Schwarzschild-Tangherlini metric from scattering amplitudes in various dimensions

Stavros Mougiakakos

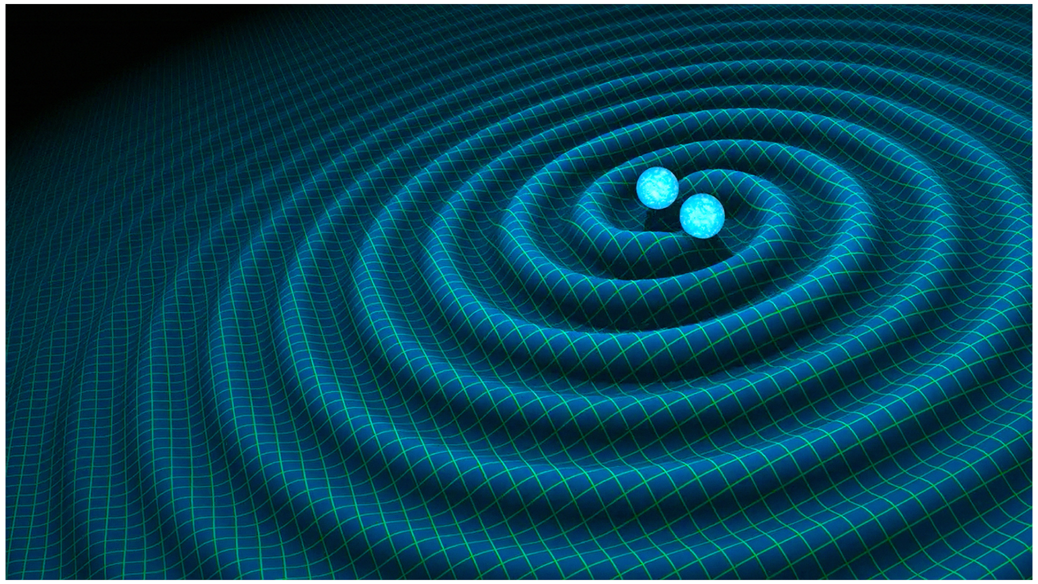
Meeting of the National Research Group on Gravitational Waves
Institut Henri Poincaré (virtual) 30/3-1/4 2021

Based on **2010.0882** with P. Vanhove

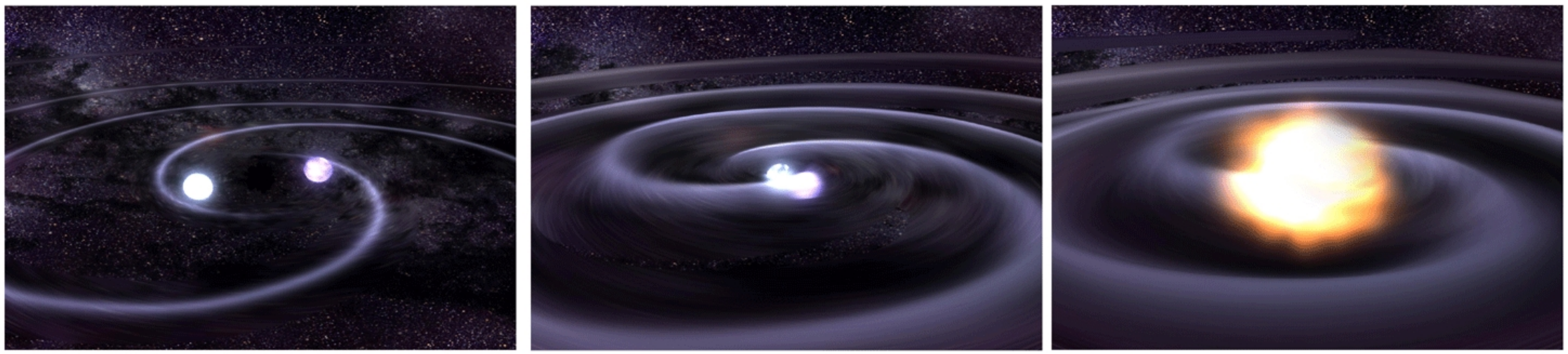
Observational Window on gravity

The detection of gravitational waves (GW150914) has opened a new window on the physics of our universe:

- For the first time detection and test of GR in the “strong” gravity coupling regime
- For the first time dynamics of BH/NS (not just static object curving space-time)



Gravitational Binary Problem

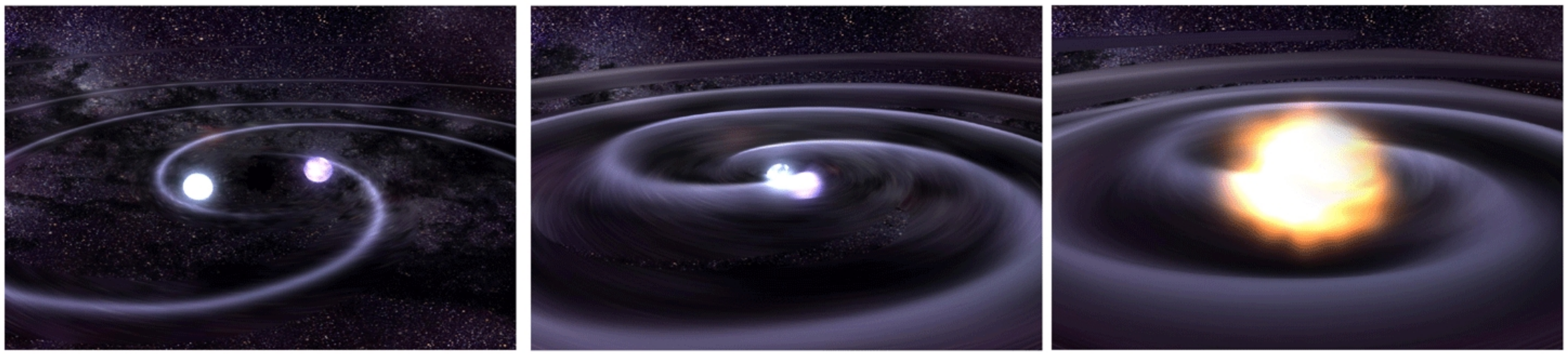


Inspiral **Merger** **Ringdown**

Renewed interest in the gravitational binary problem:

- Phenomenological importance for the **inspiral phase**
- Theoretical interest for tests of **GR extensions** and **quantum regime**

Gravitational Binary Problem



Inspiral **Merger** **Ringdown**

Renewed interest in the gravitational binary problem:

- Phenomenological importance for the **inspiral phase**
- Theoretical interest for tests of **GR extensions** and **quantum regime**

Need for analytic results:

→ **GR community (a lot of progress has been done)**

→ Particle Physics community (application of novel techniques)

Gravity as an Effective Field Theory

$$\mathcal{S}_{eff} = \mathcal{S}_{eff}^{matter} + \frac{1}{16\pi G_N}\int d^4x\sqrt{g}\mathcal{R} + \mathcal{O}(R^2, R^{\mu\nu}R_{\mu\nu}, \ldots)$$

- **Non-Renormalizable QFT**: (local, unitary, Lorentz invariant)
- **GR as a first order approximation**
- **Standard symmetries** of GR
- **Low energy dof's**: graviton + matter fields
- **Weak field approximation:** $g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n\geq 1} h_{\mu\nu}^{(n)}$

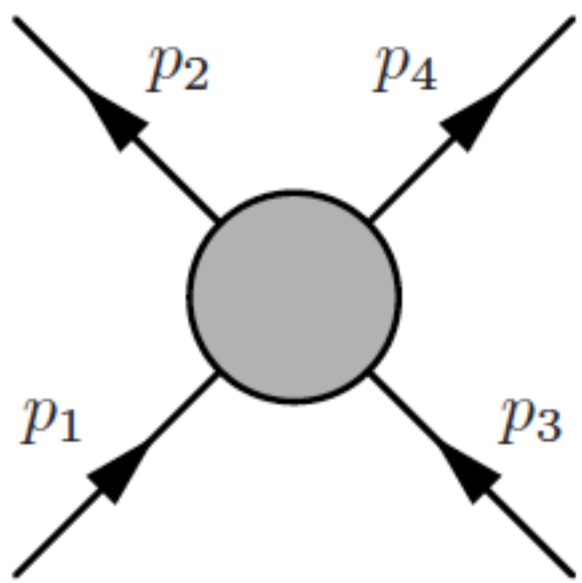
From Loops to Classical Physics

Important observation: Quantum Scattering Amplitudes give classical corrections to all loop orders

Restoring units in Klein-Gordon:

$$(\Box + \frac{m^2}{\hbar^2})\phi(x) = 0$$

→ **massive propagators suppress h/bar**



$$= \sum_{L=0}^{\infty} G_N^{L+1} \mathcal{M}^{L-\text{loop}}$$

Particle Physics approach in a nutshell for gravitational binary problem

Gravity as an Effective Field Theory + ***From Loops to Classical Physics***

On-shell methods for Amplitudes + ***Isolation of Classical Amplitude***

Classical Amplitude as a seed = ***Classical Observables***

Formalism suitable both for GR extensions and quantum contributions

Black hole as a testing lab

Black holes:

- Robust predictions of GR
- **Analytic solution** of vacuum Einstein's eq.
- Very important for gravitational waves physics

Careful treatment

Perfect testing lab of GREFT perturbative treatment

+ adaptable for Quantum effects GR extensions

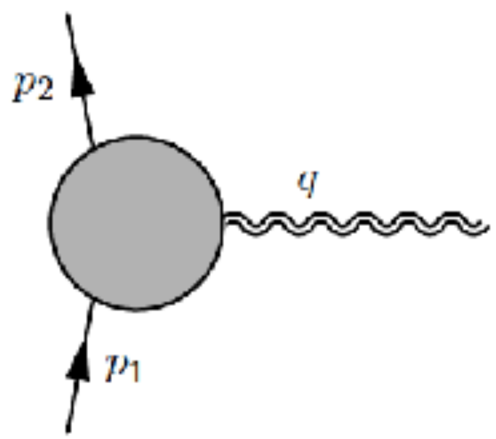
Black hole
as a testing lab

$$\rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\frac{G_N m}{r^{d-2}}$$ **, D=d+1**

GREFT

In linearised de Donder gauge:

$\mathcal{M}_3(p_1, q) =$



$$\mathcal{M}_3^{(l)}(p_1, q) = \frac{i\sqrt{32\pi G_N}}{2}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu}$$

$$h_{\mu\nu}^{(l+1)}(\vec{x}) = -16\pi G_N \int \frac{d^d q}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}}\frac{1}{\vec{q}^2}\left(\langle T_{\mu\nu}^{(l)}\rangle^{\text{class.}}(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle^{\text{class.}}(q^2)\right)$$

General Relativity

In usual coords:

$$ds^2_{\text{Schw}} = \left(1 - 4\frac{d-2}{d-1}\rho(r,d)\right)dt^2 - d\vec{x}^2 - \frac{4\frac{d-2}{d-1}\rho(r,d)}{1-4\frac{d-2}{d-1}\rho(r,d)}\frac{(\vec{x}\cdot d\vec{x})^2}{r^2}$$

$$(t,\vec{x}) \to (t, f(r,d)\vec{x})$$

Transformation to de Donder

Black hole as a testing lab

$$\rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} \frac{G_N m}{r^{d-2}}$$, **D=d+1**

General Relativity

In usual coords:

$$ds^2_{\text{Schw}} = \left(1 - 4\frac{d-2}{d-1}\rho(r,d)\right) dt^2 - d\vec{x}^2 - \frac{4\frac{d-2}{d-1}\rho(r,d)}{1-4\frac{d-2}{d-1}\rho(r,d)} \frac{(\vec{x}\cdot d\vec{x})^2}{r^2}$$

$$(t,\vec{x}) \to (t, f(r,d)\vec{x})$$

Transformation to de Donder

$$ds^2 = h_0(r,d)dt^2 - h_1(r,d)d\vec{x}^2 - h_2(r,d)\frac{(\vec{x}\cdot d\vec{x})^2}{\vec{x}^2}$$

$$h_0(r) := 1 - 4\frac{d-2}{d-1}\frac{\rho(r,d)}{f(r)^{d-2}},$$

$$h_1(r) := f(r)^2,$$

$$h_2(r) := -f(r)^2 - f(r)^{d-2}\frac{\left(f(r) + r\frac{df(r)}{dr}\right)^2}{f(r)^{d-2} - 4\frac{d-2}{d-1}\rho(r,d)}$$

Black hole
as a testing lab

$$\rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} \frac{G_N m}{r^{d-2}}$$ **, D=d+1**

General Relativity

$$ds^2 = h_0(r,d)dt^2 - h_1(r,d)d\vec{x}^2 - h_2(r,d)\frac{(\vec{x}\cdot d\vec{x})^2}{\vec{x}^2}$$

$$h_0(r) := 1 - 4\frac{d-2}{d-1}\frac{\rho(r,d)}{f(r)^{d-2}},$$

$$h_1(r) := f(r)^2,$$

$$h_2(r) := -f(r)^2 - f(r)^{d-2}\frac{(f(r) + r\frac{df(r)}{dr})^2}{f(r)^{d-2} - 4\frac{d-2}{d-1}\rho(r,d)}$$

By solving iteratively the <u>de Donder gauge</u> condition:

$$\eta^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}(g) = \eta^{\mu\nu}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) = 0 \longrightarrow 2(d-1)h_2(r) = r\frac{d}{dr}(h_0(r) + (d-2)h_1(r) - h_2(r))$$

we derive a perturbative, as expected from the linearised nature of de Donder gauge, expression for <u>f(r)</u>

Black hole as a testing lab

General Relativity

Ambiguity up to one independent integration constant

D=4:

$$f(r) = 1 + \frac{G_N m}{r} + 2\left(\frac{G_N m}{r}\right)^2 + \frac{2}{3}\log\left(\frac{rC_3}{G_N m}\right)\left(\frac{G_N m}{r}\right)^3 + \left(\frac{2}{3} - \frac{4}{3}\log\left(\frac{rC_3}{G_N m}\right)\right)\left(\frac{G_N m}{r}\right)^4 + \left(-\frac{21}{25} + \frac{32}{15}\log\left(\frac{rC_3}{G_N m}\right)\right)\left(\frac{G_N m}{r}\right)^5 + \left(\frac{112}{75} - \frac{28}{15}\log\left(\frac{rC_3}{G_N m}\right)\right)\left(\frac{G_N m}{r}\right)^6$$

D=5:

$$f(r) = 1 + \frac{2}{3}\frac{G_N m}{\pi r^2} + \frac{10}{9}\log\left(\frac{r^2 C_2}{G_N m}\right)\left(\frac{G_N m}{\pi r^2}\right)^2 - \frac{4}{81}\left(-8 + 45\log\left(\frac{r^2 C_2}{G_N m}\right)\right)\left(\frac{G_N m}{\pi r^2}\right)^3 + \frac{67 + 3780\log\left(\frac{r^2 C_2}{G_N m}\right)}{972}\left(\frac{G_N m}{\pi r^2}\right)^4 - \frac{32963 + 156420\log\left(\frac{r^2 C_2}{G_N m}\right) - 43200\log\left(\frac{r^2 C_2}{G_N m}\right)^2}{21870}\left(\frac{G_N m}{\pi r^2}\right)^5$$

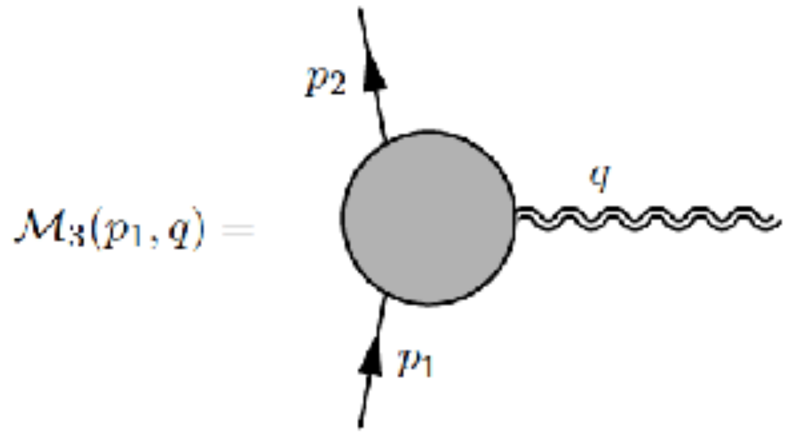
D=6:

$$f(r) = 1 + \frac{G_N m}{4\pi r^3} - \frac{5}{8}\left(\frac{G_N m}{\pi r^3}\right)^2 + \frac{2}{3}\left(\frac{G_N m}{\pi r^3}\right)^3 - \frac{775}{1344}\left(\frac{G_N m}{\pi r^3}\right)^4 + \frac{545977}{537600}\left(\frac{G_N m}{\pi r^3}\right)^5$$

Black hole
as a testing lab

GREFT

In linearised de Donder gauge:



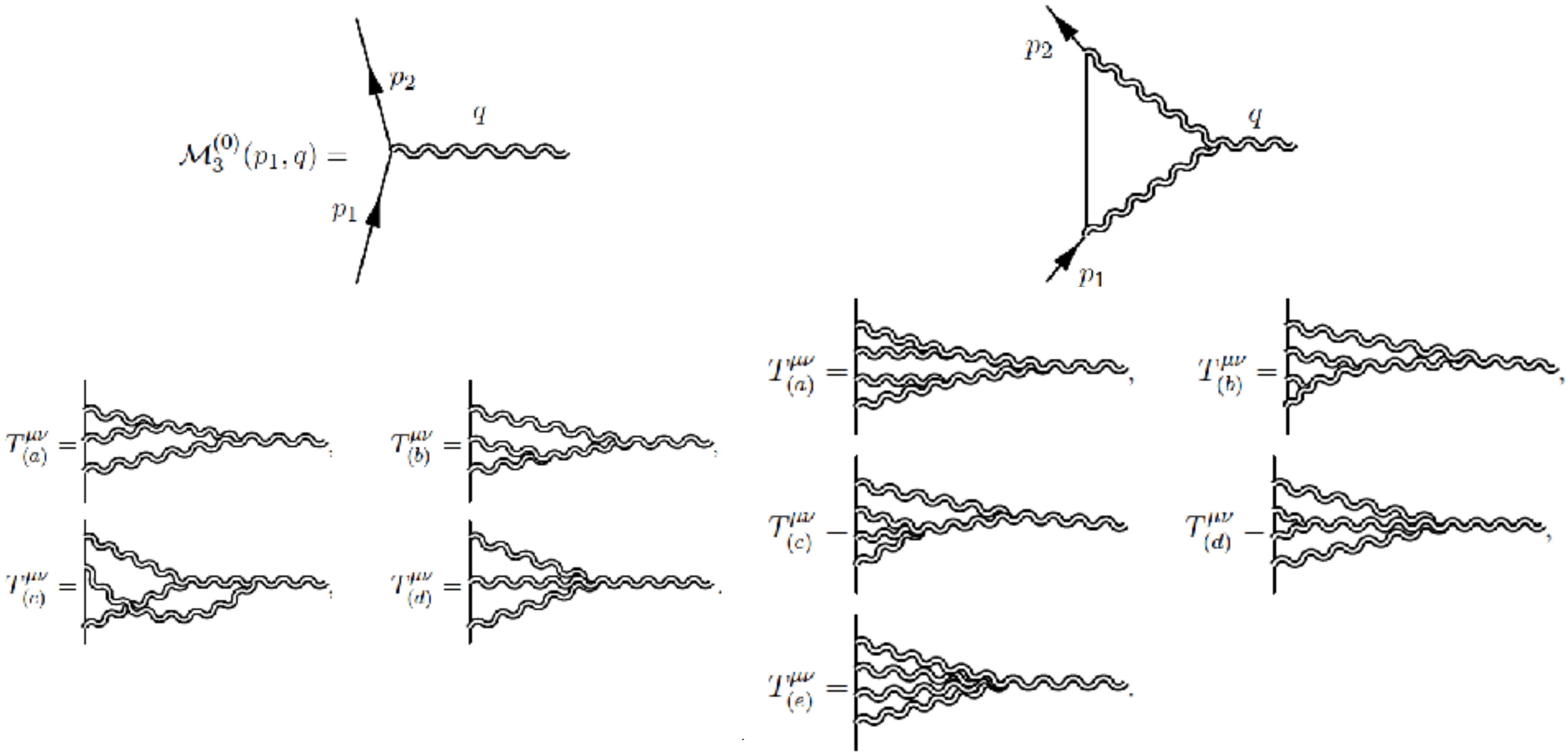
$$\mathcal{M}_3^{(l)}(p_1, q) = \frac{i\sqrt{32\pi G_N}}{2}\langle T^{(l)\,\mu\nu}(q^2)\rangle \epsilon_{\mu\nu}$$

$$h_{\mu\nu}^{(l+1)}(\vec{x}) = -16\pi G_N \int \frac{d^d q}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} \frac{1}{\vec{q}^2}\left(\langle T_{\mu\nu}^{(l)}\rangle^{\mathrm{class.}}(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle^{\mathrm{class.}}(q^2)\right)$$

Black hole as a testing lab

GREFT

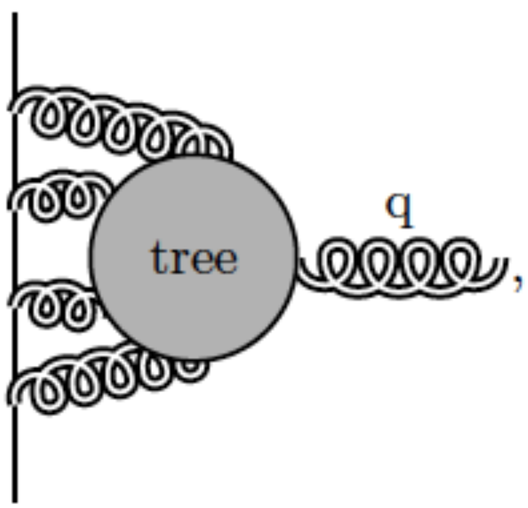
Feynman diagrams



Black hole as a testing lab

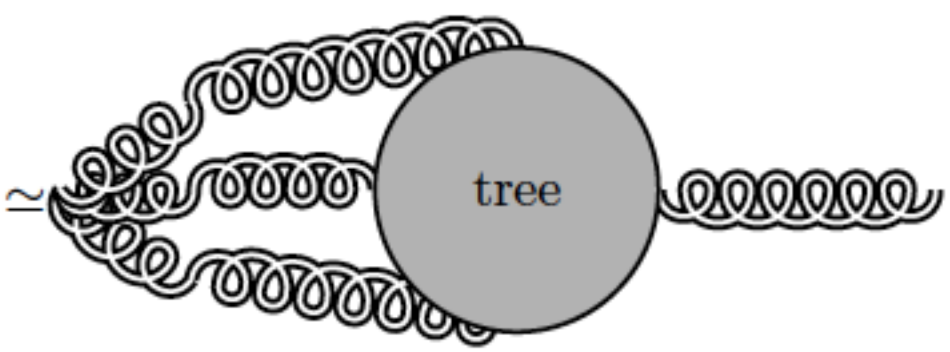
GREFT Classical limit

$\mathcal{M}_3^{(l)}(p_1,q) =$



Classical limit →

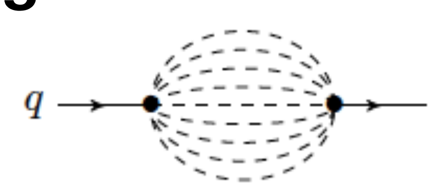
$\mathcal{M}_3^{(l)}(p_1,q) \simeq$



$$h_{\mu\nu}^{(l+1)}(\vec{q}) = -8\left(c_1^{(l)}(d)(2\delta_\mu^0\delta_\nu^0 - \eta_{\mu\nu}) + c_2^{(l)}(d)\left(2\frac{q_\mu q_\nu}{q^2} + (d-2)\eta_{\mu\nu}\right)\right) \times \frac{(\pi G_N m)^{l+1} \boxed{J_{(l)}(\vec{q}^2)}}{\vec{q}^2}$$

Metric determined by a single master integral:

$J_{(n)}(\vec{q}^2) =$



$$= \frac{(\vec{q}^2)^{\frac{n(d-2)}{2}}}{(4\pi)^{\frac{nd}{2}}} \frac{\Gamma\left(n+1-\frac{nd}{2}\right)\Gamma\left(\frac{d-2}{2}\right)^{n+1}}{\Gamma\left(\frac{(n+1)(d-2)}{2}\right)}$$

Black hole as a testing lab

GREFT Divergencies

Divergencies in stress-tensor and metric are removed by introducing non-minimal couplings in the GREFT

$$\delta^{(n)} S^{\mathrm{ct.}} = (G_N m)^{\frac{2n}{d-2}} \int d^{d+1}x \sqrt{-g} \Big(\alpha^{(n)}(d) (\nabla^2)^{n-1} R \partial_\mu \phi \partial^\mu \phi + \Big(\beta_0^{(n)}(d) \nabla_\mu \nabla_\nu (\nabla^2)^{n-2} R + \beta_1^{(n)}(d) (\nabla^2)^{n-1} R_{\mu\nu} \Big) \partial^\mu \phi \partial^\nu \phi \Big)$$

Perfect Match to 3-loops (G^4)

Constants of integration in GR coincide with renormalisation scale in GREFT

Outlook

Advanced understanding of Black Holes in GREFT

- Understanding of **classical limit of quantum Amplitudes** is consistent
- Explicit display of the importance and resolution of **gauge choice**
- Generalizable to **charged** and **spinning** black holes
- Generalizable for **quantum effects** and **GR extensions**
- Non- trivial behaviour of **gravity in higher dimensions**