

# The Kapovich-Polterovich question

joint work with Vincent Humilière and Sobhan Seyfaddini

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# The question

$(M, \omega)$  symplectic manifold.

## Hamiltonian diffeomorphisms:

- Take  $H : [0, 1] \times M \rightarrow \mathbb{R}$ . (Hamiltonian)
- Vector field  $X_H: \omega(X_H, \cdot) = dH$ .
- Hamiltonian flow:  $\varphi_H^t$ . Hamiltonian diffeo:= time-1 map  $\varphi_H^1$ .
- $\text{Ham}(M, \omega) := \{\varphi_H^1\} \triangleleft \text{Symp}(M, \omega)$ .

## Hofer norm :

- To a Hamiltonian  $H$ , define

$$\|H\|_{1,\infty} := \int_0^1 (\max_M H_t - \min_M H_t) dt.$$

- Now for  $\varphi \in \text{Ham}$ , define

$$\|\varphi\| := \inf\{\|H\|_{1,\infty} \mid \varphi = \varphi_H^1\}.$$

**Hofer metric:** For  $\varphi, \psi \in \text{Ham}$ ,

$$d_H(\varphi, \psi) := \|\varphi^{-1}\psi\|.$$

. Defines a bi-invariant metric on  $\text{Ham}(M, \omega)$ :

- bi-invariant:  $d_H(\varphi, \psi) = d_H(\theta\varphi, \theta\psi) = d_H(\varphi\theta, \psi\theta)$ .
- $d_H(\varphi, \psi) = d_H(\psi, \varphi)$ .
- $d_H(\varphi, \psi) \leq d_H(\varphi, \theta) + d_H(\theta, \psi)$ .
- non-degeneracy:  $d_H(\varphi, \psi) = 0 \iff \varphi = \psi$ . (Hofer, Polterovich, Lalonde-McDuff)

Rather remarkable due to lack of compactness

# Basic notions from large-scale geometry

$\Phi : (X_1, d_1) \rightarrow (X_2, d_2)$  a map between metric spaces.

**Quasi-isometric embedding:** if  $\exists A \geq 1, B > 0$  s.t.

$$\frac{1}{A}d_1(x, y) - B \leq d_2(\Phi(x), \Phi(y)) \leq Ad_1(x, y) + B.$$

Eg: 1.  $\mathbb{Z} \hookrightarrow \mathbb{R}$ , 2.  $\mathbb{R} \rightarrow \mathbb{Z}, x \mapsto \lfloor x \rfloor$ .

**Quasi-isometry:**  $\Phi$  QI embedding and  $\exists C$  s.t.  $\forall y \in X_2$

$$d_2(y, \Phi(X_1)) \leq C.$$

Eg: 1.  $\mathbb{Z} \stackrel{\text{QI}}{\simeq} \mathbb{R}$ , 2.  $\mathbb{R} \not\stackrel{\text{QI}}{\simeq} \mathbb{R}^2$ , 3.  $X$  bdd  $\implies X \stackrel{\text{QI}}{\simeq} pt$ . "space viewed from far away"

# The Kapovich-Polterovich Question

Theorem (Polterovich 1998)

$\text{Ham}(\mathbb{S}^2)$  admits a QI embedding of  $\mathbb{R}$ .

Question (Kapovich-Polterovich 2006, McDuff-Salamon: Problem 21)

$\text{Ham}(\mathbb{S}^2) \stackrel{QI}{\simeq} \mathbb{R}$ ?

## Theorem (CG., Humilière, Seyfaddini; Polterovich-Shelukhin)

$$\text{Ham}(S^2) \stackrel{QI}{\not\sim} \mathbb{R}.$$

We can say more:

Recall **quasi-flat rank**:  $\text{rank}(X, d) = \max\{n : \mathbb{R}^n \xrightarrow{QI} X\}$ . QI Invariant:

$$X \stackrel{QI}{\sim} Y \implies \text{rank}(X) = \text{rank}(Y).$$

- Example:  $\text{rank}(\mathbb{R}^n) = n$ ,  $\text{rank}(G) < \infty$  for  $G$  connected finite-dim Lie group (Bell-Dranishnikov).
- We show:  $\text{rank}(\text{Ham}(S^2)) = \infty$

More:

# More about the QI type of $\text{Ham}(\mathbb{S}^2)$

Recall a metric space is **coarsely proper** if there exists  $R_0 > 0$  such that any bounded subset can be covered by finitely many balls of size  $R_0$ .

- Example: any finitely generated group, or more generally any locally compact, compactly generated group is coarsely proper. An infinite-dimensional Banach space is not coarsely proper.
- We show:  $\text{Ham}(\mathbb{S}^2)$  is not coarsely proper

Upshot:  $\text{Ham}(\mathbb{S}^2)$  is quite “large”! Polterovich-Shelukhin:  $\text{Ham}(\mathbb{S}^2)$  contains an (isometric) copy of  $C_c^\infty([0, 1])$ .

**Remark:** Polterovich-Shelukhin’s proof uses very different methods — Lagrangian spectral invariants via Mak-Smith; ours goes via Periodic Floer Homology (PFH). Interesting to compare this method with ours: some computations seem to agree!



$\Sigma$  surface of positive genus:

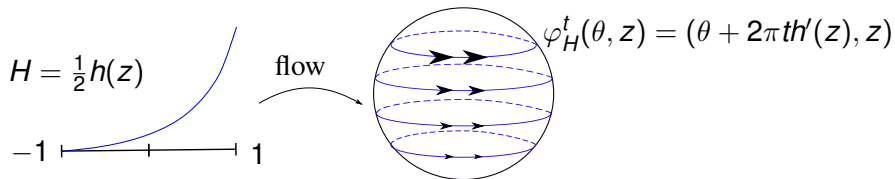
- Lalonde-McDuff:  $\mathbb{R}^n \xrightarrow{QI} \text{Ham}(\Sigma)$ , for every  $n$ . (1995)
- Polterovich:  $(C([0, 1]), \|\cdot\|_\infty) \xrightarrow{QI} \text{Ham}(\Sigma)$ . (1998).
- Other results: Polterovich-Shelukhin (2014), Alvarez-Gavela-Kaminker-Kislev-Kliakhandler-Polterovich-Rigolli-Rosen-Shabtai-Stevenson-Zhang (2016).

More general manifolds: Entov-Polterovich, Kawamoto, Khanevsky, Lalonde-Polterovich, Lalonde-McDuff, McDuff, Ostrover, Polterovich-Shelukhin, Py, Schwarz, Usher, Stojisavljevic-Zhang, ...

Open question: Is  $\text{Ham}(S^2) \stackrel{QI}{\approx} \text{Ham}(\Sigma)$ ?

# Idea of the proof

**Monotone twist Hamiltonians:**  $H : \mathbb{S}^2 \rightarrow \mathbb{R}$  of the form  $H(\theta, z) = \frac{1}{2}h(z)$ , where  $h \geq 0, h' \geq 0, h'' \geq 0$ .



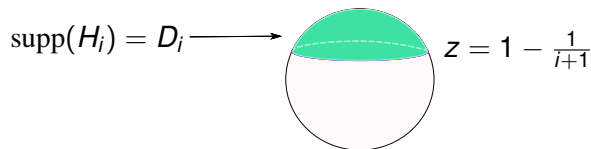
# Our QI embeddings

To prove our first theorem, suffices to produce QI embedding of  $\mathbb{R}_{\geq 0}^n = \{(t_1, \dots, t_n) : t_i \geq 0\}$ .

**Our embedding:**

Discs:  $D_i = \{(z, \theta) : 1 - \frac{1}{i+1} \leq z \leq 1\}$ . Note:  $D_i \supset D_{i+1}$ ,  $\text{Area}(D_i) = \frac{1}{2(i+1)}$ .

$H_i$ : monotone twists such that  $\text{supp}(H_i) = D_i$ .



Define

$$\Phi : \mathbb{R}_{\geq 0, \|\cdot\|_\infty}^n \rightarrow \text{Ham}(\mathbb{S}^2, d_{\text{hofer}}), (t_1, \dots, t_n) \longrightarrow \varphi_{H_1}^{t_1} \circ \dots \circ \varphi_{H_n}^{t_n}.$$

Claim:  $\Phi$  is a QI embedding.

Hard part of the proof of the claim: bounding the Hofer distance from below.  
To do this, we use Hutchings' Periodic Floer Homology (PFH) to construct

$$\mu_d : \text{Ham}(\mathbb{S}^2) \longrightarrow \mathbb{R},$$

every  $d \in \mathbb{N}$ .

We show:

- The  $\mu_d$  are **Hofer Lipschitz**, eg

$$|\mu_d(\varphi) - \mu_d(\psi)| \leq C_d d_H(\varphi, \psi), C_d = 2d$$

so bound Hofer's distance from below.

- They can be computed for Monotone twists.

# Comparison with previous work

We also used PFH spectral invariants to show  $\text{Homeo}_c(D^2, \omega)$  is not simple in previous work.

**New challenge here:** need invariants that depend only on the time-1 map, not the choice of Hamiltonian. In the disc case, can restrict to Hamiltonians that vanish near boundary; can't do this here.

Our solution to get around this: homogenize and restrict to mean normalized Hamiltonians.

# More about the proofs

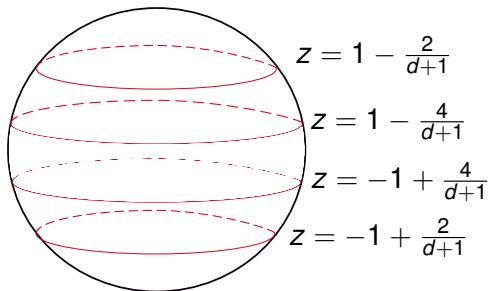
# Computing the $\mu_d$

We first establish:

**Monotone twist formula:**

$$\mu_d(\varphi_H^1) \approx \sum_{i=1}^d H\left(-1 + \frac{2i}{d+1}\right) - dH(0).$$

$d = 4$



**Linearity for monotone twists:**  $\mu_d(\varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}) = t_1 \mu_d(\varphi_{H_1}^1) + t_2 \mu_d(\varphi_{H_2}^1).$



## Sketch of Proof ( $n = 2$ case)

$$\Phi : \mathbb{R}_{\geq 0}^2 \rightarrow \text{Ham}(\mathbb{S}^2), (t_1, t_2) \longrightarrow \varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}.$$

(Recall,  $H_i$ : monotone twist,  $\text{supp}(H_i) = \{(\theta, z) : 1 - \frac{2}{d_i} \leq z \leq 1\}$ ,  $d_i = 2(i + 1)$ .)

Let  $\mathbf{t} = (t_1, t_2)$ . Recall  $\Phi(\mathbf{t}) = \varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}$ . Goal: show  $\exists C_1, C_2$  st

$$C_1 \|\mathbf{t} - \mathbf{s}\|_{\infty} \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s})) \leq C_2 \|\mathbf{t} - \mathbf{s}\|_{\infty}.$$

We'll just do the **lower bound** (the hard part): By Hofer Lipschitz property (which says  $|\mu_d(\varphi) - \mu_d(\psi)| \leq 2d d_H(\varphi, \psi)$ )

$$\max_i \left| \frac{\mu_{d_i}(\Phi(\mathbf{t}))}{2d_i} - \frac{\mu_{d_i}(\Phi(\mathbf{s}))}{2d_i} \right| \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s})).$$

From previous slide:

$$\max_i \left| \frac{\mu_{d_i}(\Phi(\mathbf{t}))}{2d_i} - \frac{\mu_{d_i}(\Phi(\mathbf{s}))}{2d_i} \right| \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s})).$$

Claim: LHS =  $\|A(\mathbf{t} - \mathbf{s})\|_\infty$  where  $A = \begin{bmatrix} \frac{\mu_{d_1}(\varphi_{H_1}^1)}{2d_1} & \frac{\mu_{d_1}(\varphi_{H_2}^1)}{2d_1} \\ \frac{\mu_{d_2}(\varphi_{H_1}^1)}{2d_2} & \frac{\mu_{d_2}(\varphi_{H_2}^1)}{2d_2} \end{bmatrix}$  Proof: By Linearity of  $\mu_d$

(details left as an exercise.)

Claim:  $A$  is invertible. Proof: see next slide.

Since  $A$  is invertible can write

$$\frac{\|\mathbf{t} - \mathbf{s}\|_\infty}{\|A^{-1}\|_{op}} \leq \|A(\mathbf{t} - \mathbf{s})\|_\infty,$$

where  $\|A^{-1}\|_{op}$  = denotes the operator norm of  $A^{-1} : (\mathbb{R}^2, \|\cdot\|_\infty) \rightarrow (\mathbb{R}^2, \|\cdot\|_\infty)$ .

So, take  $C_1 = \frac{1}{\|A^{-1}\|_{op}}$ , hence the lower bound. Remark: We can arrange that our QI embedding is in the kernel of Calabi, answering a 2012 question of Polterovich.

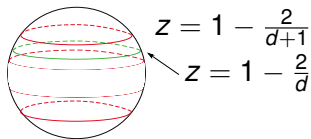
# Why is $A$ invertible?

Claim:  $A$  is invertible. Recall from previous slide:  $A = \begin{bmatrix} \frac{\mu_{d_1}(\varphi_{H_1}^1)}{2d_1} & \frac{\mu_{d_1}(\varphi_{H_2}^1)}{2d_1} \\ \frac{\mu_{d_2}(\varphi_{H_1}^1)}{2d_2} & \frac{\mu_{d_2}(\varphi_{H_2}^1)}{2d_2} \end{bmatrix}$

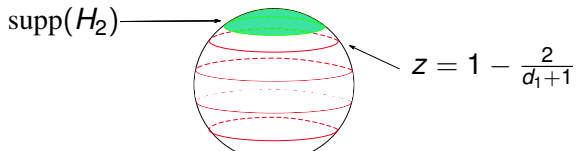
Proof: follows from the next two observations.

Observation 1:  $\mu_{d_i}(\varphi_{H_i}^1) > 0$ . Proof:

$$\mu_{d_i}(\varphi_{H_i}^1) = H_i \left(1 - \frac{2}{d_i+1}\right) > 0$$



Observation 2:  $\mu_{d_1}(\varphi_{H_2}^1) = 0$ . Proof:



# **PFH spectral invariants**

(impressionistic sketch of the construction)

# The PFH of $\varphi$ : the setup

Let  $\varphi \in \text{Ham}(\mathbb{S}^2, \omega)$ . Recall the **mapping torus**

$$Y_\varphi = \mathbb{S}_x^2 \times [0, 1]_t / \sim, \quad (x, 1) \sim (\varphi(x), 0).$$

Canonical two-form  $\omega_\varphi$  induced by  $\omega$ .

Canonical vector field  $R := \partial_t$ . Captures the dynamics of  $\varphi$ .

$$\{\text{Periodic Points of } \varphi\} \xleftrightarrow{1:1} \{\text{Closed Orbits of } R\}$$

$R$  is the "Reeb" vector field of the Stable Hamiltonian Structure  $(dt, \omega_\varphi)$ .

PFH = ECH in this setting. (Hutchings)

There exists PFH spectral invariants  $c_d$  "=" ECH spectral invariants in this setting.  
(Hutchings)

$PFH(\varphi)$  is homology of a chain complex  $PFC(\varphi)$ . ( $\varphi$  non-degenerate)

$PFC(\varphi)$ : generated by (certain) "Reeb orbit sets"  $\{(\alpha_i, m_i)\}$

- $\alpha_i$  distinct, embedded closed orbits of  $R$
- $m_i$  positive integer. ( $m_i = 1$  if  $\alpha_i$  is hyperbolic)

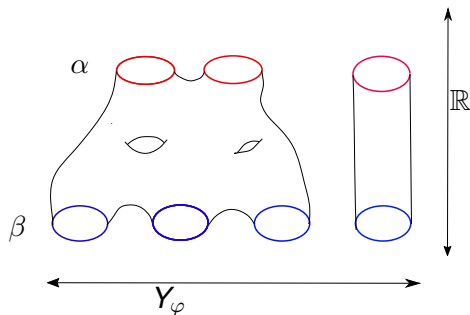
$\partial$ : counts certain  $J$ -holomorphic curves in  $\mathbb{R} \times Y_\varphi$ .

$PFH(\varphi)$  is the homology of this chain complex.

Lee-Taubes:  $PFH(\varphi)$  independent of choices of  $J, \varphi$ .

(Remark: for technical reasons, we work with a twisted version of PFH.)

# A $J$ -hol curve contributing to $\langle \partial\alpha, \beta \rangle$



$\langle \partial\alpha, \beta \rangle := \# \text{ maps } u : (\Sigma, j) \rightarrow (\mathbb{R} \times Y_\varphi, J) \text{ such that}$

- $J$  holomorphic:  $du \circ j = J(u)du$ .
- Asymptotic to  $\alpha$  and  $\beta$ .
- “ECH index”  $l = 1$ .

To construct spectral invariants need two ingredients:

1.  $PFH(\varphi)$  has an action filtration. (depends on a choice of Hamiltonian!)
  - $PFH^a(\varphi)$ : what you see up to action level  $a \in \mathbb{R}$ .
2. There exist (more or less) distinguished nonzero classes  $\sigma_d \in PFH(\varphi)$  for  $d \in \mathbb{N}$ .

Define:

$$c_d(\varphi) := \inf\{a \in \mathbb{R} : \sigma_d \in PFH^a(\varphi)\}.$$

In words:  $c_d(\varphi)$  is the action level at which you first see  $\sigma_d$ .

Remark:  $d$  corresponds to the degree of the class.



The numbers  $c_d(\varphi)$  as defined depend on the choice of generating Hamiltonian (because the action does). First step to remedy this: restrict to **mean-normalized** Hamiltonians, that is

$$\int_{S^2} H_t \omega = 0$$

for all  $t$ . We show this gives a well-defined invariant  $c_d$  on  $\widetilde{\text{Ham}}$ .

We next **homogenize** to get invariants  $\mu_d$  on Ham:

$$\mu_d(\varphi) := \lim_{n \rightarrow \infty} \frac{c_d(\tilde{\varphi}^n)}{n}$$

where  $\tilde{\varphi}$  is any lift of  $\varphi$ .

**Thank you!**

**Bonus: twisted PFH.**

# The reference cycle

To define twisted PFH, need in addition a (trivialized) reference cycle  $\gamma \subset Y_\varphi$ .  
Using this we proceed as follows:

- A twisted PFH generator is a pair  $(\alpha, Z)$ , where  $\alpha$  is a PFH generator and  $Z \in H_2(\alpha, \gamma^d)$  is a “capping”.
- The differential counts  $l = 1$  curves  $C$  from  $(\alpha, Z)$  to  $(\beta, Z')$ : that is  $[C] + Z' = Z \in H_2(\alpha, \gamma^d)$ .
- The action is given by:  $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$ .

We produce  $\gamma$  by trivializing  $Y_\varphi$  via

$$S^1 \times S^2 \longrightarrow Y_\varphi, \quad (t, x) \longrightarrow (t, (\varphi_H^t)^{-1}(x)),$$

and taking  $\gamma$  to be the push-forward of the invariant cycle over  $p_-$ .