currents and the scalar operator $\phi^i\phi^i$ with $\Delta = 1d-2$ are all the "single trace" primaries of the CFT_3.

Higher Spin Holography

The standard AdS/CFT dictionary identifies single trace primary operators in the CFT with single particle states in AdS.

The conserved currents are dual to gauge fields (massless Higher Spin fields) with spin $s = 2, 4, 6, \cdots$.

The scalar operator is dual to a bulk scalar with $m^2 = \Delta(\Delta - d)$.

This precisely matches the spectrum of Vasiliev's minimal bosonic theory in AdS_4.

Interacting Higher Spin Holography

The dual (gravitational) higher spin fields must be interacting in order to reproduce the non-vanishing correlation functions of Higher Spin currents in the CFT.

The large N limit corresponds, as usual, to weak interactions in the bulk.

So AdS/CFT implies that consistent theories of interacting massless higher spins in AdS indeed have to exist, as they should provide AdS duals to free vector like CFT's (in the singlet sector).

Bi-local Holography

Free vector model in original variables

$$\langle \cdots \rangle = \int [d\phi^i] \; e^{iS} \; \cdots \tag{20}$$

Collective variable is the $O(N)$ singlet $\sigma(x, y)$

$$\sigma(x, y) = \frac{1}{N} \sum_{i=1}^{N} \phi^i(x)\phi^i(y) \tag{21}$$

After changing variables

$$\langle \cdots \rangle = \int [d\sigma] \; J[\sigma] e^{iS[\sigma]} \; \cdots \tag{22}$$

$\sigma(x, y)$ is a function of $2d$ variables: AdS_{d+1}, S_{d-1}

Jacobian

The Jacobian is easily evaluated

$$\log J = (N - L^d \delta^d(0)) \mathrm{Tr} \log \sigma \tag{23}$$

The bilocal is expanded as

$$\sigma(x^\mu, y^\mu) \to H(x_{\mathrm{AdS}_{d+1}}, y_{\mathrm{S}^d}) = \sum_n \phi^{\alpha_1 \cdots \alpha_n}(x_{\mathrm{AdS}_{d+1}}) y_{\alpha_1} \cdots y_{\alpha_n}$$

Quadratic bilocal collective field theory reproduces Fronsdal dynamics. Cubic terms reproduce the correct correlation functions. To improve the B grade, we'd like to see some of the non-linear terms in Vasilliev's theory reproduced.

References for bilocal holography

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Tensor Model Holography

We have seen that single matrix models and vector models are simple enough that we can solve them. They do not give much insight into the multi-matrix problem.

The advertised hope was that tensor models would be a useful "middle ground" or "sweet spot".

New example - maybe new mechanisms for holography? Perhaps useful to go beyond a single matrix?

Tensor Model Holography

Rank 3 complex tensors with a global symmetry $U(N_1) \times U(N_2) \times U(N_3)$ acting as

$$\phi^{abc} \rightarrow (U_1)^a_d (U_2)^b_e (U_3)^c_f \phi^{def} \qquad \bar{\phi}_{abc} \rightarrow (U_1^\dagger)^d_a (U_2^\dagger)^e_b (U_3^\dagger)^f_c \bar{\phi}_{def}$$

where $U_i \in U(N_i)$

How many invariants? (e.g. $\phi^{a_1 b_1 c_1} \phi^{a_2 b_2 c_2} \bar{\phi}_{a_2 b_1 c_1} \bar{\phi}_{a_1 b_2 c_2}$)

$$1, 4, 11, 43, 161, 901, \cdots$$

Compare to 2-matrix model invariants:

$$2, 3, 4, 6, 8, 14, \cdots$$

Tensor Model Holography

Number of invariants in a tensor model grows much more rapidly that the number of invariants in matrix models. This leads us to suspect that the dynamics is extremely rich.

Also makes the problem formidable and once again there is as yet no nice way to package these invariants into a nice collective field. We are stuck...

Instead of giving up, lets look for some subset of invariants that is dynamically closed. There are interesting non-trivial sectors.

Tensor Model: Matrix Like Holography

Study quantum mechanics:

$$H = \Pi_{abc}\bar{\Pi}^{abc} + \omega^2 \phi^{abc}\bar{\phi}_{abc} = -\frac{\partial}{\partial \phi^{abc}}\frac{\partial}{\partial \bar{\phi}_{abc}} + \omega^2 \phi^{abc}\bar{\phi}_{abc}$$

Introduce the variable

$$T^i_j = \phi^{abi}\bar{\phi}_{abj}$$

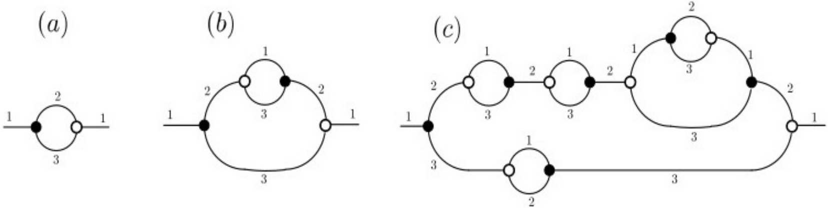
that transforms in the adjoint of $U(N_3)$. As long as we restrict ourselves to tensor models with potentials $\sum_n \mathrm{Tr}(T^n)$, and we scale $N_1, N_2, N_3 \to \infty$ with $N_3/(N_1 N_2)$ fixed then there is a subsector of the dynamics with a single matrix-like holography, with collective variable

$$\phi(k) = \mathrm{Tr}(e^{kT})$$

The eigenvalues of T^i_j reproduces the eigenvalue densities of random matrix models.

Tensor Model: Bilocal Holography

A very different limit is the melonic limit. To reach it we could take $N_1 = N_2 = N_3 \to \infty$. Use melonic matrices:



Density of eigenvalues localizes to a single point, so no extra dimension. Experience with SYK suggests bilocal holography is relevant.

Tensor Model Holography References

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Future Directions

Make the holography of the melonic limit precise.

Genuinely new types of holography?

Lessons from tensor models to go beyond one matrix? Loop spaces?

Dynamics of "heavy operators"?

Thanks for your attention!