Spinning black holes fall in Love

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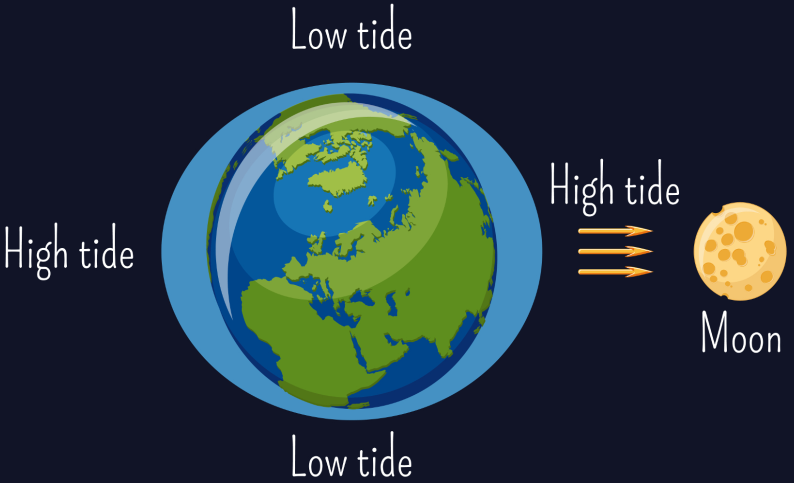
Collaborators: M. Casals & E. Franzin

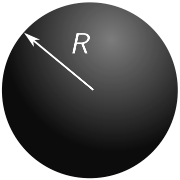
gr-qc/2007.00214 — PRL (to appear)
gr-qc/2010.15795 — PRD (to appear)

TIDES



Newtonian theory of Love numbers

M



$$U = \frac{M}{r}$$

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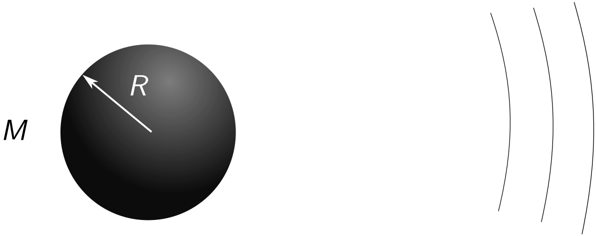

$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$$

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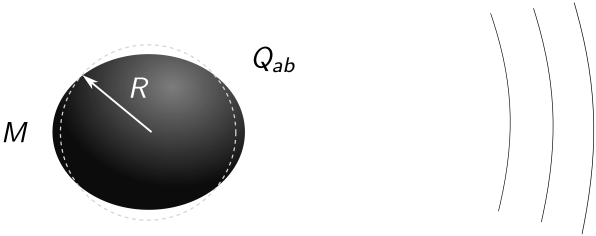

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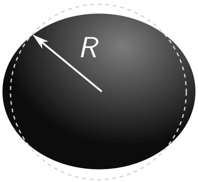

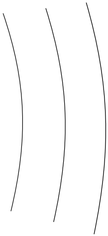

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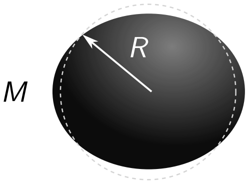

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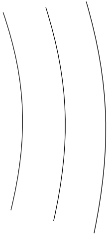

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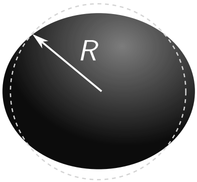

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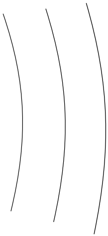

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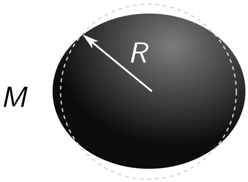
Newtonian theory of Love numbers



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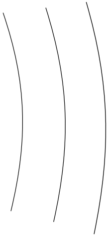

$$\mathcal{E}_{ab} = -\partial_a \partial_b U_{\rm ext}(\mathbf{0})$$

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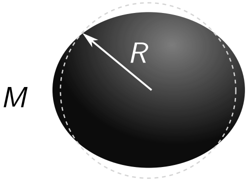

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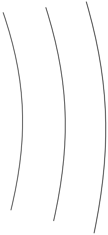

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Newtonian theory of Love numbers

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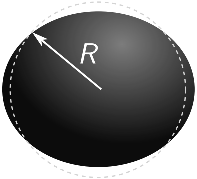

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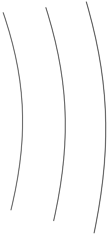

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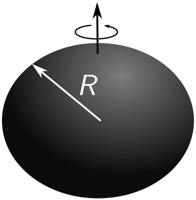
Newtonian theory of Love numbers

$S = \chi M^2$



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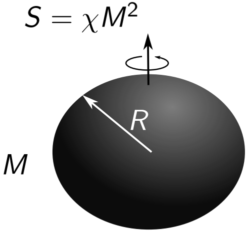
Newtonian theory of Love numbers



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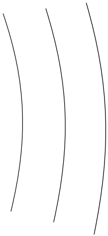

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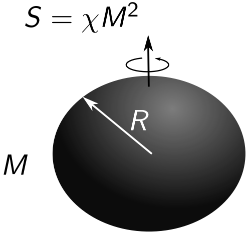
$$k_{\ell m} = k_\ell^{(0)} + i m \chi\, k_\ell^{(1)} + O(\chi^2)$$

Newtonian theory of Love numbers



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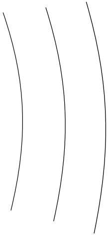

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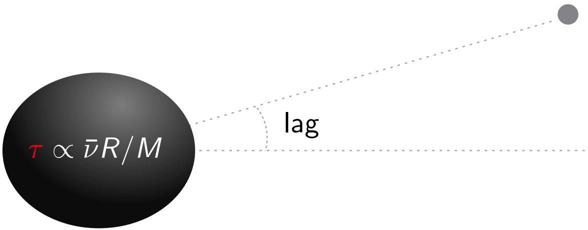
Tidal Love numbers $k_{\ell m}$ $\longleftrightarrow$ **body's internal structure**

Tidal dissipation: lag, heating and torquing



$$Q_{ab}(t) = -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t) - \tau \dot{\mathcal{E}}_{ab}(t) + \cdots \right]$$
$$= -\frac{2}{3} k_2 R^5 \left[\mathcal{E}_{ab}(t - \tau) + \cdots \right]$$

Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_{a_1\cdots a_\ell} \propto [C_{0a_10a_2;a_3\cdots a_\ell}]^{\mathsf{STF}}\,, \quad \mathcal{B}_{a_1\cdots a_\ell} \propto [\varepsilon_{a_1bc}C_{a_20bc;a_3\cdots a_\ell}]^{\mathsf{STF}}$$

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- Metric and Geroch-Hansen multipole moments:

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + \underbrace{h^{\mathrm{tidal}}_{\alpha\beta}}_{\sim r^\ell} + \underbrace{h^{\mathrm{resp}}_{\alpha\beta}}_{\sim r^{-(\ell+1)}} \quad \longrightarrow \quad \begin{cases} M_{\ell m} = \mathring{M}_{\ell m} + \delta M_{\ell m} \\ S_{\ell m} = \mathring{S}_{\ell m} + \delta S_{\ell m} \end{cases}$$

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- *Four* families of tidal deformability parameters:

$$\lambda^{M\mathcal{E}}_{\ell m} \equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell m}} \qquad \lambda^{S\mathcal{B}}_{\ell m} \equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell m}}$$

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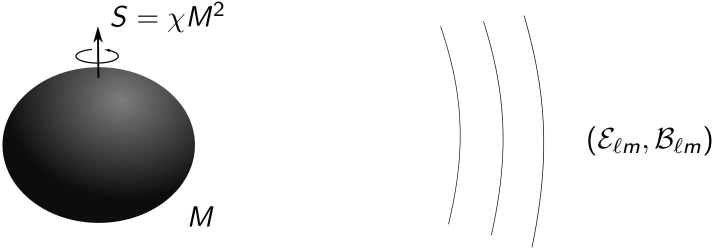
Investigating Kerr's Love



$$(\mathcal{E}_{\ell m}, \mathcal{B}_{\ell m}) \to \psi_0 \to \Psi \to h_{\alpha\beta} \to (M_{\ell m}, S_{\ell m}) \to \lambda_{\ell m}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Love numbers of a Kerr black hole

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- The associated dimensionless tidal Love numbers are

$$k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0$$

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- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are } \textbf{“rigid”}$$

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\mathrm{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\mathrm{mag}} \mathcal{B}_{ab}$$

- For a spinning black hole we have the more general tensorial relations

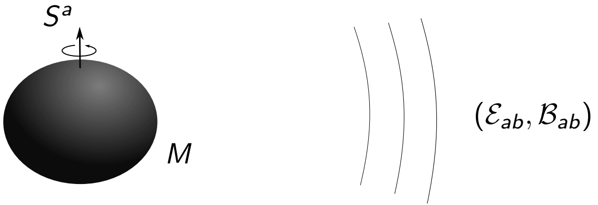
$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

- To linear order in the black hole spin vector S^a we find

$$\delta M_{ab} \doteq \frac{16}{45} M^3 \, S^c \mathcal{E}^d{}_{(a} \varepsilon_{b)cd}$$
$$\delta S_{ab} \doteq \frac{16}{45} M^3 \, S^c \mathcal{B}^d{}_{(a} \varepsilon_{b)cd}$$

Tidal torquing of a spinning black hole

[Thorne & Hartle 1980; Poisson 2004]



- An arbitrary spinning body interacting with a tidal environment suffers a tidal torquing:

$$\langle \dot{S}^a \rangle = -\varepsilon^{abc} \langle M_{bd} \mathcal{E}^d{}_c + S_{bd} \mathcal{B}^d{}_c \rangle$$

- Applied to a spinning black hole this yields

$$\langle \dot{S} \rangle \doteq -\frac{8}{45} M^5 \chi \left[2 \langle \mathcal{E}^{ab} \mathcal{E}_{ab} \rangle - 3 \langle \mathcal{E}_{ab} s^b \mathcal{E}^{ac} s_c \rangle + (\mathcal{E} \to \mathcal{B}) \right]$$

Summary

- Love numbers of Kerr black holes do not vanish in general
- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin
- Kerr black holes deform like any other self-gravitating body, despite being particularly “rigid” compact objects
- This is closely related to the phenomenon of tidal torquing
- New black hole test of the Kerr-like nature of the massive compact objects at the center of galaxies?

Spinning black holes fall in Love!

Black holes have *zero* Love numbers

Reference	**Background**	**Tidal field**
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(\ell, m) = (2, 0)$
[Chia 2020]	Exact Kerr	weak, generic ℓ
[Goldberger & Rothstein 2020]	Exact Kerr	weak, generic ℓ
[Charalambous et al. 2021]	Exact Kerr	weak, generic ℓ

Problem of **fine-tuning** from an Effective-Field-Theory perspective

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c\to\infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m}\, r^{\ell-2} \left[1 + 2k_{\ell m} \left(R/r\right)^{2\ell+1}\right] {}_2Y_{\ell m}(\theta,\phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto \left[\mathcal{E}_{\ell m} + \tfrac{3i}{\ell+1}\, \mathcal{B}_{\ell m}\right] R_{\ell m}(r)\, {}_2Y_{\ell m}(\theta,\phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2}\left(1+\cdots\right)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m}\, \underbrace{r^{-\ell-3}\left(1+\cdots\right)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Kerr black hole linear response

$$R_{\ell m}(r) = \underbrace{R^{\text{tidal}}_{\ell m}(r)}_{\sim r^{\ell-2}} + 2k_{\ell m}\underbrace{R^{\text{resp}}_{\ell m}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients $k_{\ell m}$ can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{\ell m} = -im\chi\,\frac{(\ell+2)!(\ell-2)!}{4(2\ell+1)!(2\ell)!}\prod_{n=1}^{\ell}\left[n^2(1-\chi^2)+m^2\chi^2\right]$$

- The linear response vanishes identically when:
  - the black hole spin vanishes ($\chi = 0$)
  - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h^{\text{resp}}_{\alpha\beta}$ via Ψ^{resp}

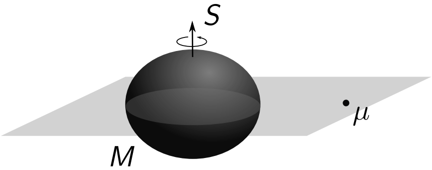
Love tensor of a Kerr black hole

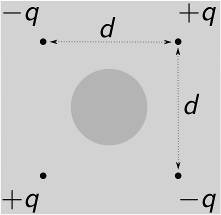
$$(\lambda_{abcd}) \doteq \frac{\chi}{180}\,(2M)^5 \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & -\mathbf{I}_{11} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{I}_{11} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{I}_{12} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_{13} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \mathbf{I}_{23} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Newtonian static quadrupolar tide





$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\delta M_{ab} \doteq \underset{\substack{\uparrow \\ \frac{\chi}{180} (2M)^5 \frac{\mu}{r^3} = qd^2}}{3Q} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$