Scaling limits of random maps and a proposal for random geometry in higher dimensions

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Joint work with J.F. Marckert

13/05/20 - Tensor Journal Club.. from your couch

1 – Planar maps, etc

2 – Notion of scaling limit

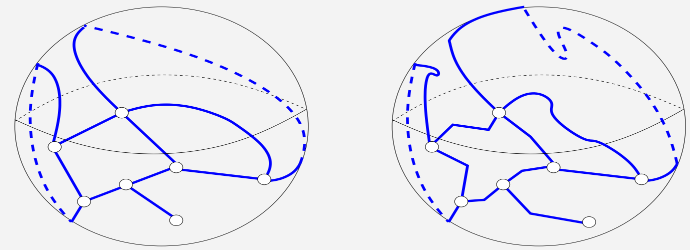
3 – The problem in higher dimensions

4 – The Cori-Vauquelin-Schaeffer bijection, etc

5 – Random feuilletages

1 – Planar maps, etc

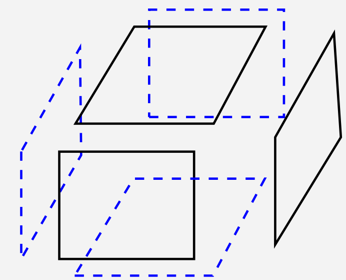
Combinatorial maps:



Drawing on a surface without crossings

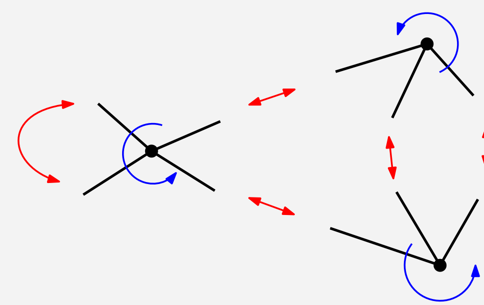
Gluings of polygons

Triangulations, quadrangulations, no restrictions…



Gluings of vertices

combinatorial encoding using permutations

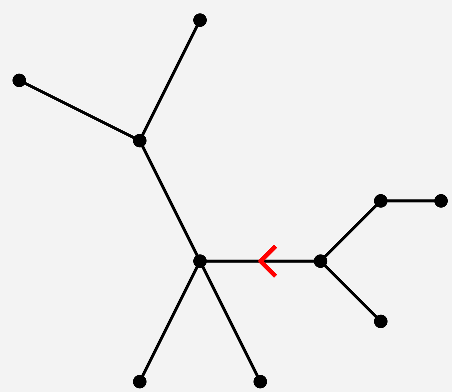


→ **Planar maps**: spherical topology V-E+F=2

→ **Plane trees**: V=E+1; F=1

→ We consider all maps rooted (no symmetries).

→ ``size'' = number of edges, or vertices, or faces…



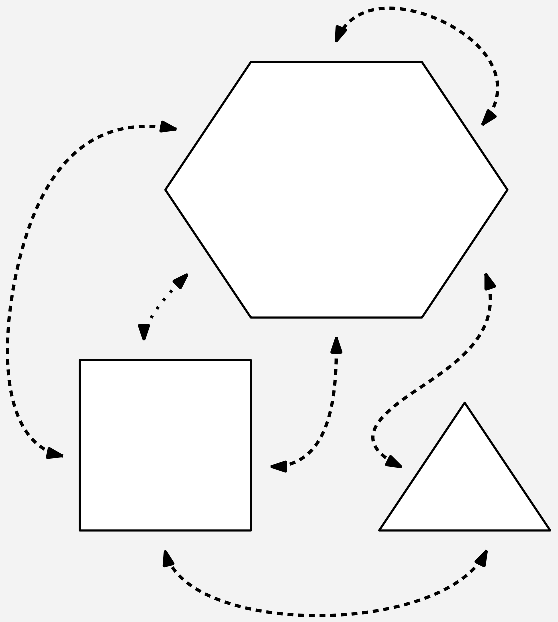
Random maps:

Choose a distribution for a given set of maps of the same size,

→ Uniform distribution on binary trees with n edges
→ Uniform distribution on all trees with n edges.

→ Uniform distribution on planar maps made of n squares (quadrangulations)
→ Same but for all planar maps with n faces

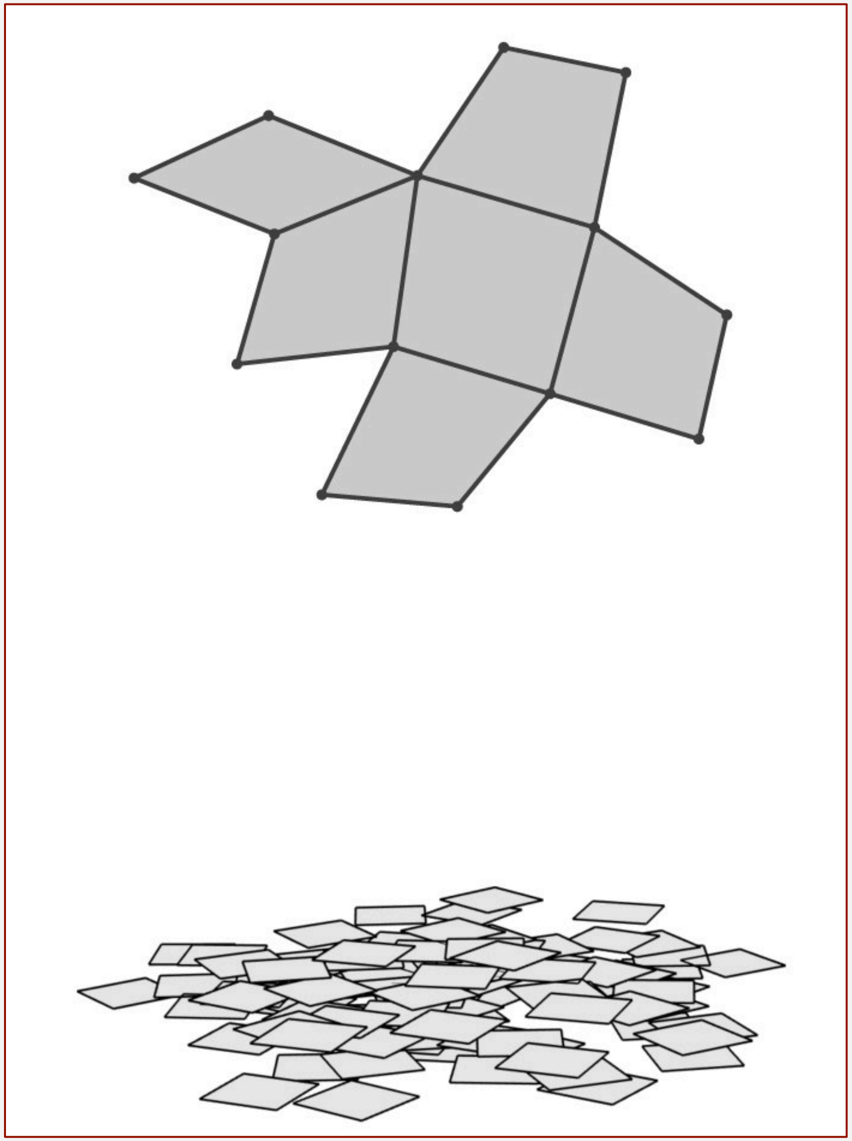
→ Other kinds of distributions…



Random maps:

→ Uniform distribution on planar maps made of n squares (quadrangulations)

→ What does it typically look like??

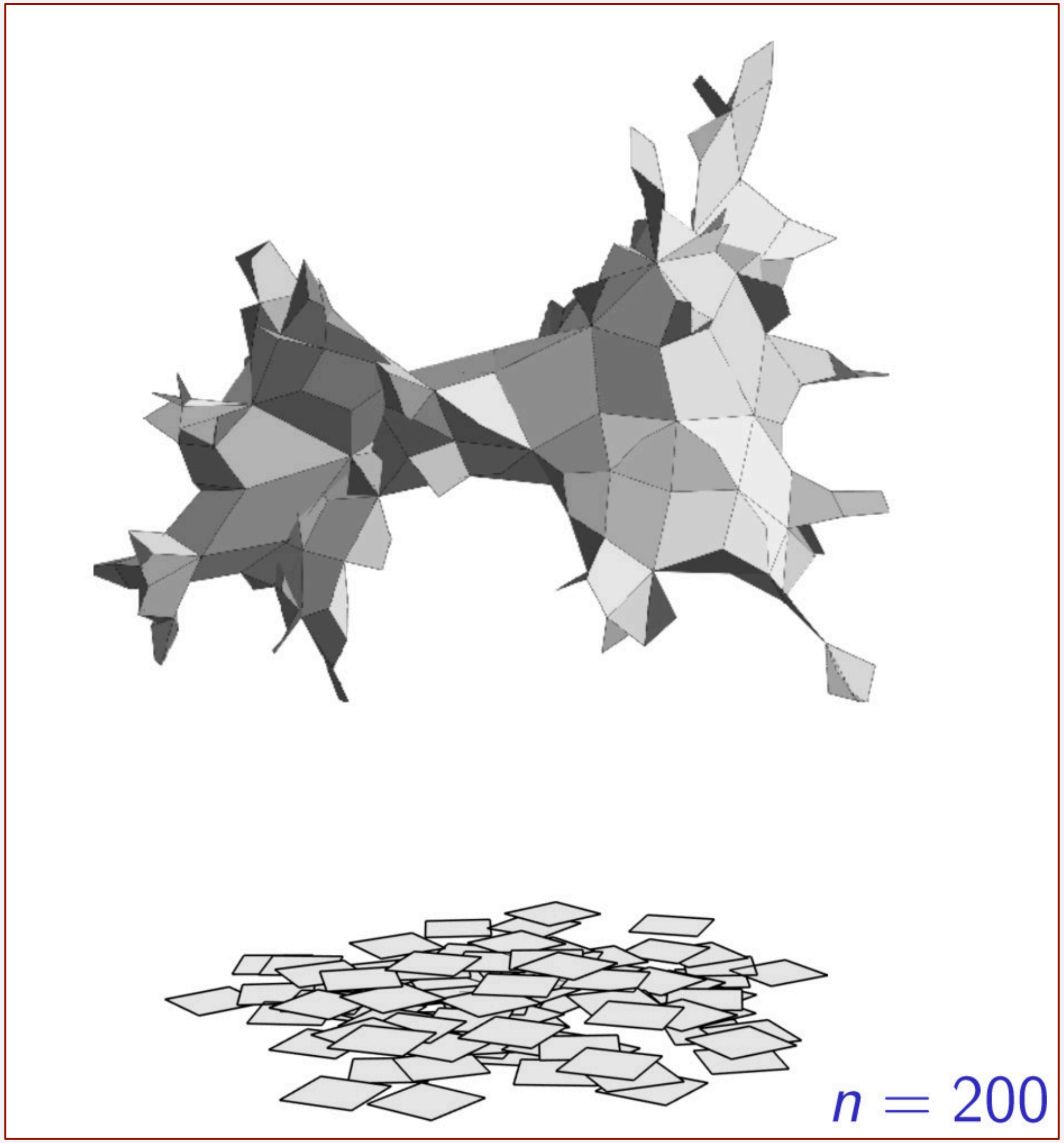


Picture
T. Budd

Random maps:

→ Uniform distribution on planar maps made of *n* squares (quadrangulations)

→ What does it typically look like??



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Random maps:

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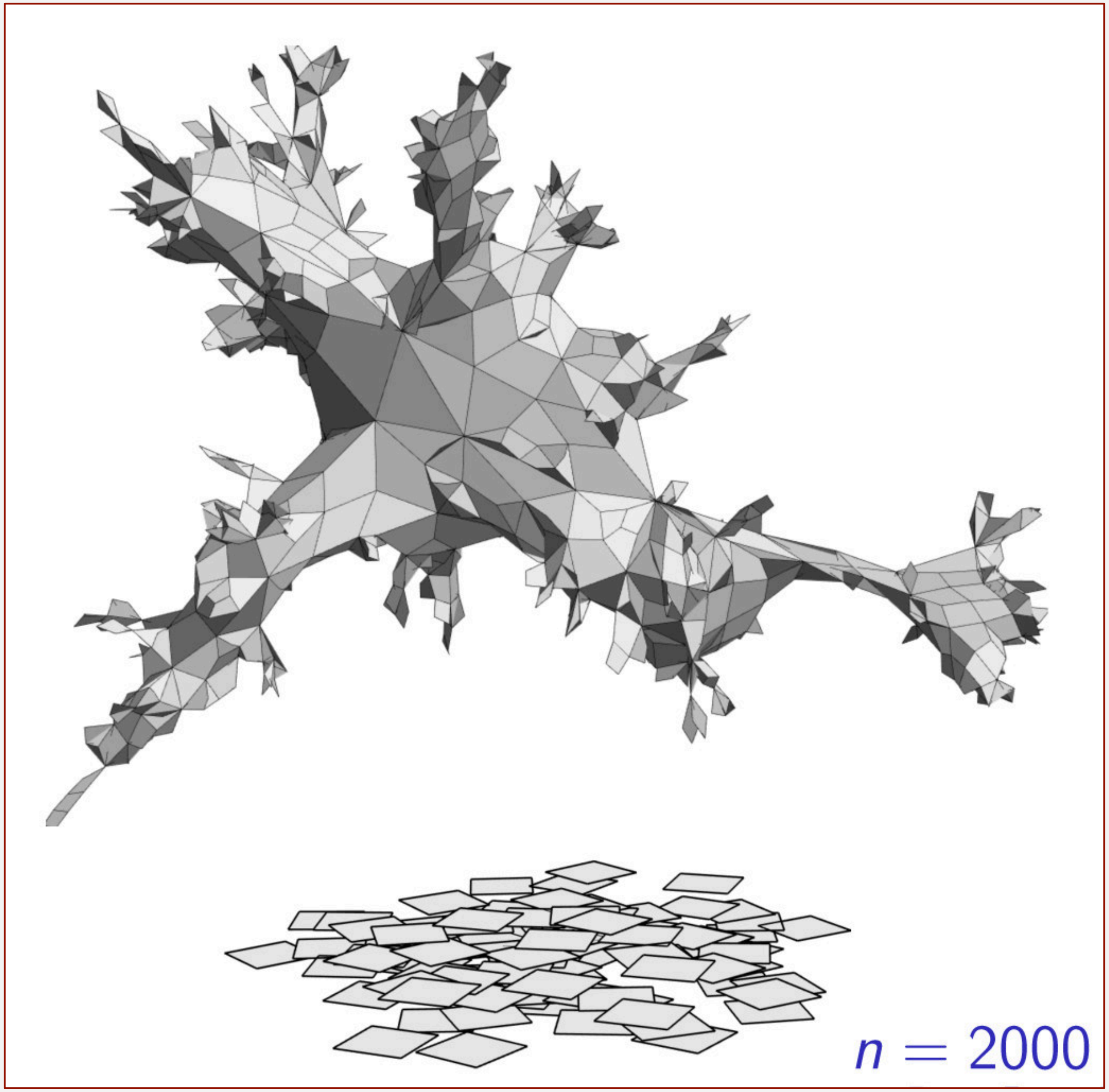

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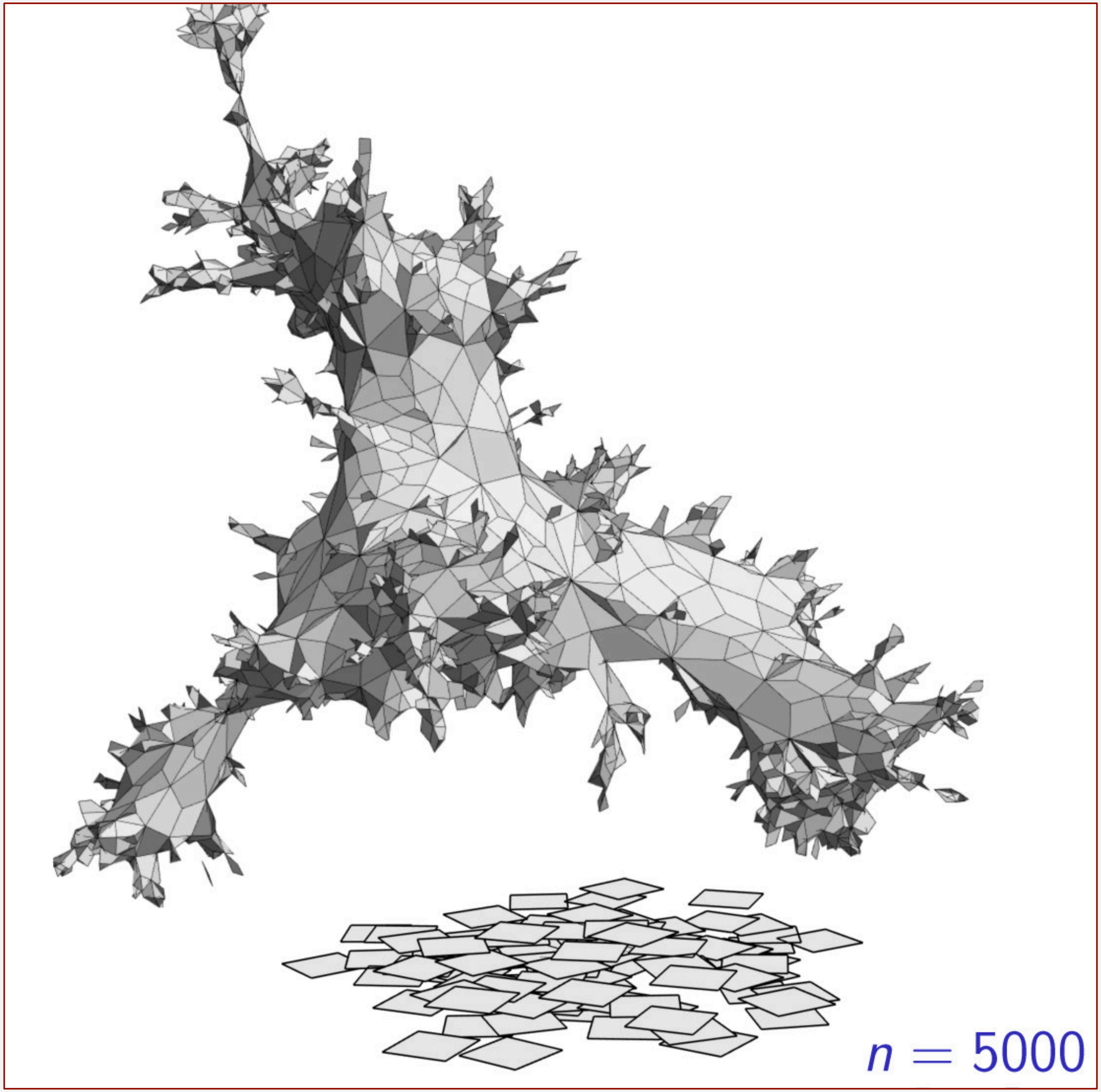
T. Budd



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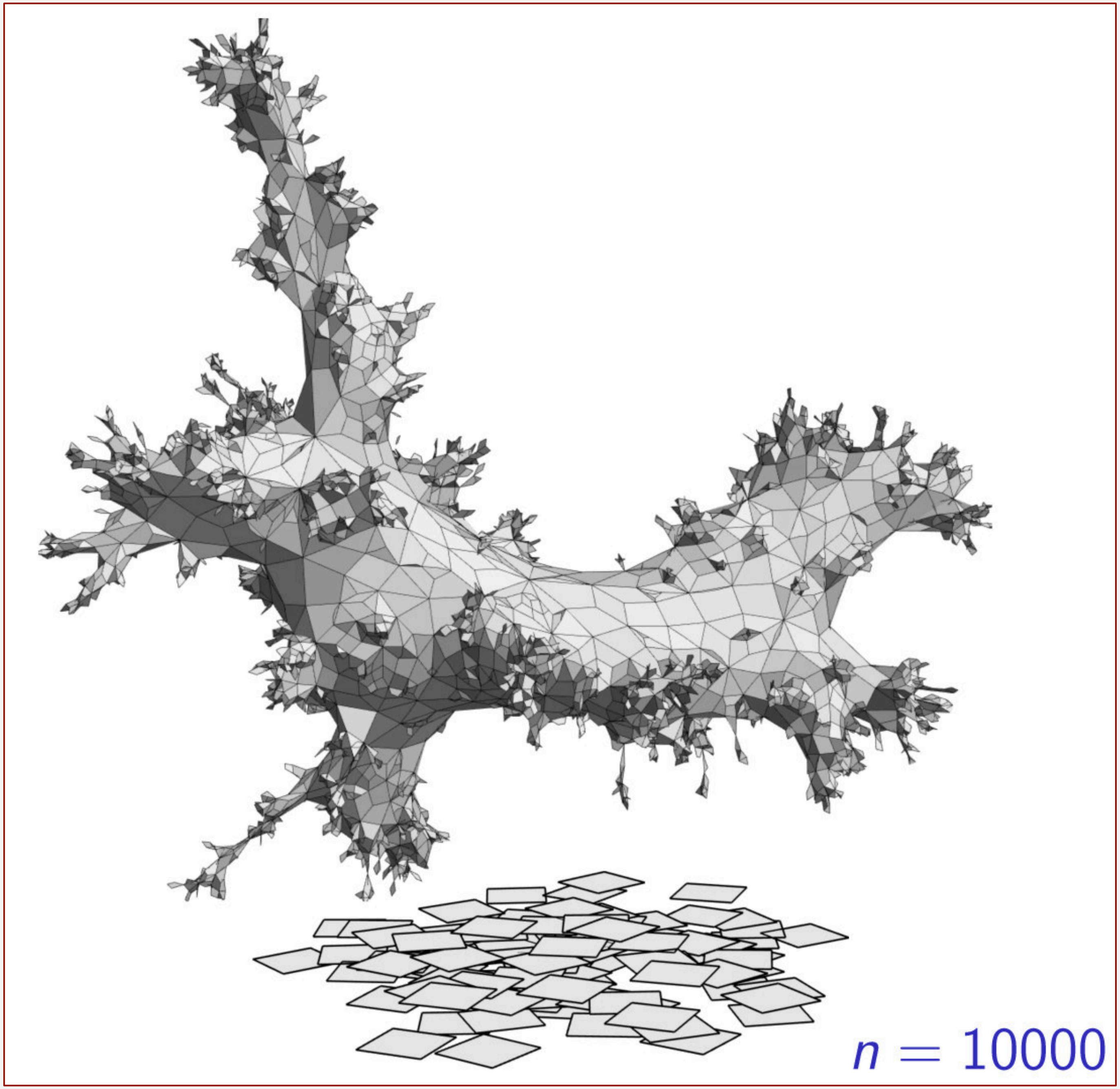

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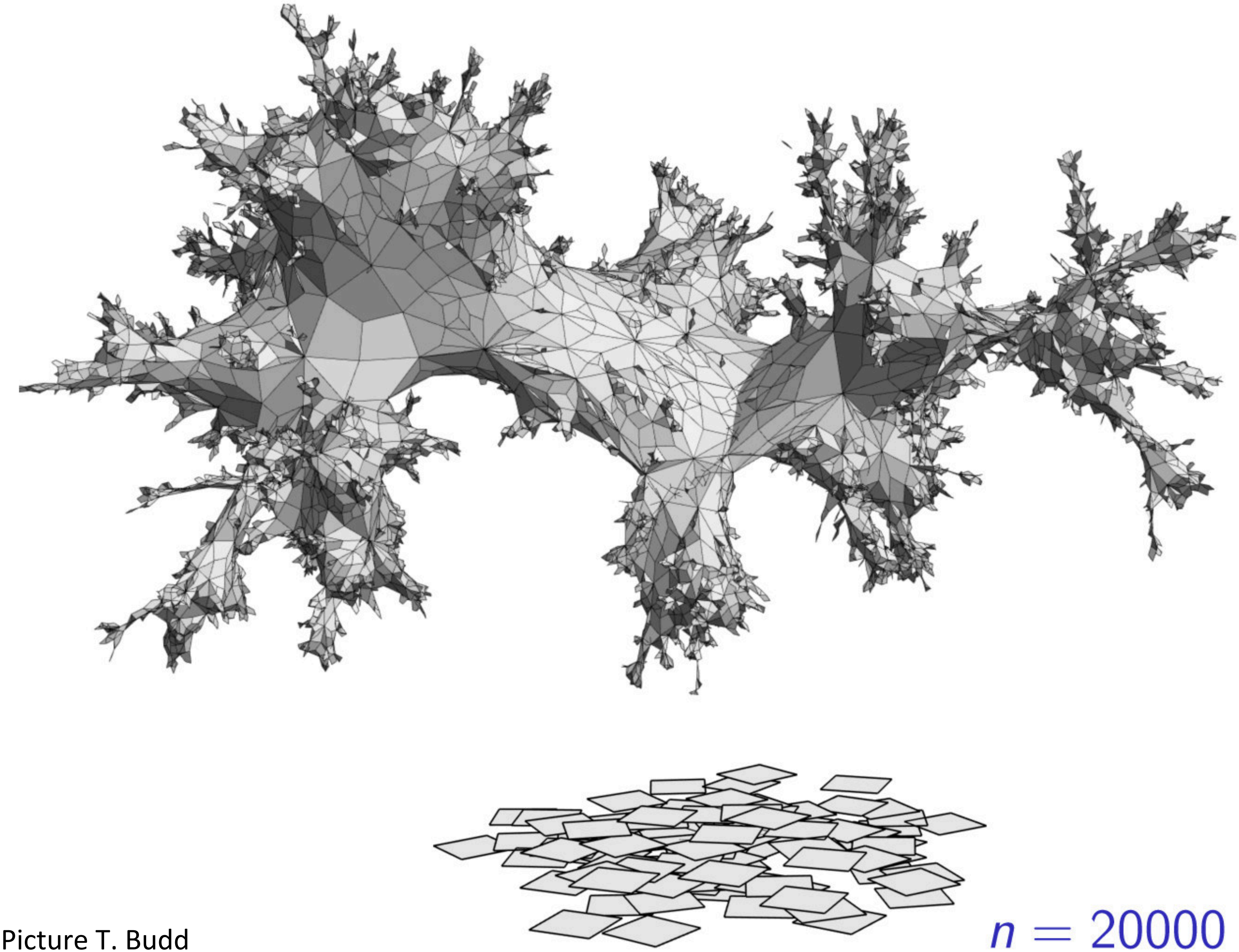
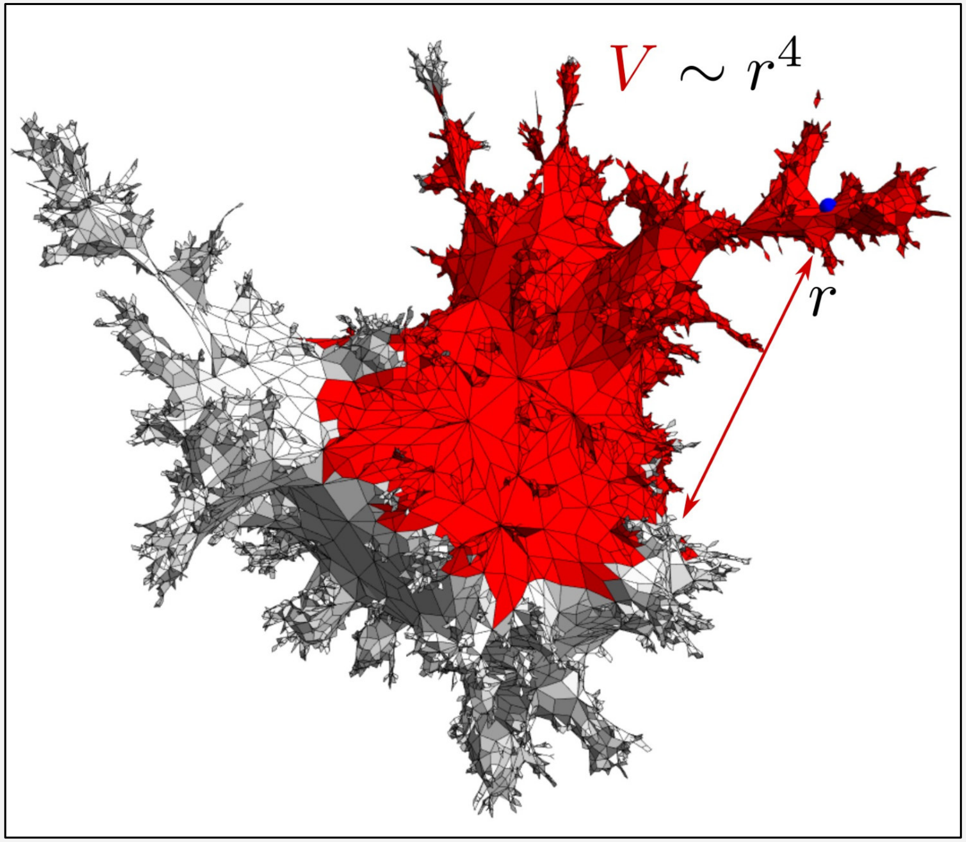


Picture T. Budd

Random maps of large size exhibit common features → Universality classes

	Asymptotic enumeration	Typical diameter
Rooted plane trees	$N_T(n) \sim k_T\, r_T^{-n}\, n^{-3/2}$	$n^{1/2}$ → $d_H=2$
Rooted planar maps	$N_M(n) \sim k_M\, r_M^{-n}\, n^{-5/2}$	$n^{1/4}$ → $d_H=4$



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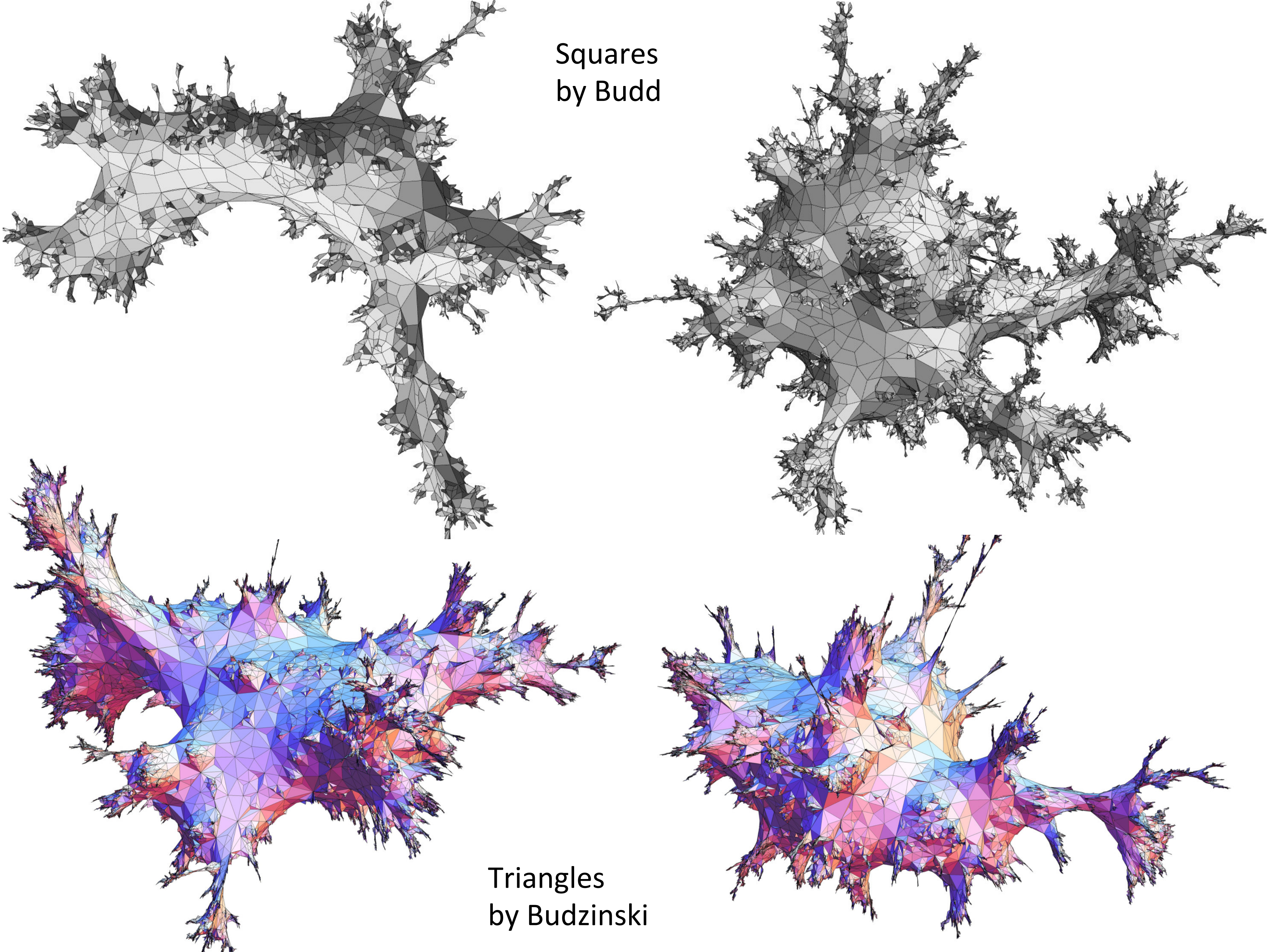

Squares
by Budd
Triangles
by Budzinski

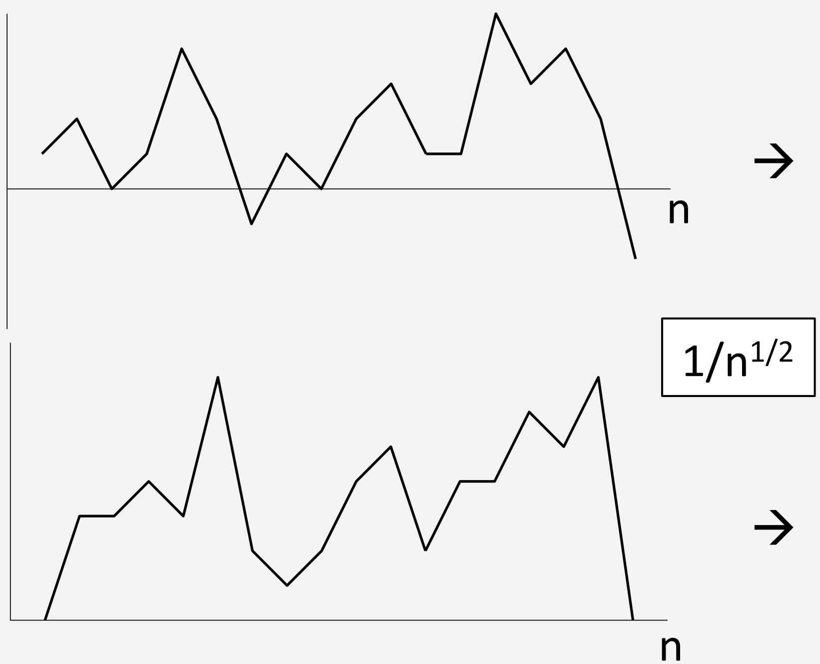
2 – Notion of scaling limit

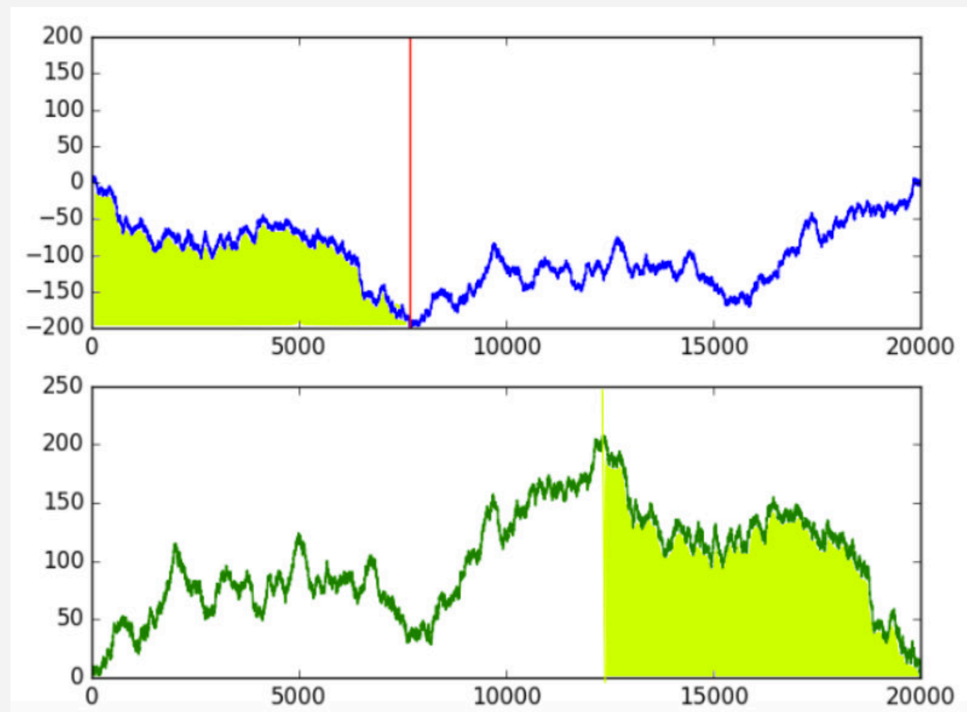
Random walks and excursions: Donsker's theorem

Random walk $S_n = \Sigma_i X_i$, where $X_1, \ldots, X_n$ are i.i.d. random variables with mean 0 and variance 1

$W_n(t) = S_{[nt]} / n^{1/2}$ t in [0,1]

→ W_n converges in distribution to a standard Brownian motion as $n \to \infty$





Picture O. Safsari

Scaling limit:

Take a random map of size n
+ A notion of distance *d* on the maps (e.g. *graph distance* but not necessarily)
→ Random metric space G_n

Suppose that the diameter behaves asymptotically as $d(G_n) \sim_{n \to \infty} d_n$ (e.g. $d_n = n^{\alpha}$), and *normalize* the distances in the map by d_n.

→ *Limit of the random metric space* $(G_n, d/d_n)$ *is* a **random compact continuum metric space**

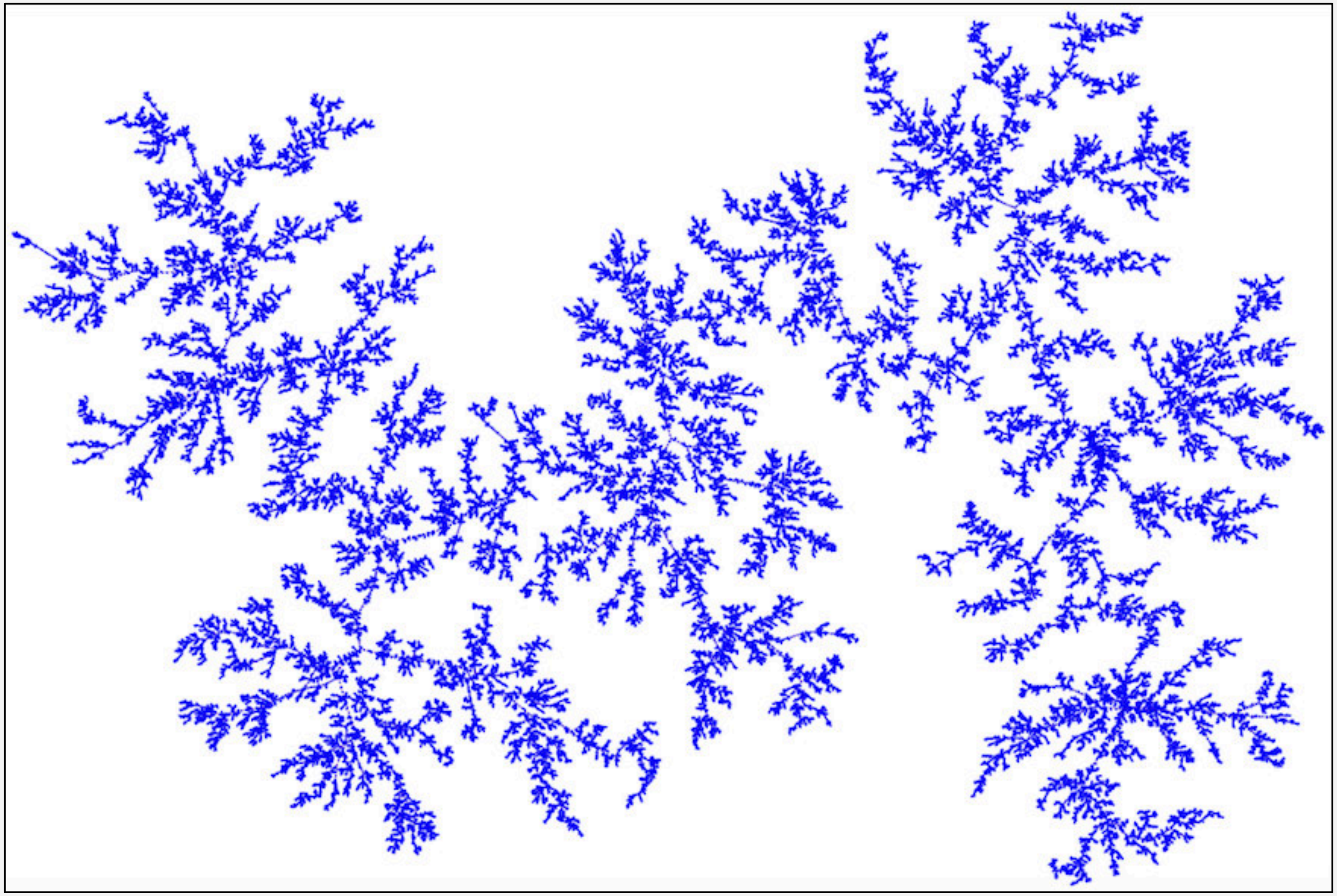
Two levels:

- Definition of the limit using limits of random walks + gluing procedure

- Limit for the Gromov-Hausdorff distance, in the space of metric spaces

→ Example: trees

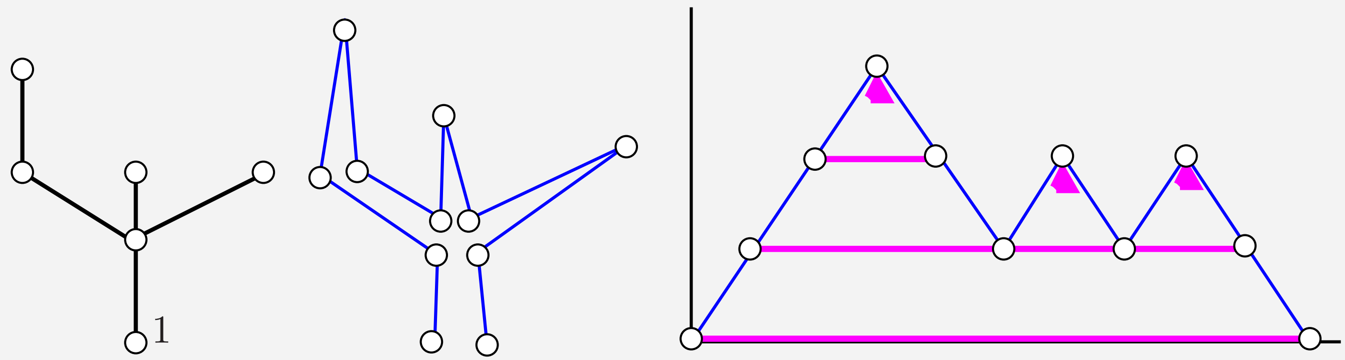
Scaling limit of uniform trees: Aldous' continuum random tree (CRT)



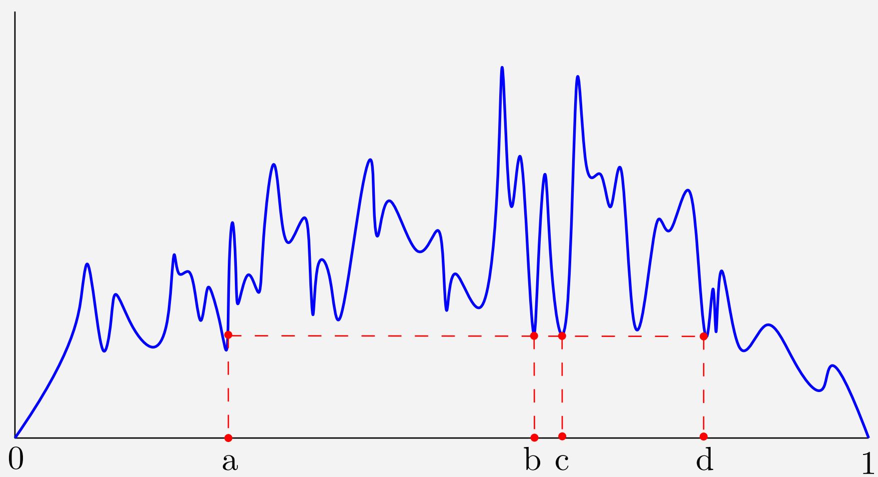
Picture:
L. Ménard
100000 edges

To define the scaling limit of random rooted plane trees, we need contour processes + real trees.

Contour process:



Real trees: notion of tree for any continuous positive function f on [0,1] with f(0)=f(1)=0



Vertices:

$$x \sim_g y \Leftrightarrow g(x) = g(y) = \min_{[x,y]} g$$

Distance:

$$D_f(x,y) = f(x) + f(y) - 2\min_{[x,y]} f$$

Scaling limit of random trees: the continuum random tree (CRT)

Consider a uniform random rooted plane tree $\mathbf{T}_n$ with n edges, $\mathbf{c}_n$ its contour process

Two levels:

1. CV (in law) of the normalized contour process to the Brownian excursion **e** (Donsker)

   $\mathbf{c}_n(2nt)/n^{1/2} \to 2\mathbf{e}(t)$

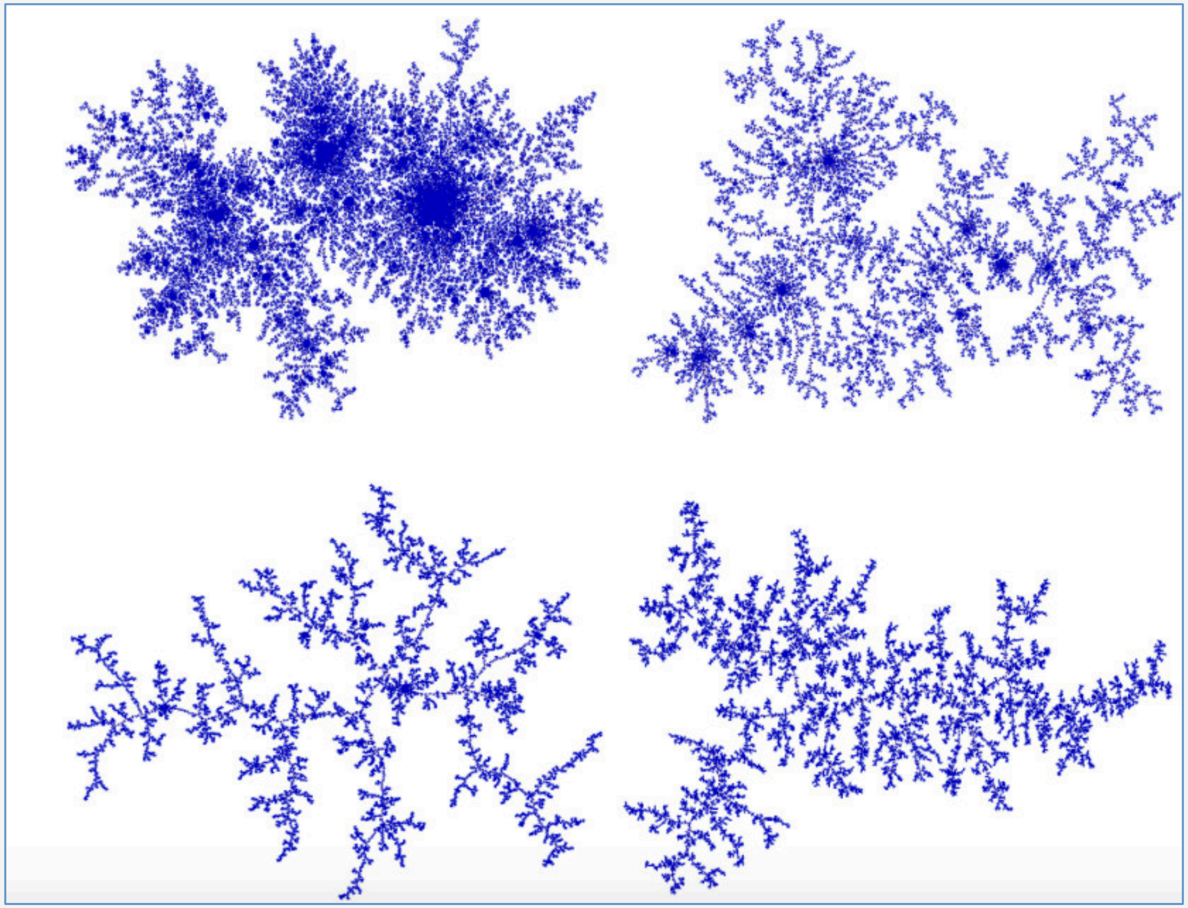
→ Definition of the CRT as real tree with contour 2**e**, and CV of (uniform) discrete random trees to the CRT in the space of real trees. [Aldous 97]

2. CV in law of ($\mathbf{T}_n$, $d/n^{1/2}$) to the CRT in the space of metric-spaces for the Gromov Hausdorff distance [Aldous 97, Le Gall 10]

Other scaling limits of ``non-uniform'' random trees

e.g. α-stable trees.

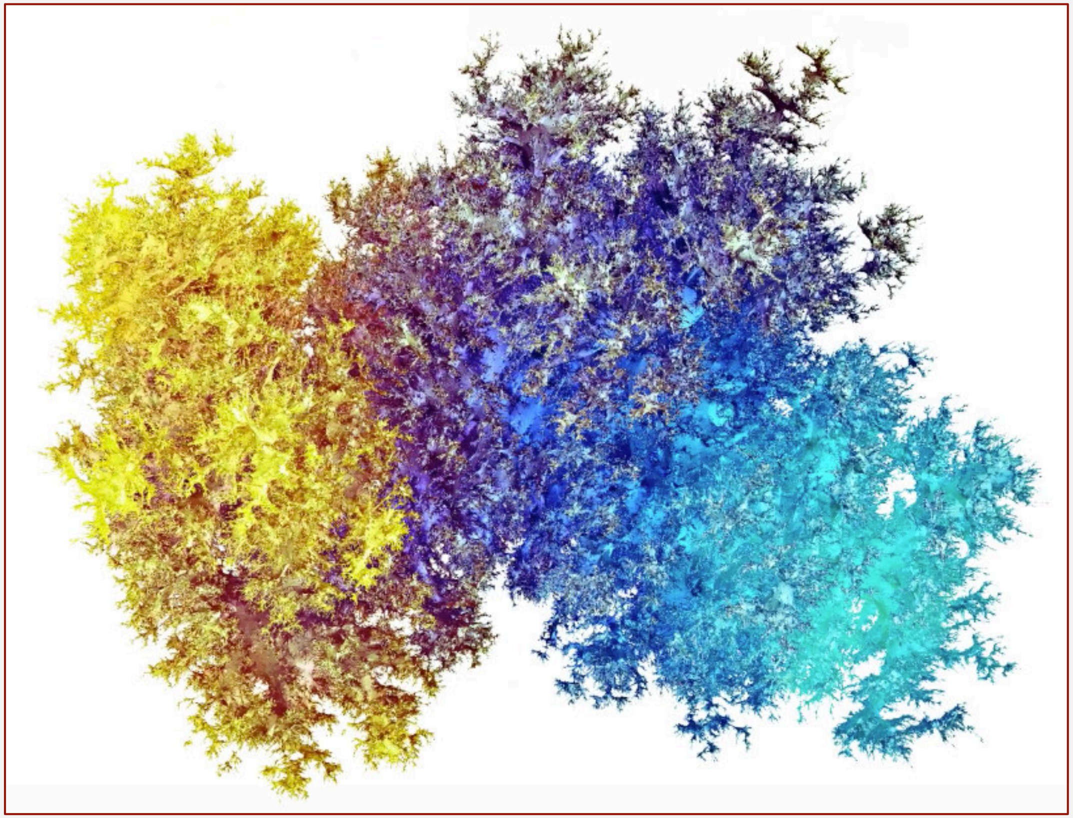
For α in (1, 2), consider Galton-Watson trees with offspring distribution $\eta(k)$ of mean 1 and such that $\eta(k) \sim Ck^{-1-\alpha}$ as $k \to \infty$. Normalizing the graph distance by $n^{1-1/\alpha}$, they converge towards a compact random metric space called the α-stable tree, of Hausdorff dimension $d_H=\alpha/\alpha\text{-}1$ in $(2,\infty)$.



Approximations of α-stable trees for α=1.1, 1.5, 1.9, and 2. The α=2 case corresponds to the CRT.
Pictures: Kortchemski.

→ Example: planar maps

Scaling limit of uniform random maps: the Brownian map



Movies by Benedikt Stufler

Scaling limit of uniform random maps: the Brownian map

Two levels:

1. Random planar maps ⇔ pairs of random trees + gluing procedure + distance *(details later)*.

→ Definition of the Brownian map as random real trees + limit of gluing procedure + limit of distance. [Marckert, Mokkadem 2006]

2. CV in the space of metric-spaces in the Gromov-Hausdorff sense:

[Le Gall 13; Miermont 13; Bettinelli et al. 14; Abraham 16; Addario-Berry & Albenque 19]

Consider a uniform random rooted planar map $\mathbf{M}_n$ with n faces and the graph distance d normalized by $n^{1/4}$. Then ($\mathbf{M}_n$, $cd/n^{1/4}$) converges in law to the the Brownian map for the Gromov-Hausdorff distance.

The same is true (for a different *c* but the *same* distance on the Brownian map) for any p-angulation and for bipartite maps.

Scaling limit of uniform random maps: the Brownian map

Different notions of dimensions…

- Hausdorff dimension 4 (a.s.) [Kawai *et al*. 93; Ambjørn & Watabiki 95; Le Gall 07]
- Homeomorphic to the 2-sphere (a.s.) [Le Gall & Paulin 08; Miermont 08]
- Spectral dimension 2: [Ambjørn *et al.* 98; Rhodes & Vargas 13, Gwynne & Miller 17]

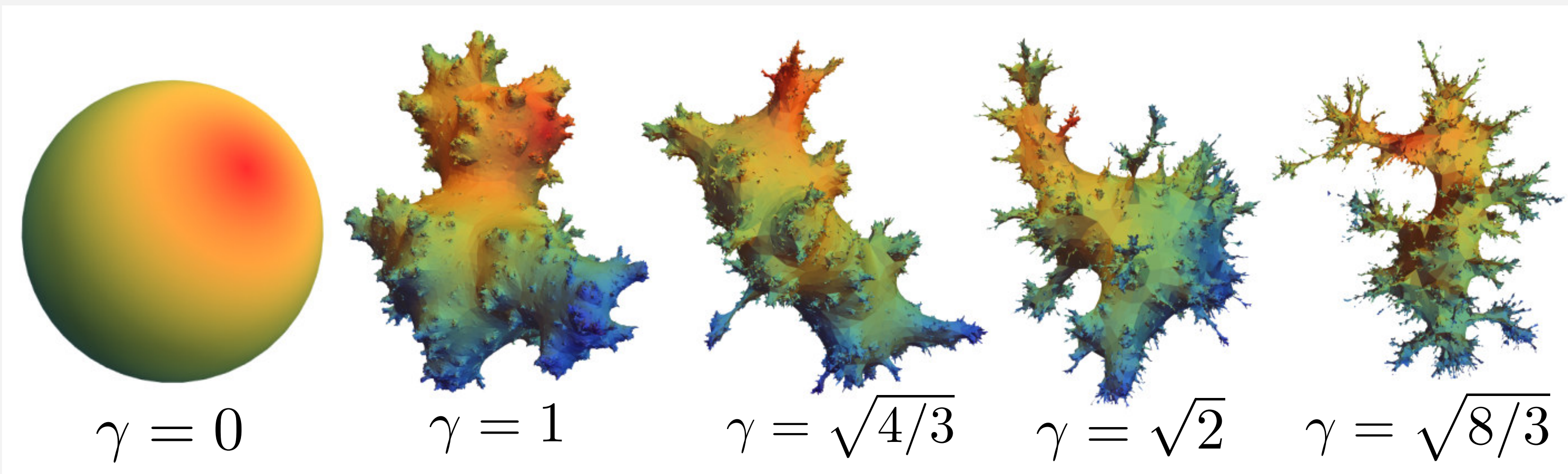
Other universality classes of ``non-uniform'' random spheres

Without entering into details:

- Stable spheres [Le Gall, Miermont 09]. Scaling limits of random planar maps with large faces. Distances ~$n^{1/2\alpha}$ for α in (1,2), Hausdorff dimension 2α.

- Statistical physics models on random planar maps (Ising, tree-decorated, bipolar orientations…)

Pic. T. Budd



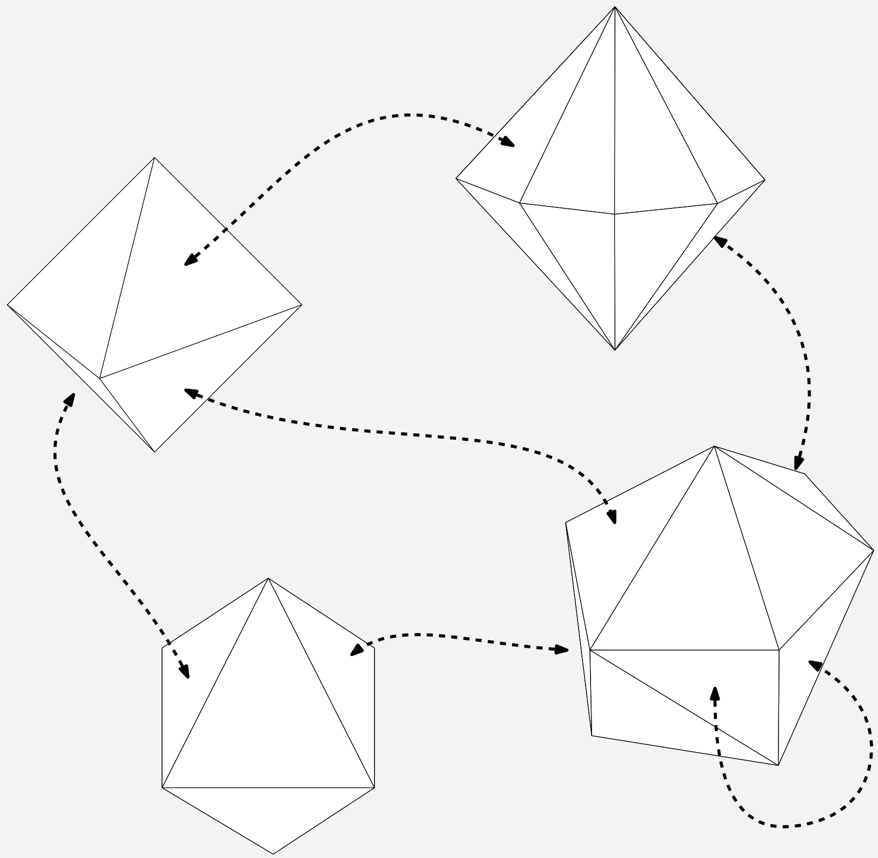
Maps selected according to number of spanning trees for $\gamma=\sqrt{2}$, Ising for $\gamma=\sqrt{3}$, uniform for $\gamma=\sqrt{8/3}$…

- Liouville quantum gravity measures on the unit sphere, equivalent to the previous kind.

3 – The problem in higher dimensions

We want to build a compact random uniform continuum volume.

→ Natural to start from random gluings of n ``polytopes'' (tetrahedra…) with *uniform* distribution and topology of the 3-sphere:



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[numerical: Ambjørn, Jurkiewicz 95]

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But this leads to the continuum random tree as a scaling limit (dimensional reduction)
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In 2D, spherical topology and maximizing the curvature is equivalent (Gauss-Bonet) but not in higher D

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→ No known way of producing anything else than random continuum **trees** or **surfaces** from random gluings of building blocks.

→ **We can try to explore new universality classes of random geometry by taking limits of well-motivated more general random graphs.**

→ We then face the difficult question of characterizing what we want…

What are we looking for?

- Ability to define a scaling limit (diameter grows as a power law)?
- Typical diameter smaller than $n^{1/4}$?
- Finite Haus. dim., larger than 4? (we have 2 for trees, 4 for planar maps)

- Spectral dimension, larger than 2? (we have 4/3 for trees, 2 for planar maps)

- Scaling limit: not a tree, not a surface…
- Locally looks like a 3-ball
- Well defined topology (topological dimension 3)?

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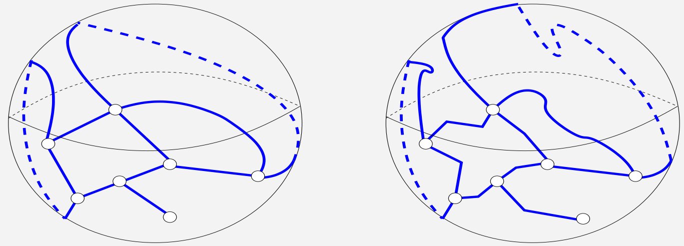
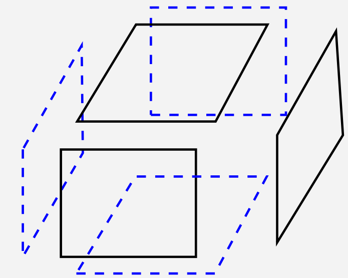
A proposal: we built a family of random graphs obtained by identifying many points on some random discrete 2-spheres [L. & Marckert 19].

There are different ways of encoding maps:



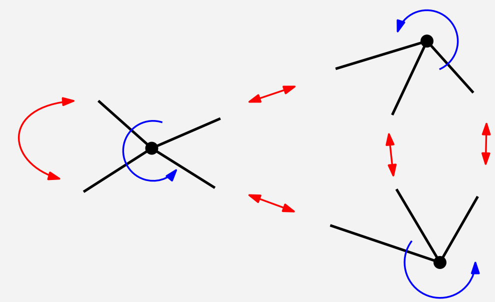
Drawing on a surface

→ Renders the fact that you have surfaces obvious



Gluings of polygons

→ Same, and uniformity



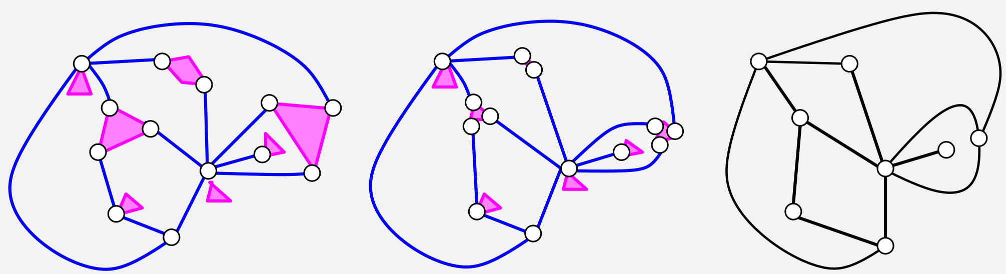
Gluings of vertices

→ Nice combinatorial encoding using permutations

No known way of producing new scaling limits in higher dimensions

A tree + some corner identifications

⇔ a map + splitting some vertices



→ Nice bijections: enumeration but also distances in some cases

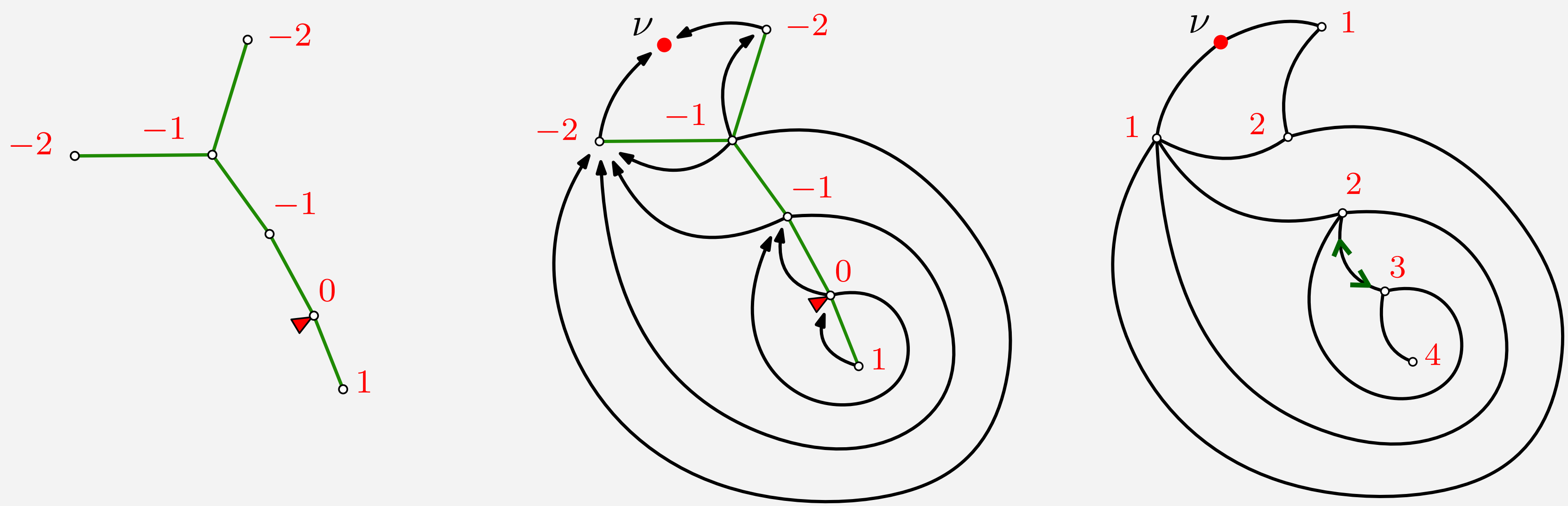
→ But: more difficult to see that it's a surface, to track the topology

→ Some other things are also less obvious e.g. invariance by change of root

4 - The Cori-Vauquelin-Schaeffer bijection:

Distances in planar maps and construction of the Brownian map

Cori-Vauquelin-Schaeffer's bijection between labeled trees and (rooted pointed) planar quadrangulations:

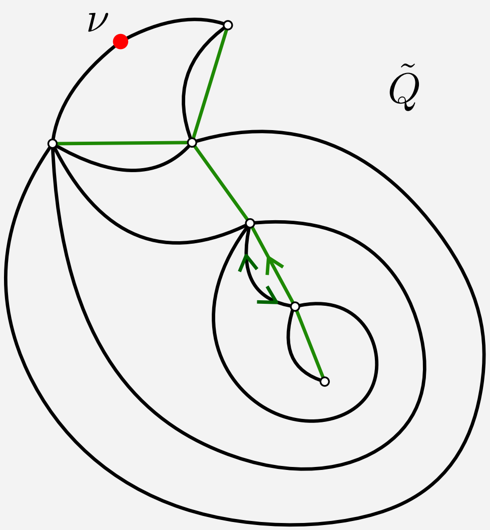


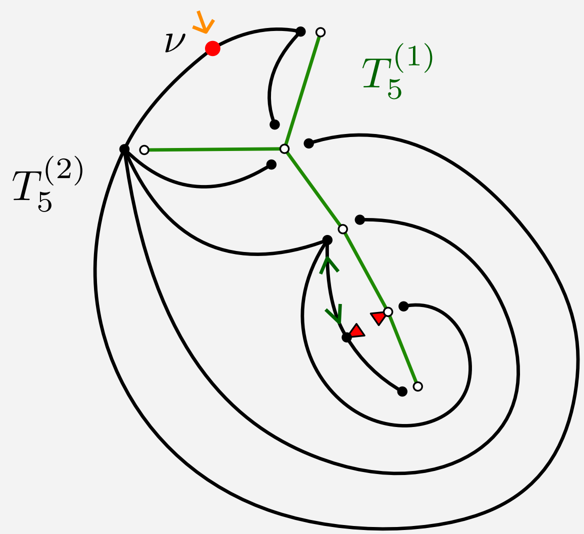
The CVS bijection is a powerful tool giving control on the distances in planar maps: it allows proving that the diameter of random planar quadrangulations is ~$n^{1/4}$.

It also allows constructing the Brownian map.

Cori-Vauquelin-Schaeffer's bijection between labeled trees and (rooted pointed) planar quadrangulations.

→ It can be reformulated in terms of identifications on a second tree:
[Marckert, Mokkadem 2006]





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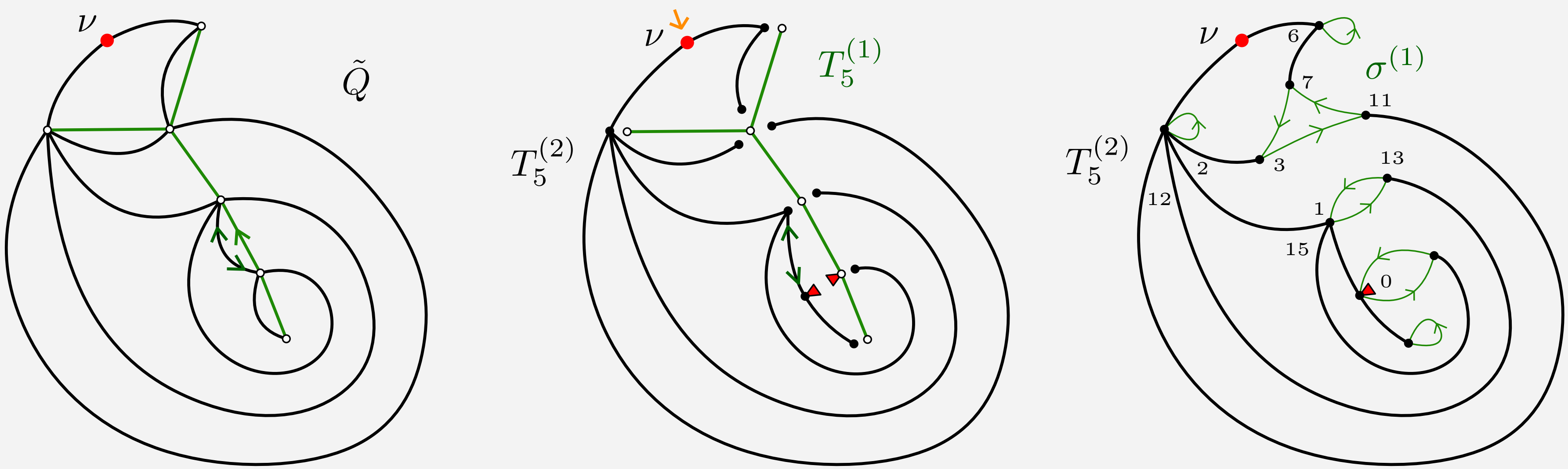
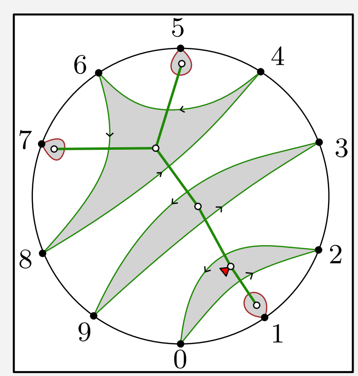
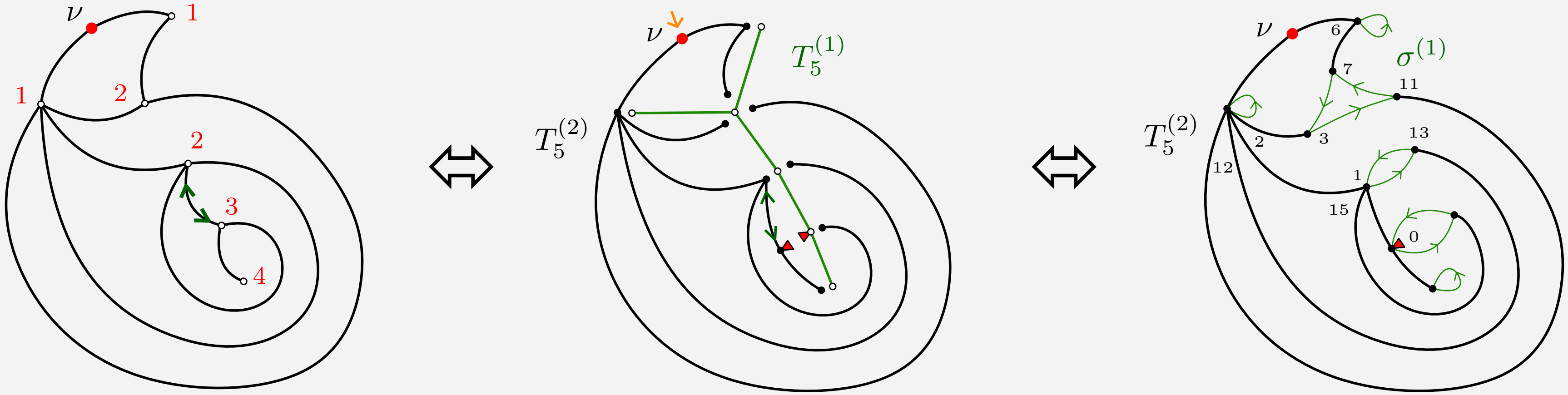


Gluing the vertices of T_2 using T_1 **preserves the distances** to the pointed vertex:

Distances to v in this second tree are the distances to v in the quadrangulation





A Uniform random planar quadrangulation (root. point.) is

- A uniform tree T_1, vertices give the vertices of the map
- A non-uniform tree T_2, diameter ~$n^{1/4}$, edges give the edges of the map
- Distance: distance in T_2 + free jumps on the vertices of T_1

Distances: typical diameter of the quadrangulation is that of T_2 : ~$n^{1/4}$

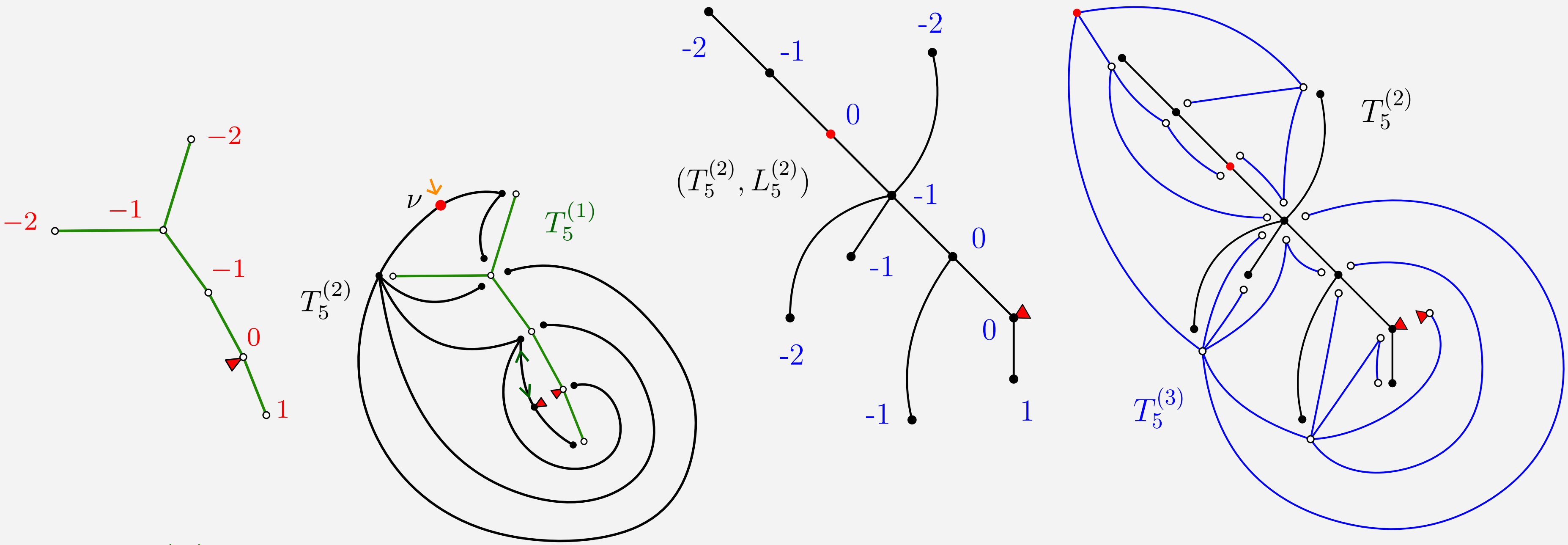
Construction of the Brownian map: Random trees → random walks

- Gluing procedure defined on the walks
- Distances defined on the walks

→ The Brownian map defined by taking their limits, using the limiting gluing procedure (T_2 quotiented by the vertices of T_1), and using the limiting distance.

4 – Random feuilletages

Iterate this construction: label the second tree, build a third tree, quotiented by the vertices of the two first trees!

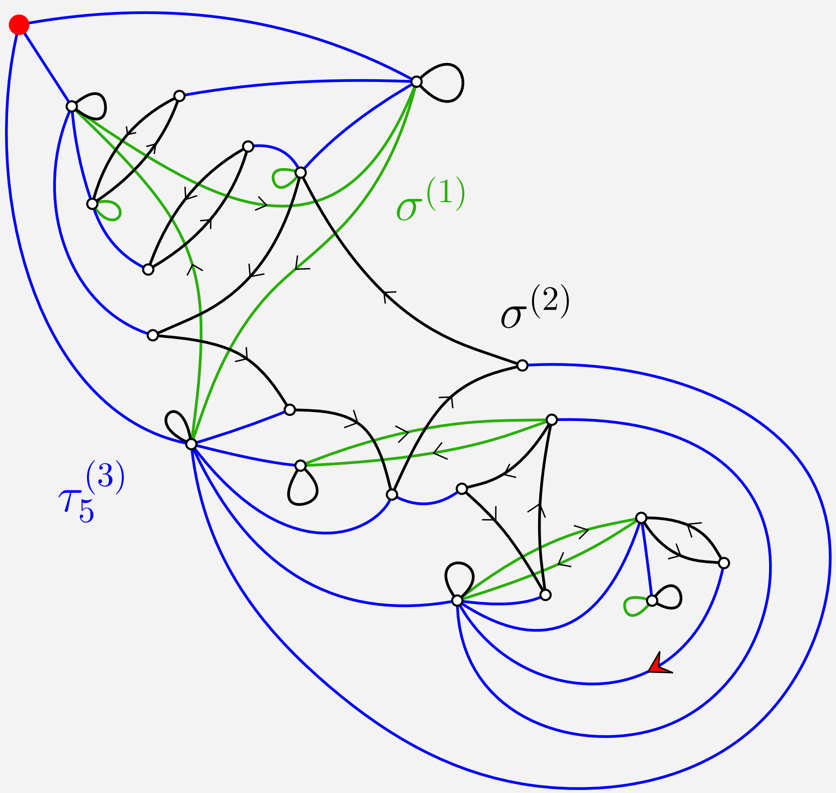


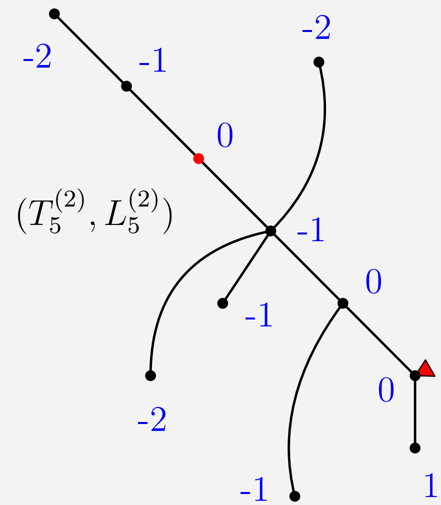
$\mathbf{T}_n^{(1)}$ $\quad$ $\mathbf{T}_n^{(2)}$ $\quad$ $\mathbf{T}_n^{(3)}$

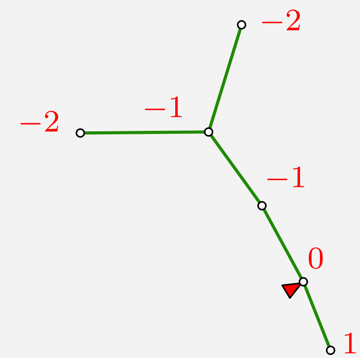
$n^{1/2}$ $\quad$ $n^{1/4}$ $\quad$ $n^{1/8}$

Typical asymptotic diameter

<u>Iterate this construction</u>: label the second tree, build a third tree, quotiented by the vertices of the two first trees!







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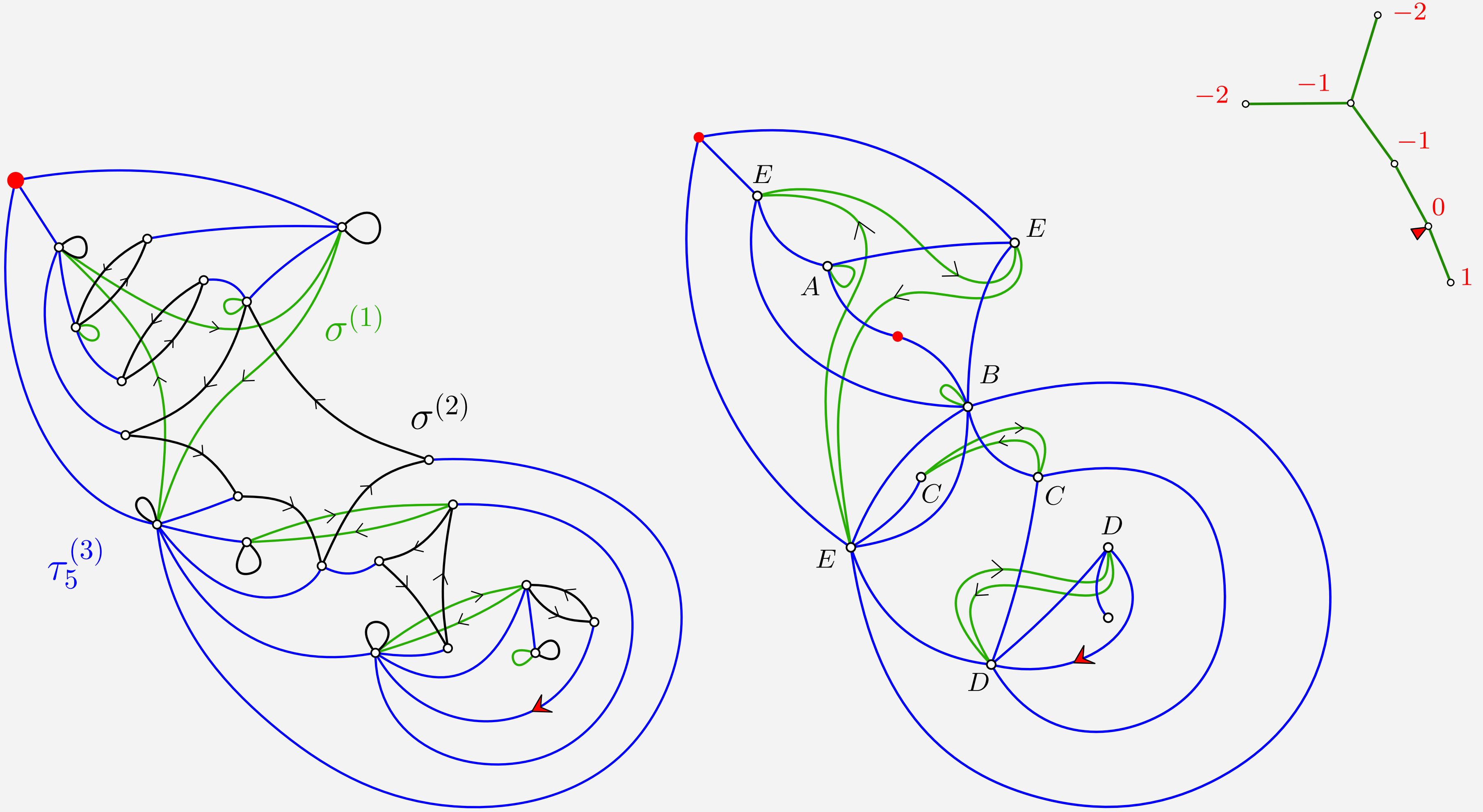
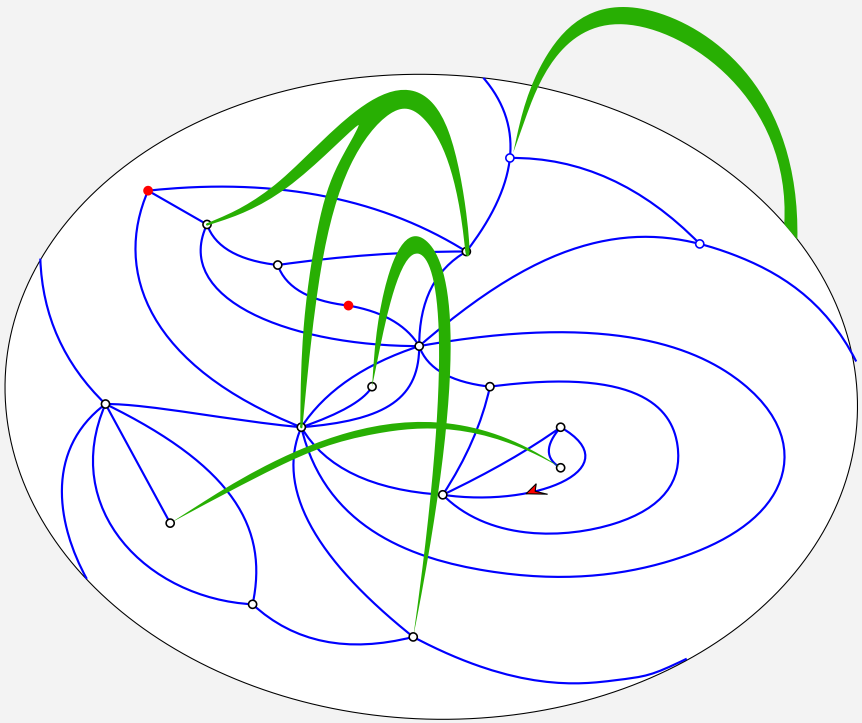
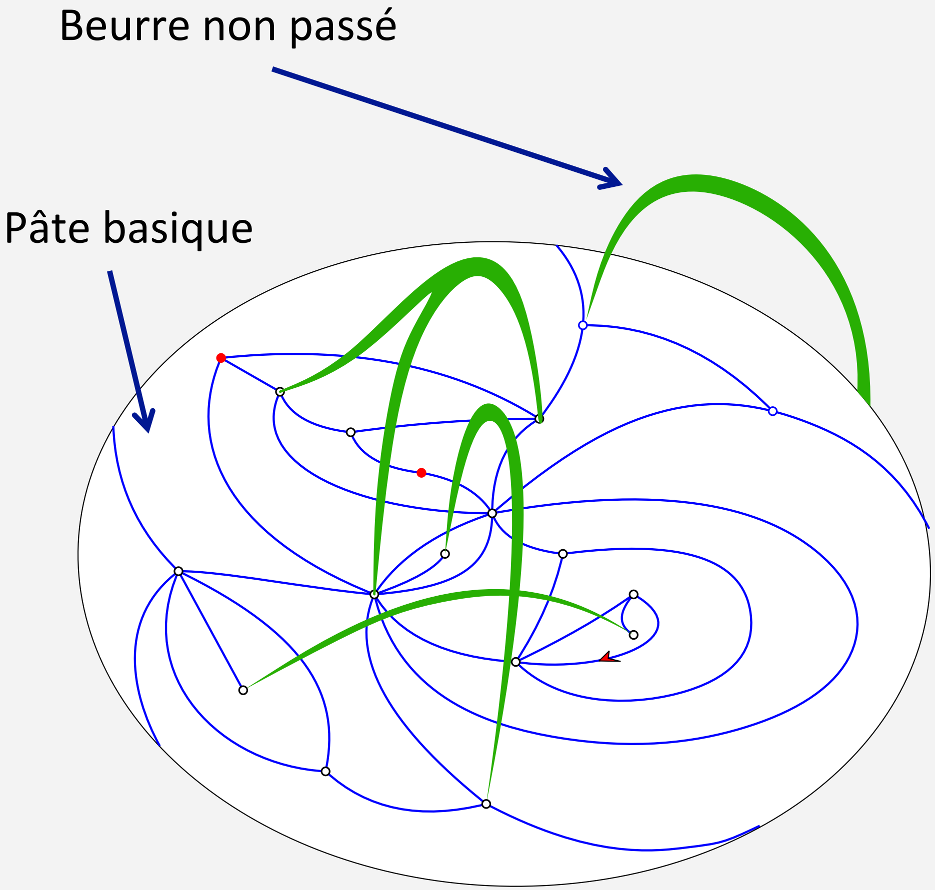




"La pâte feuilletée, ou feuilletage, est un type de pâte obtenue par abaissage et pliages successifs de couches alternant une pâte basique (farine, eau et sel) et du beurre."



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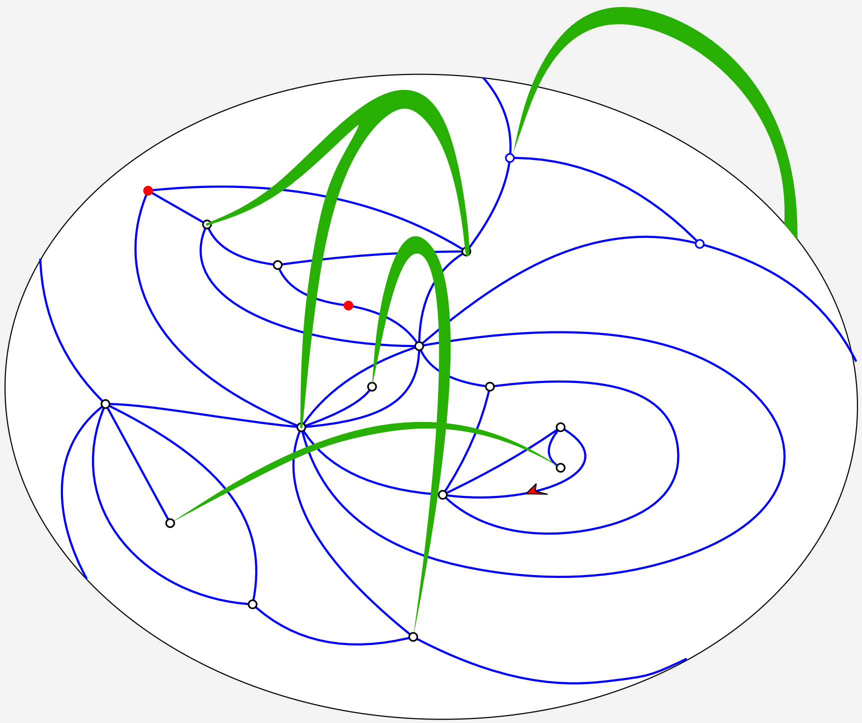
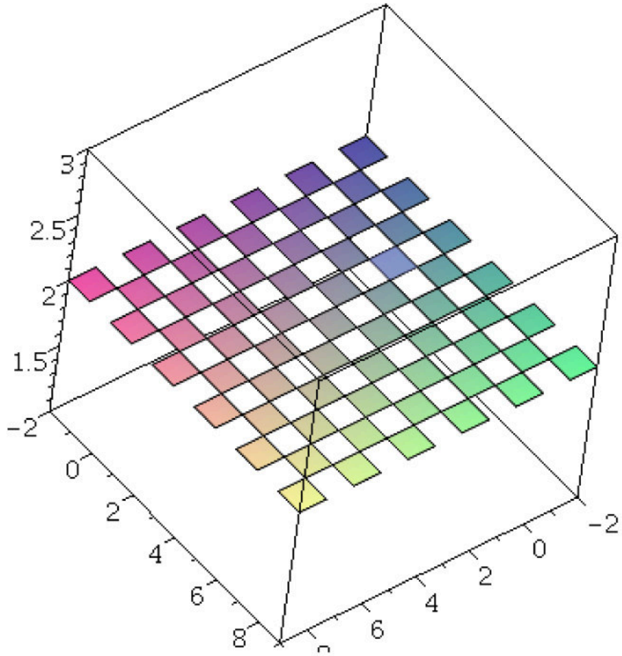


→ Sequence of random graphs with asymptotic diameter $O(n^{1/8})$

→ Asymptotically not going to give random trees or surfaces of any genus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1.5
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0
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4
6
8
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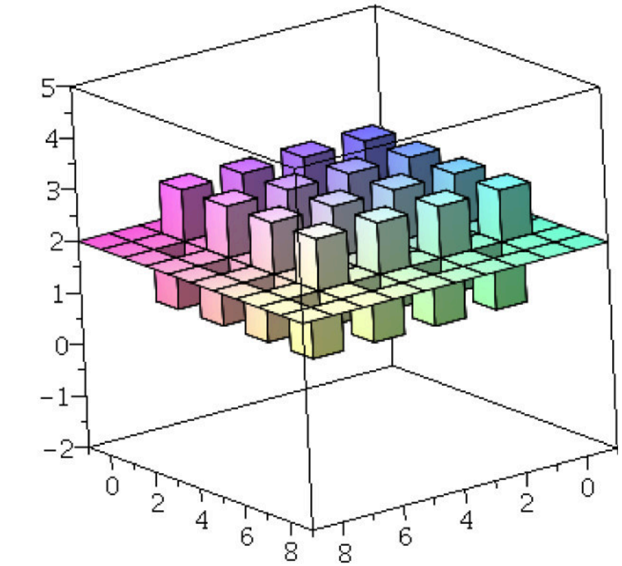
0
2
4
6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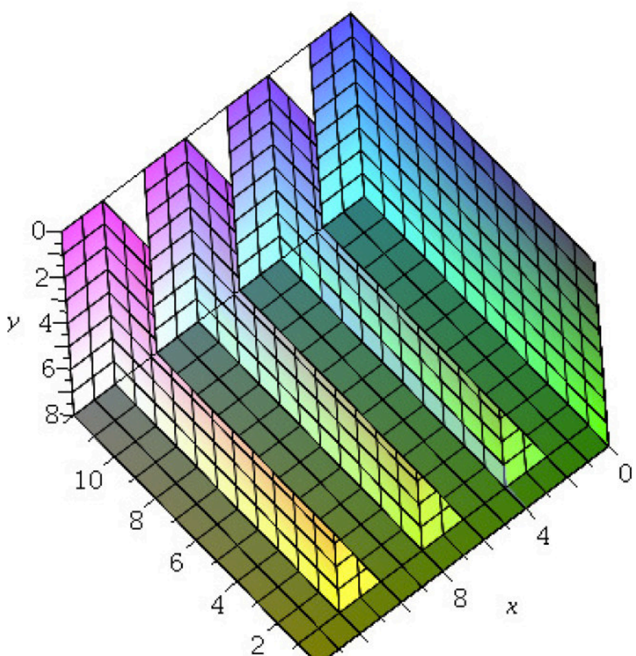
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y
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2
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x
4
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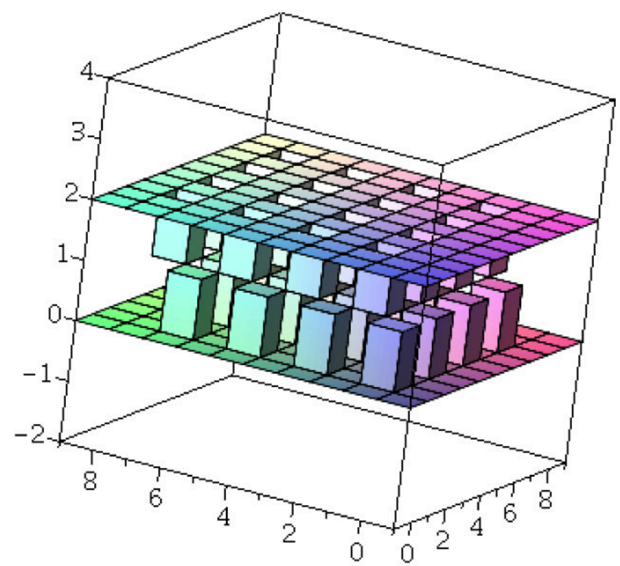
4
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2
1
0
−1
−2
8
6
4
2
0
0
2
4
6
8

→ Iteration

D-Random feuilletages:

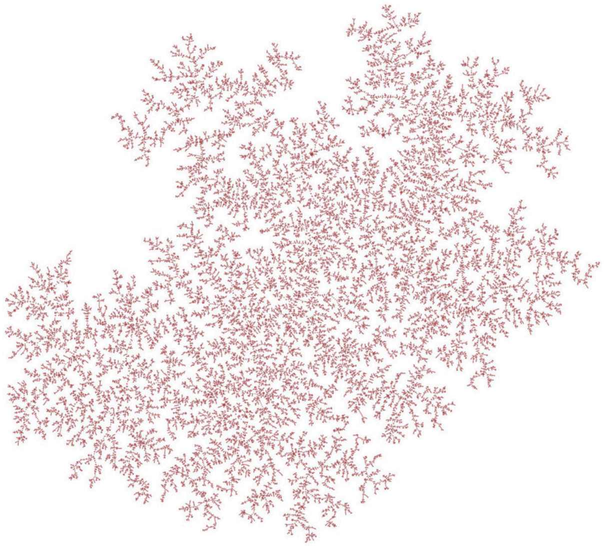
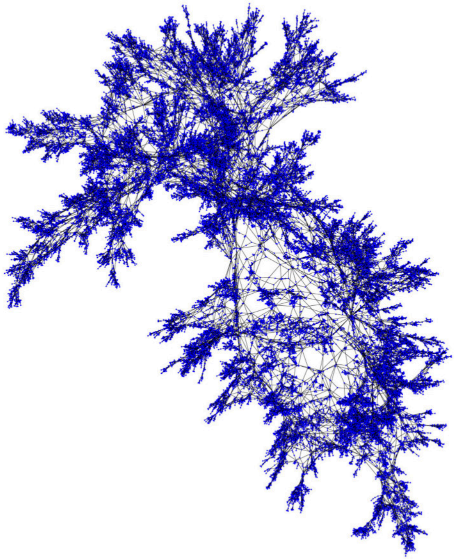
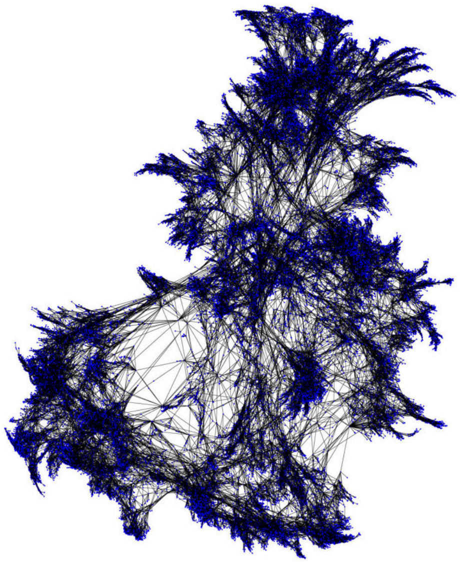
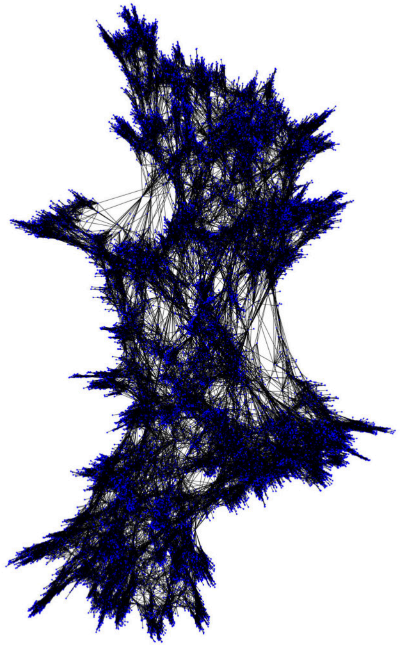
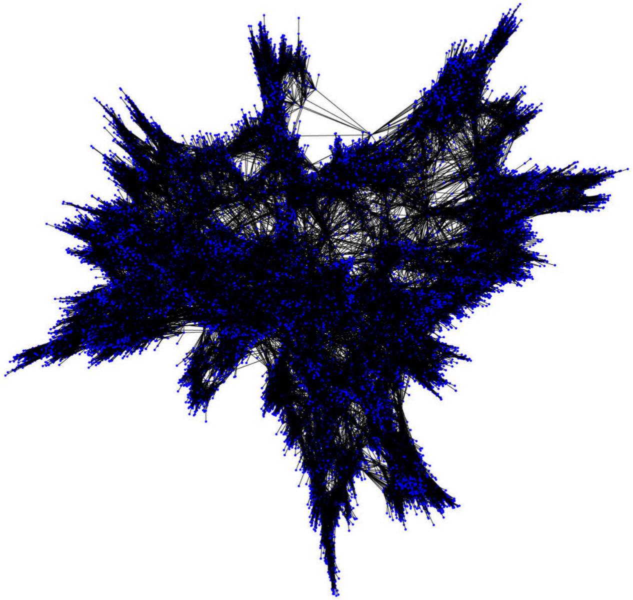
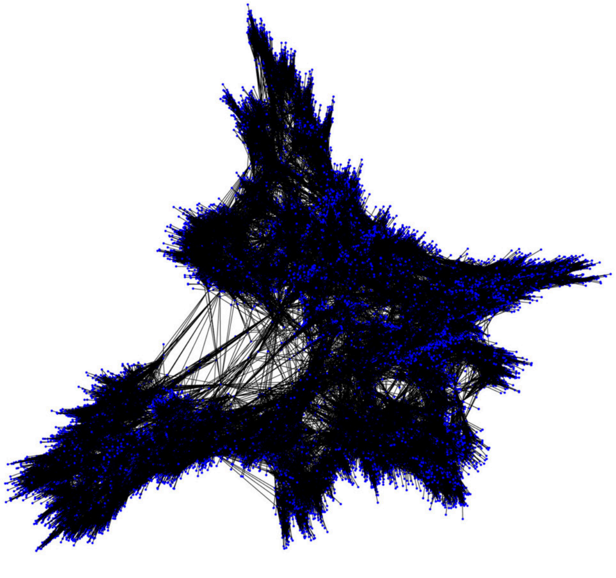
→ Iterative construction to obtain a <u>family indexed by a parameter *D*</u>

- ***D=1***: discrete objects are uniform rooted plane trees with n+1 vertices.
  CV to the continuum random tree in terms of Gromov-Hausdorff.
  Typical diameter ~ $n^{1/2}$ (Hausdorff dimension 2).

- ***D=2***: discrete objects are uniform root. point. planar quadrangulations with n+2 vertices.
  CV to the Brownian map in terms of Gromov-Hausdorff.
  Typical diameter ~ $n^{1/4}$ (Hausdorff dimension 4).

- ***D=3***: discrete objects obtained by a series of foldings of a discrete 2-sphere with n+3 vertices, rooted and pointed 2 times.
  CV to a continuum space with typical diameter $O(n^{1/8})$ (Hausdorff dimension 8 or more).

- ***D***: discrete objects obtained by *D-2* series of foldings of a discrete 2-sphere with n+D vertices, rooted and pointed D-1 times.
  CV to a continuum space with Hausdorff dimension 2^D or more).

Asymptotic enumeration:

From the CVS bijection, for rooted pointed and non-pointed planar quadrangulations:

$$m_n^{\bullet(2)} = 2 \times 3^n C_n \qquad \text{and} \qquad (n+2)m_n^{(2)} = 2 \times 3^n C_n$$

$$m_n^{\bullet(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2 - 1} \qquad \text{and} \qquad m_n^{(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2 - 2}$$

$$\gamma_2 = -1/2$$

Not entirely clear for instance how the feuilletages are pointed, but in any case, this factor is asymptotically of order n^{D-1}, so that we have asymptotically:

$$m_n^{\bullet(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D + D - 3} \qquad \text{and} \qquad m_n^{(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D - 2}$$

$$\propto 4 \times 3^{\sum_{j=0}^{D-2} 2^j}$$

$$\gamma_D = \frac{3}{2} - D$$

This generalizes the $\gamma_1 = 1/2$ for trees and $\gamma_2 = -1/2$ for planar maps and more generally:

$$\gamma_{2,L} = \frac{3}{2}L + \frac{1}{2} \qquad \gamma_{2,g} = \frac{5}{2}g - \frac{1}{2}$$

A combinatorial encoding:

$$\mathcal{F}_N[D] = \left(C_N, \sigma^{(D)}, \ldots, \sigma^{(1)}\right)$$

$C_N = \{0 < 1 < \ldots < N-1\}$

$\sigma^{(j)}$ nested non-crossing partitions on C_N

$\sigma^{(j)}$ + parity gives a tree ($\mathbf{T}_n^{(j)}$ of scale $n^{1/2^j}$, for the right distributions)

$\left(\sigma^{(j)}, \sigma^{(j-1)}\right)$+ parity gives a planar map (random planar quad. of scale $n^{1/2^j}$ for the right dist.

Questions…

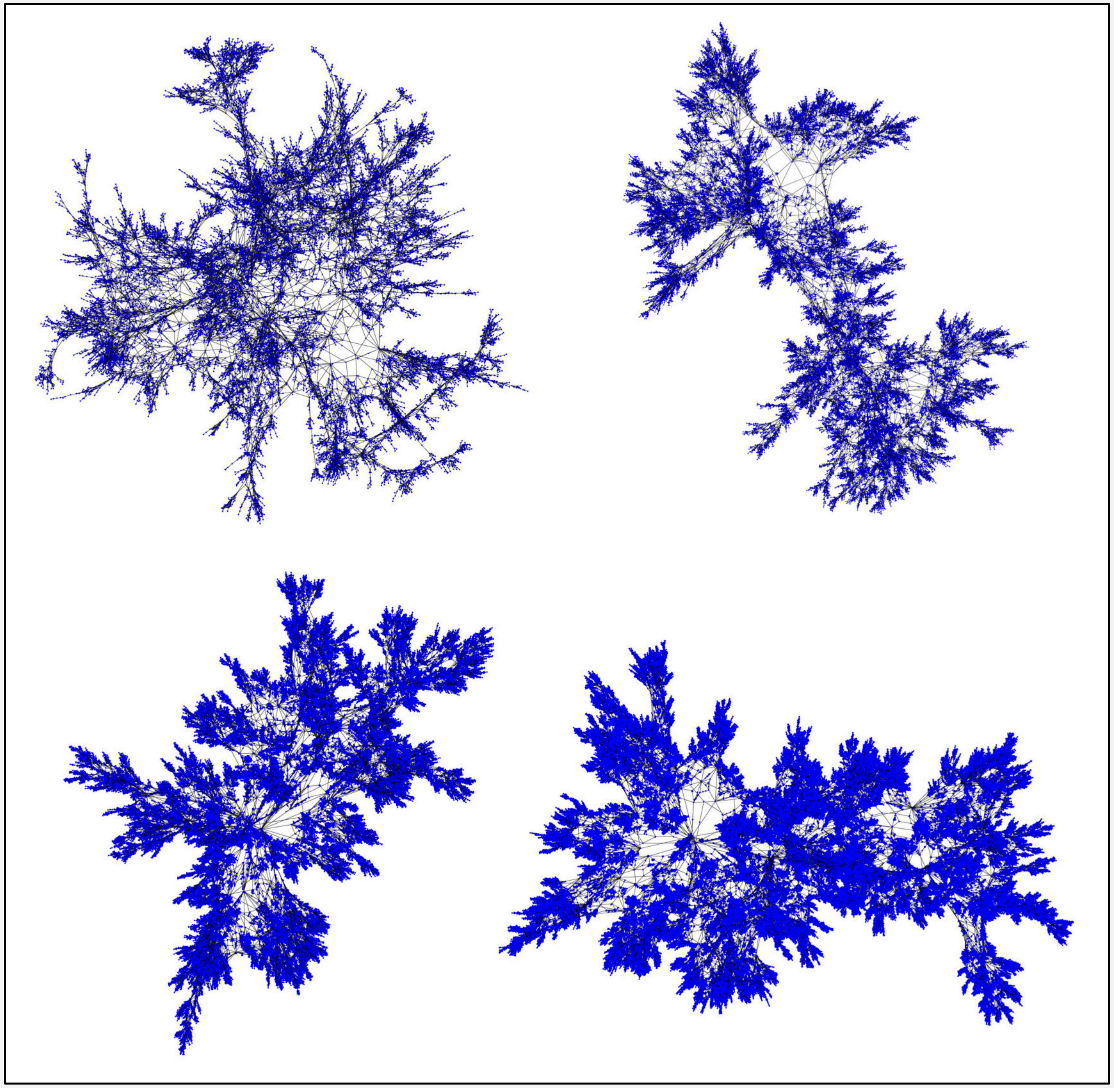
→ We conjecture that the asymptotic diameter is of order $n^{1/2^D}$ (we know it is ≤) and that the Hausdorff dimension is *2^D (we know it is ≥)* $\ldots$ needs to be proven.

→ Convergence in the sense of Gromov-Hausdorff (second level)?

→ What can we say about the topology? (hard)

→ Can we obtain the same scaling limit from a model of random *D*-dimensional triangulations? (Universality class?)

→ Spectral dimension?



Thank you!