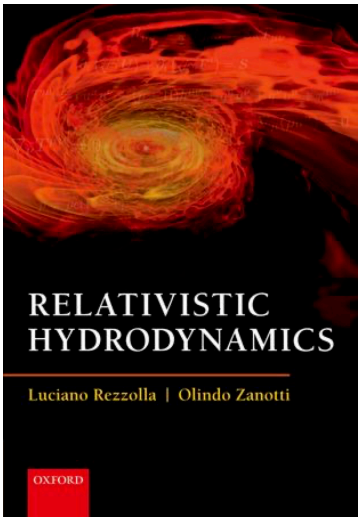
Relativistic Hydrodynamics

Luciano Rezzolla
rezzolla@itp.uni-frankfurt.de

Material covered:

- Basic concepts on fluids ($\underline{u}$, $\underline{a}$, ...)
- Energy-momentum and current tensors
- Hydrodynamics as conservation laws
- The beauty and curse of nonlinearity
- The importance of conservative formulations



- Much of what I will present can be found in this book (Chap. 3, 4, 7).
- Copies of these chapters are available on request.
- Time plan: 16:00 - 17:00
  17:00 - 17:15 Q/A
  17:15 - 18:15
  18:15 - 18:30 Q/A

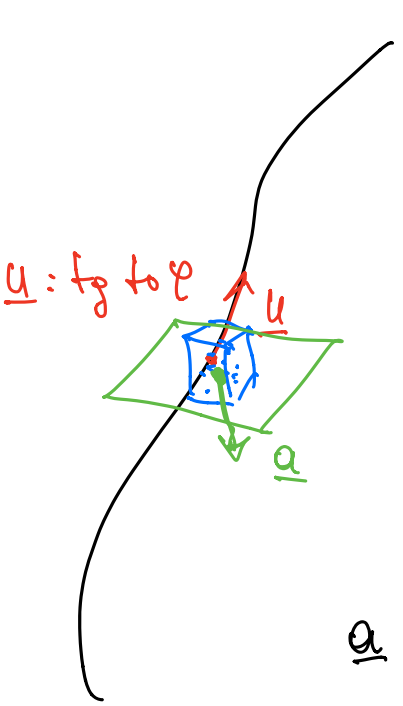
interrupt me if useful!

- kinematics

fluid's four-velocity $\underline{u}$

$u^\mu := \frac{dx^\mu}{d\tau}$: τ: proper time, ie the time in the fluid's ref. frame

$u^\mu u_\mu = -1$



Another important kinematic quantity is the acceleration

$a^\mu := u^\nu \nabla_\nu u^\mu$: convective derivative of $\underline{u}$

$\underline{a} := \nabla_{\underline{u}} \underline{u}$

There is a class of integral curves of $\underline{u}$ s.t. $\nabla_{\underline{u}} \underline{u} = 0$: curves that // transport the tangent vector

Geodesics curves $\Rightarrow \underline{a} = 0$ always geod.

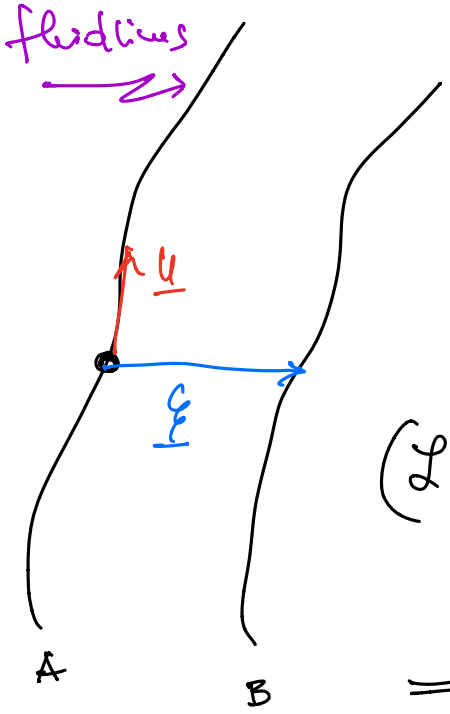
Note $\underline{a} \cdot \underline{u} = 0$

Proof $u^\nu (\nabla_\mu u^\nu) = 0$

Since $\underline{u}$ is timelike $\Rightarrow$ $\underline{a}$ is fully spatial

□

Let $\underline{\xi}$: displacement vector between two fluid elements.



Naturally $\underline{\xi}$ is Lie dragged along $\underline{u}$ $\Longleftrightarrow$

$$\mathcal{L}_{\underline{u}}\,\underline{\xi} = 0 \iff \mathcal{L}_{\underline{u}}\,\underline{\xi} = [\underline{u}, \underline{\xi}]$$

$$\left(\mathcal{L}_{\underline{u}}\,\underline{\xi}\right)^{\nu} = u^{\mu}\nabla_{\mu}\xi^{\nu} - \xi^{\mu}\nabla_{\mu}u^{\nu} = 0$$

$\Longrightarrow$

$$\dot{\xi}^{\mu} := u^{\nu}\nabla_{\nu}\xi^{\mu} = \xi^{\nu}\nabla_{\nu}u^{\mu}$$

fluid lines

Remember that $\dot{\xi}^{\mu} = \text{const.}$ are fluid lines.

It's possible to show that

$$\xi^{\nu}\nabla_{\nu}u_{\mu} = \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - a_{\mu}u_{\nu}\right)\xi^{\nu}$$

Irreducible decomposition of $\nabla_{\nu}u_{\mu}$ and hence

of the fluid lines.

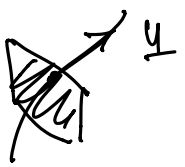
$\underline{\omega}$, $\underline{\sigma}$, $\underline{\theta}$ contain all the information on the kinematics of the fluid.

Note

h: projector operator orthogonal to $\underline{u}$

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \quad ; \quad \underline{h} \cdot \underline{u} = 0$$

$$h_{\mu\nu} u^\nu = g_{\mu\nu} u^\nu + \overbrace{u_\mu u_\nu u^\nu}^{-1} = u_\nu - u_\nu = 0$$



$\underline{\omega}$: kinematic vorticity tensor (antisym. tensor)

$$\omega_{\mu\nu} := h^\alpha_\mu h^\beta_\nu \nabla_{[\beta} u_{\alpha]}$$

$$= h^\alpha_\mu h^\beta_\nu \frac{1}{2} \left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta \right)$$

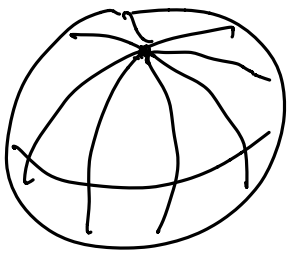
$\underline{\sigma}$: shear tensor (symmetric tensor)

$$\sigma_{\mu\nu} := \nabla_{\langle\mu} u_{\nu\rangle} =: \nabla_{(\mu} u_{\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{3}\Theta h_{\mu\nu}$$

$$= \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu + a_\mu u_\nu + a_\nu u_\mu - \frac{2}{3}\nabla_\alpha u^\alpha h_{\mu\nu}\right)$$

where $\Theta := \nabla_\alpha u^\alpha$: expansion scalar

In order to see what is the physical interpretation of these tensors, let's consider a fluid element as represented by an ellipsoid

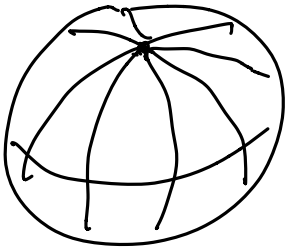


: reference fluid element

$\omega_{\mu\nu} \rightarrow \omega_\mu$: vorticity vector

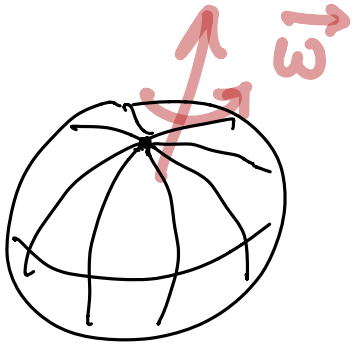
$u^\mu := {}^*\omega^{\mu\nu} u_\nu$: dual of $\underline{\omega}$ in fluid frame

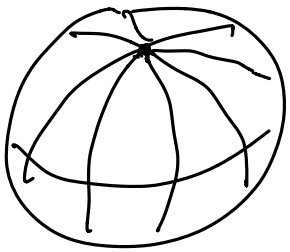
$= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta} u_\nu$ $\quad \epsilon^{\mu\nu\alpha\beta}$: Levi-civita tensor



vorticity

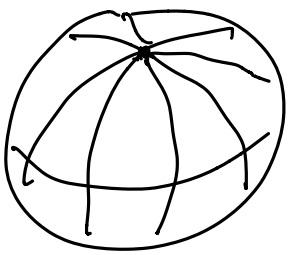
measures rigid rotations





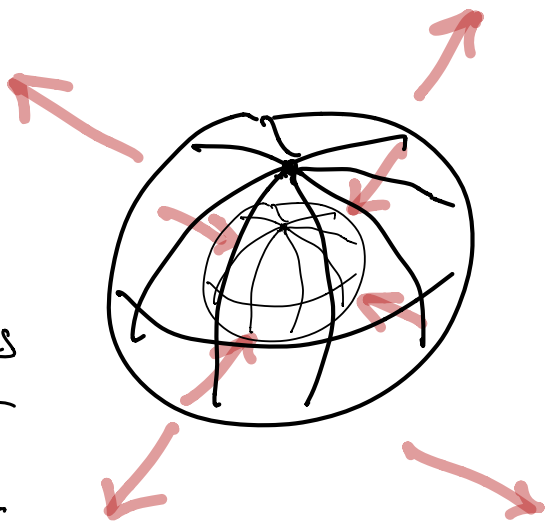
shear

measures changes in the ellipsoid's axes while preserving the volume



expansion

measures changes in volume but not in shape



$\underline{u}$, $\underline{a}$ are basic building blocks of fluid kinematics.

□

In dynamics there two more building block tensors : $\underline{J}$ $\underline{T}$ ⟵ el. mom. tensor.

$\underline{J}$ — 4-current density

I. Boltzmann eq.

$$p^\mu \nabla_\mu f(x^\mu, p^\mu) = \Pi(f)$$

$\Pi(f)$ — collisional integral

f — distribution function of a set of particles with rest-mass m

$p^\mu := m u^\mu$ — 4-momentum

If f is known, then one can build a hierarchy of moments

$$p^0 = p^t$$

$$= nmu^\mu$$

$$J^\mu := m\int p^\mu f \frac{d^3p}{p^0} \quad \Big| \quad = nN^\mu \qquad N^\mu = mU^\mu$$

$$T^{\mu\nu} := m\int p^\mu p^\nu f \frac{d^3p}{p^0}$$

$$\vdots$$

2. I need a symmetric rank-2 tensor that collects every information on the energy-mom. of the fluid ($\underline{\underline{T}}$); rank-1 tensor tells me about currents ($\underline{J}$).

$$J^\mu = \rho u^\mu \qquad = \rho: \text{rest-mass density}$$

$$\Big| \quad = nm$$

$T^{\mu\nu}$: flux of the μ-momentum density in ν-direction

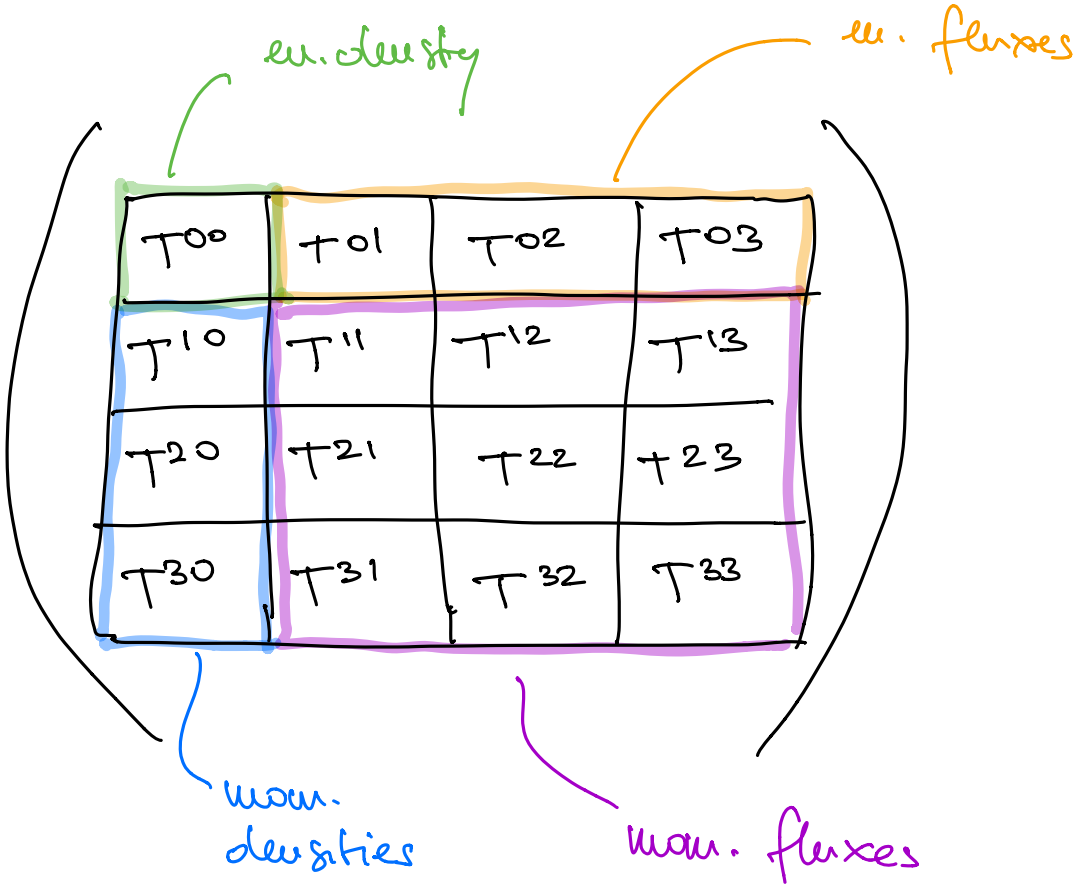
T^{00} : energy density (0-th component in time direction)

T^{0i} : energy flux (0-th component in i-th direction)

T^{i0} : mom. density (i-th component in time direction)

T^{ij} : mom. flux (i-th component in j-th direction)

$T^{\mu\nu} =$



To fill in these slots we can think of going in the <u>only</u> frame where all of these quantities are well-defined: fluid rest frame (comoving frame).

I know that the energy of a particle is

$$E = p^0$$

$T^{\hat{0}\hat{0}} = n\langle p^{\hat{0}} \rangle$: energy density, ie average energy of all particles

$= e$

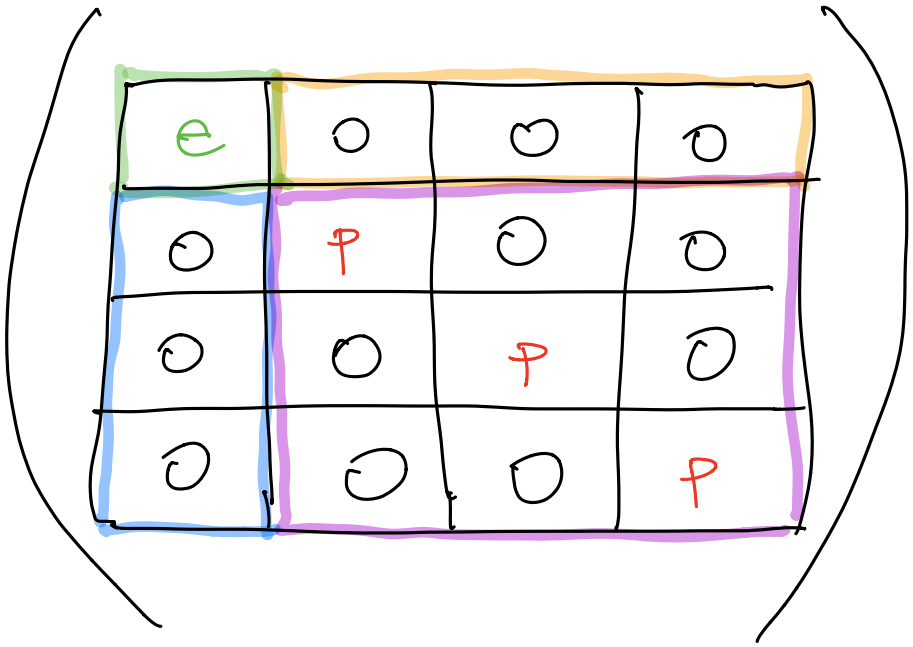
$T^{\hat{0}\hat{i}} = T^{\hat{i}\hat{0}} = 0$: these quantities are zero for a perfect fluid

$T^{\hat{i}\hat{j}} = 0$: zero in a perfect fluid

$\hat{i} \neq \hat{j}$

$T^{\hat{i}\hat{i}} =: p$ isotropic pressure

$T^{\mu\nu} =$



A tensor representation is not difficult to obtain : $u^\mu = (u^0, 0, 0, 0) = (1, 0, 0, 0)$

↑ spec. rel.

$$\boxed{T^{\hat{\mu}\hat{\nu}} = e\, u^{\hat{\mu}} u^{\hat{\nu}} + p\left(\eta^{\hat{\mu}\hat{\nu}} + u^{\hat{\mu}} u^{\hat{\nu}}\right)}$$

this tensor equation has to be valid in any frame (not only in SR)

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

$$J^\mu = \rho u^\mu$$

These are the building blocks of the dynamics of fluids, ie of fluid dynamics.

□

what do we do with these tensors?

We know, from other considerations, that the rest-mass, the energy and mom. of these tensors, are conserved. Hence, it's natural to assume that also $\underline{J}$ and $\underline{T}$ need to satisfy conservation laws.

$$\begin{cases} \nabla_\mu J^\mu = 0 & (1) : \text{rest-mass conserv.} \\ \nabla_\mu T^{\mu\nu} = 0 & (2) : \text{en.-mom. conserv.} \end{cases}$$

Dynamics of fluids.

(2) expresses conservation of energy and momentum; these 4 equations

Q: How do I distinguish en. cons. from mom conservation.

Energy cons. is naturally defined in the comoving frame, ie when I move with fg vector $\underline{u}$

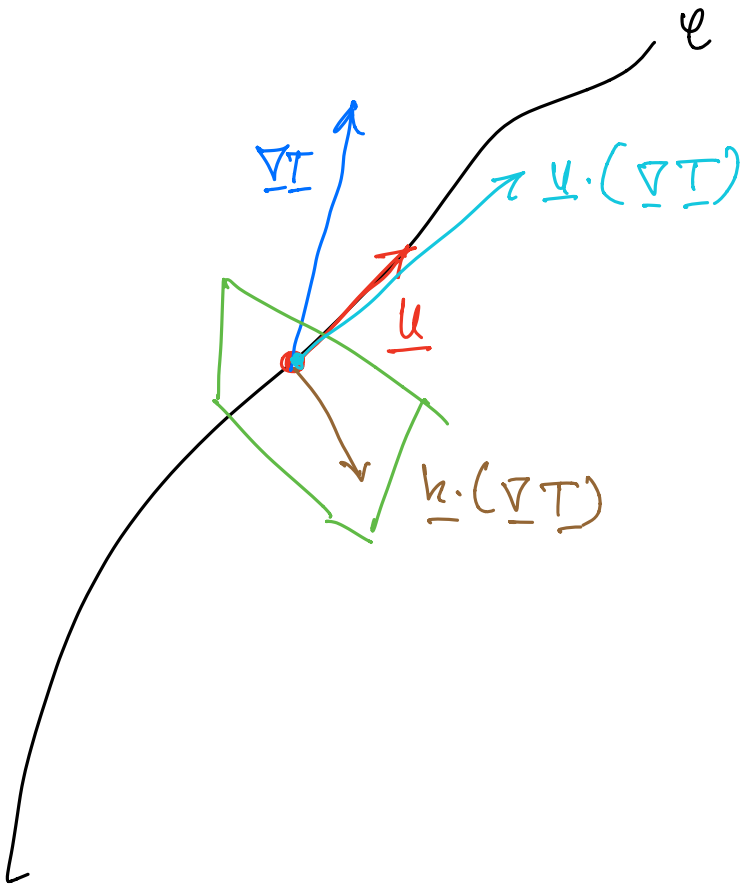
(3) $\underline{u} \cdot (\underline{\nabla} \underline{T}) = 0$ energy conservation law

For the complementary reasons the momentum conservation eqs are those in a frame that is orthogonal to $\underline{u}$

(4) $\underline{h}\cdot(\underline{\nabla}\underline{T}) = 0$: momentum cons. laws

(3) $\Leftrightarrow$ $u_\nu \nabla_\mu T^{\mu\nu} = 0$ (1 eq)

(4) $\Leftrightarrow$ $h_{\alpha\nu} \nabla_\mu T^{\mu\nu} = 0$ (3 eqs. $\alpha = 0$)



let's look at more detail to (1)

$$\nabla_\mu J^\mu = 0 \qquad u^\mu \nabla_\mu \varrho + \varrho \underbrace{\nabla_\mu u^\mu}_{\Theta} = 0$$

(5) $$\boxed{u^\mu \nabla_\mu \varrho = -\varrho\,\Theta}$$: continuity equation

$$(2) \iff \nabla_\mu \left(\varrho h\, u^\mu u^\nu + p\, g^{\mu\nu} \right) = 0$$

h : specif enthalpy $\qquad h := \dfrac{\varepsilon + p}{\varrho} = 1 + \epsilon + \dfrac{p}{\varrho}$

$\varepsilon = \varrho + \varrho\epsilon$: en. density

$\varrho\epsilon$: internal en. density $\qquad \epsilon$: specific internal energy (cf temperature)

$$\underline{h} \cdot (\underline{\nabla}\, \underline{T}) = 0 \iff$$

$$h^\nu{}_\lambda \nabla_\mu T^{\mu\lambda} = h^\nu{}_\lambda \left[u^\lambda u_\mu \nabla_\mu (\varrho h) + \varrho h\, u^\mu \nabla_\mu u^\lambda + \varrho h \Theta u^\lambda + g^{\lambda\mu} \nabla_\mu p \right]$$

$\underline{h} \cdot \underline{u} = 0$

$$= \varrho h\, u^\mu \nabla_\mu u^\nu + h^\nu{}_\lambda\, g^{\lambda\mu} \nabla_\mu p = 0$$

$$\boxed{u^\mu \nabla_\mu u^\nu = -\frac{h^\mu{}_\nu \nabla_\mu p}{\rho h}} \qquad (6)$$

$$a^\nu = -\frac{1}{\rho h}(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu p$$

$\rho h = 0$; $p = 0$; $\nabla_\nu p = 0$

(dust fluid)

$$\underline{u}\cdot(\nabla \underline{\underline{T}}) = 0 \iff u_\mu \nabla_\nu T^{\mu\nu} = 0$$

$$(7) \quad \boxed{u^\mu \nabla_\mu e + \rho h \Theta = 0} \iff \text{using (5)}$$

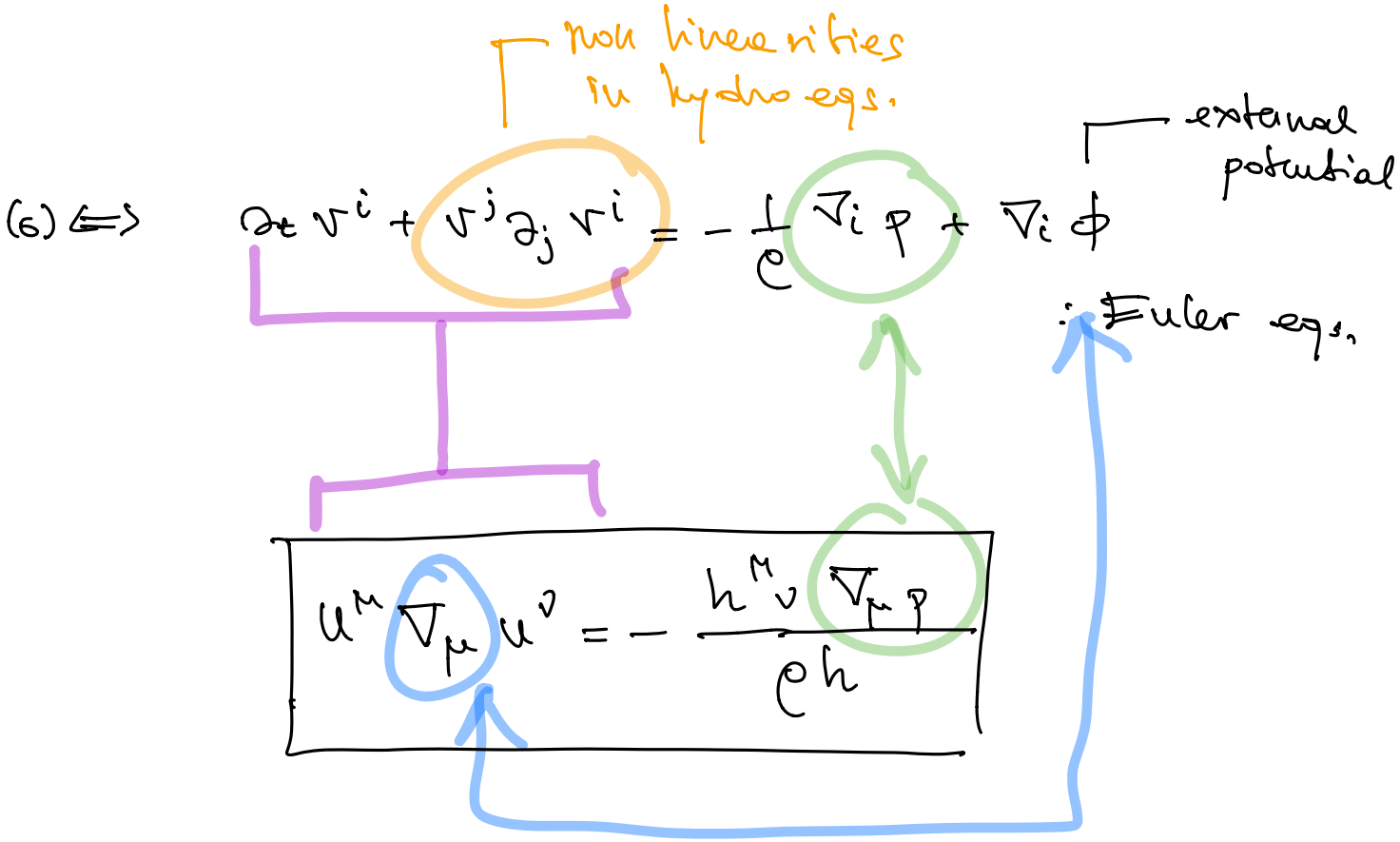
$$u^\mu \nabla_\mu e = -h u^\mu \nabla_\mu \rho$$

□

What are the equivalent equations to (5)-(7) in Newtonian physics?

(5) $\Leftrightarrow$ $\partial_t \rho + \nabla_i (\rho v^i) = 0$: contin. eq.

$$u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0$$



(7) $\Leftrightarrow$ $\partial_t \left(\frac{1}{2}\rho v^i v_i + \rho\epsilon\right) + \nabla_i \left[\left(\frac{1}{2}\rho v^i v_i + \rho\epsilon + p\right) v^i\right] = -\rho v^i \nabla\phi$

$$u^\mu \nabla_\mu \rho + \rho h \Theta = 0$$

: en. eq.

(5) - (7) are $1 + 1 + 3 = 5$ equations

How many quantities do we have?

ρ, v^i, p, ϵ

$1 + 3 + 1 + 1 : 6$ quantities

This means that the system (5)-(7) is missing one equation that needs to be provided extra:

equation of state:

$$\boxed{p = p(\rho, \epsilon, Y_e, \dots)} \qquad (8)$$

In this way the system can be closed!

The hydrodynamic eqs are <u>non linear</u> and this implies that specific behaviours (eg shock formation) are inevitable. To see this, let's consider first a classification of the hydrodynamic eqs.

$$\begin{cases} \nabla_\mu J^\mu = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{cases} \quad \Longleftrightarrow \quad \text{true for a large class of PDEs}$$

$$\boxed{\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S}} \qquad (9)$$

$\underline{U}$: state vector $\qquad \underline{U} = \{U_1, U_2, U \dots U_5\}$

$\underline{S}$: source vector $\qquad \underline{S} = \{S_1, S_2 \dots S_5\}$

$$\partial_t \left(\underline{U}\right) + A \nabla \left(\underline{U}\right) = \left(\underline{S}\right)$$

A: matrix of coefficients

The set of PDEs (9) can be classified into different classes by looking at the properties of A. In particular (9) is said to be:

1. <u>linear with const. coeff.</u> if elements of A are const.

$$a_{jk} = \text{cons.}$$

2. <u>linear with variable coefficients</u> if elements of A are const.

$$a_{jk} = a_{jk}(x, t)$$

3. <u>quasi-linear</u> if elements of A are function of the components of $\underline{u}$

$$a_{jk} = a_{jk}(\underline{u})$$

[if $a_{jk} = a_{jk}(\partial_x^n u)$: (9) is <u>nonlinear</u>]

Furthermore, (9) is said to be (strongly) <u>hyperbolic</u> if A is diagonalizable with a set of real eigenvalues and a set of linearly independent right eigenvectors.

$$\Lambda = R^{-1} A R = \text{diag}\left(\lambda_1, \lambda_2 \dots \lambda_N\right)$$

R: matrix of right eigenvectors

$$A R^{(i)} = \lambda_i R^{(i)} \qquad \boxed{\lambda_i \in \mathbb{R}}$$

It is possible to show that the set of hydrodynamic equation is hyperbolic in suitable forms.

The λ_i are called the <u>characteristic speeds</u>, ie speeds at which signals (perturbations) propagate. They are finite as reasonable for physical signals.

For example Einstein eqs can be shown to be hyperbolic in suitable coordinate system.

(*) is weakly hyperbolic if A is not diagonalizable

Theor. A hyperbolic system is <u>well posed</u> ©

⊙ A system is said to be well-posed if

$$\|u(x,t)\| \leq k e^{at} \|u(x,0)\|$$

$\|u(x,t)\|$: norm of solution at any time

$\|u(x,0)\|$: norm of solution at initial time

The system of equations can at most blow up exponentially.

▷

Summary:

1. Hydro equations are nonlinear
2. Hydro equations can be written in a hyperbolic form
3. Hyperbolic set of PDEs is well-posed

Examples of linear / nonlinear hyperbolic eqs.

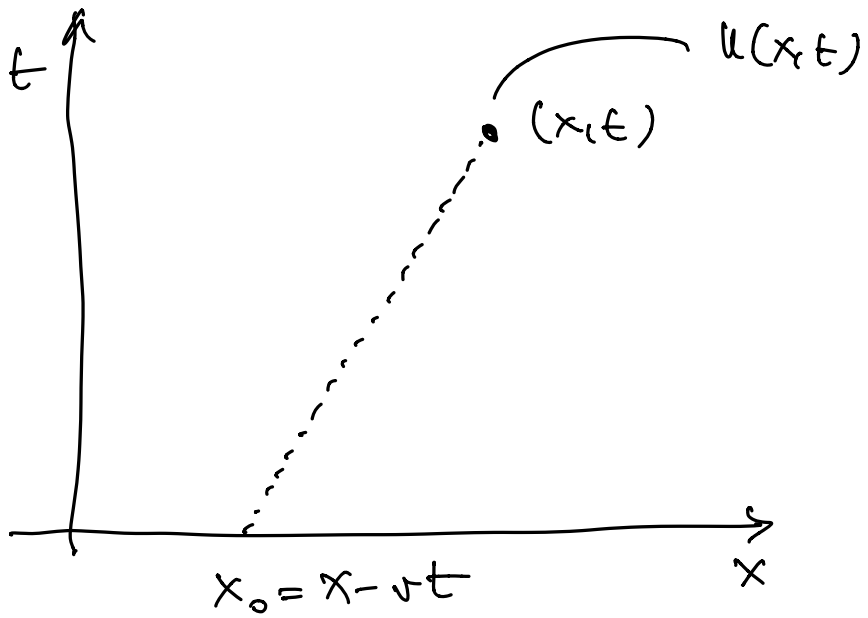
Advection equation in 1+1 dim.

$$\boxed{\partial_t u + v \partial_x u = 0} \qquad (10)$$

v: const.

The solution to (10) is very simple:

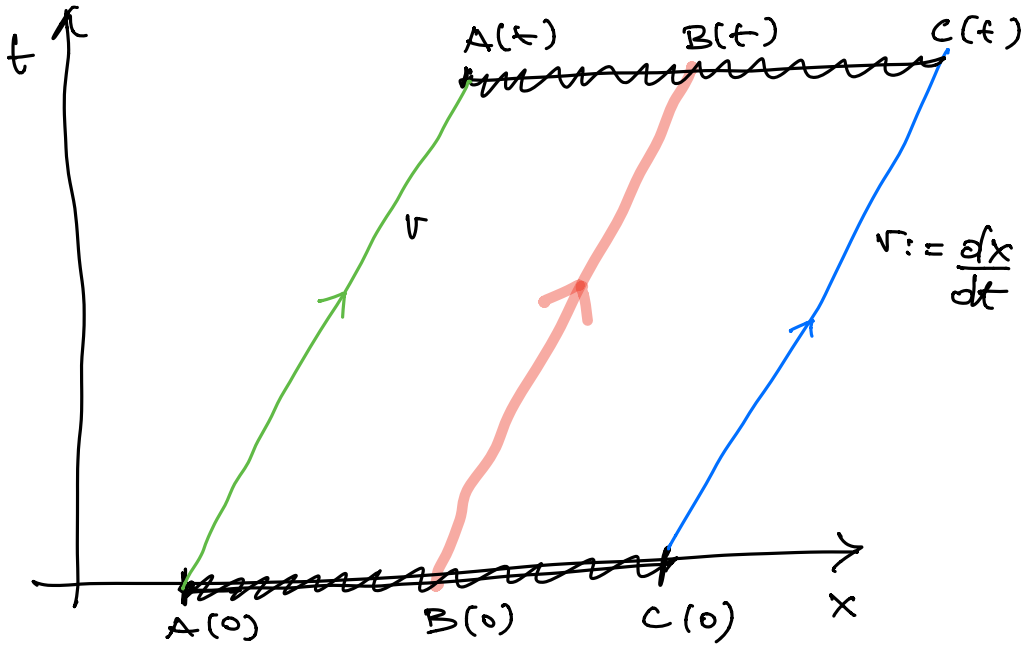
$$u(x,t) = u(x_0, 0) =: u_0(x_0) = u_0(x - vt)$$



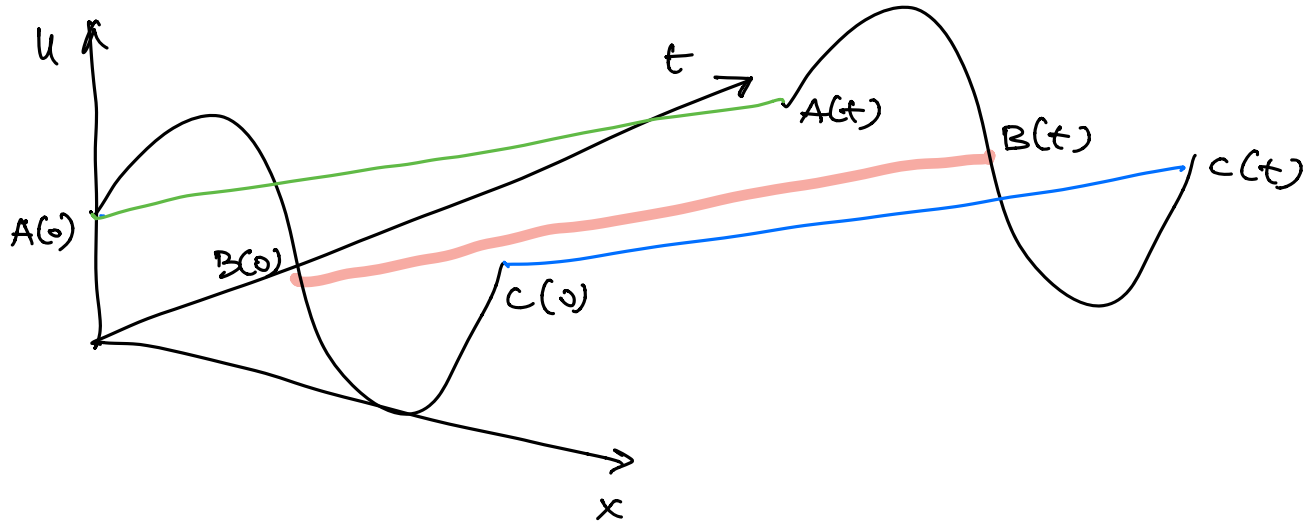
In other words, the solution is simply <u>advected</u> in spacetime along the characteristic directions

$$d_t u = \frac{du}{dt} = \partial_t u + \frac{dx}{dt}\partial_x u = 0 \quad \text{if} \quad v = \frac{dx}{dt} = \lambda$$

$= \text{const}$



Let u: sine wave



Example of nonlinear equation.

$$\partial_t u + u\,\partial_x u = 0$$

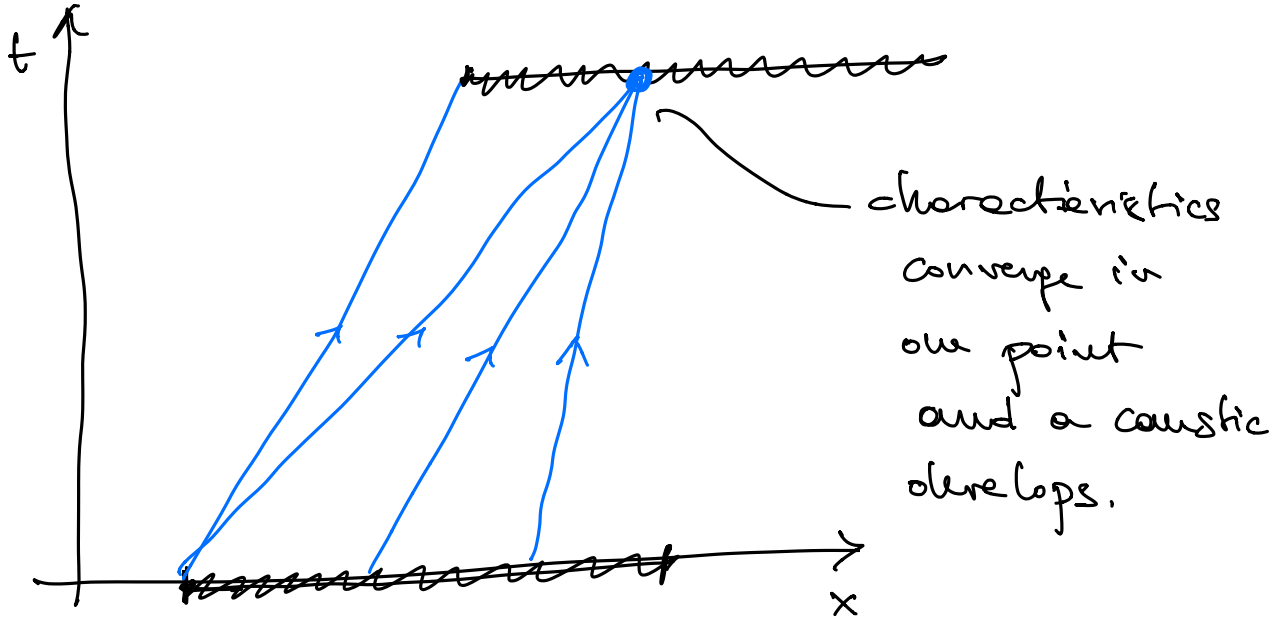
: Burgers equation (nonlinear because $a_{11} = u$)

only difference!

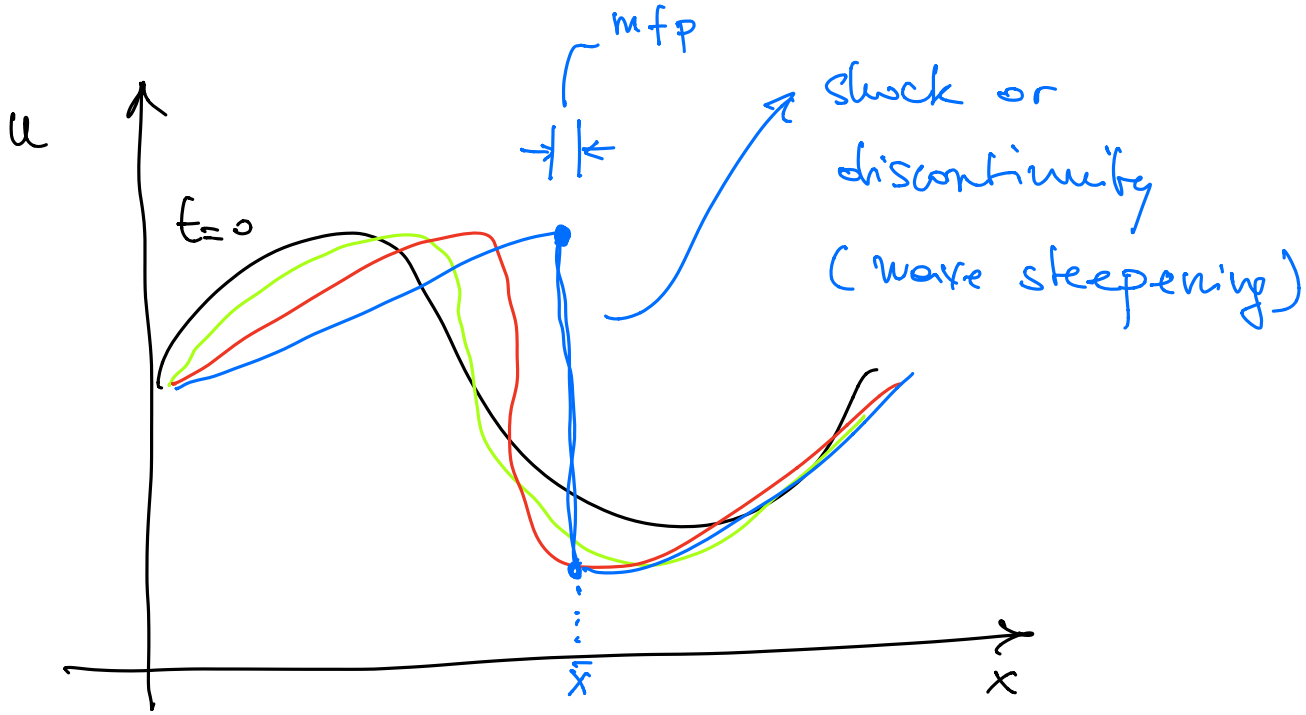
$$\frac{d}{dt}u = \partial_t u + \frac{dx}{dt}\partial_x u = 0$$

along directions

$\lambda = u(x,t)$: functions of position and time



When a caustic develops, then the solution is no longer unique and discontinuities can develop



In non linear equations, caustics can develop even from smooth initial data; ie shocks are inevitable and numerical schemes have to be developed to cope with this problem: HRSC (high-resolution, shock capturing) methods.

There is a way to avoid shocks and nature knows it well: through dissipative terms (diffusion)

$$\partial_t u + u \partial_x u = \underbrace{\varepsilon \partial_x^2 u}_{\text{dissipative terms}}$$

viscous Burgers equation

1. when u smooth $\quad \partial_x^2 u \ll \partial_x u$: nonlin. hyp. eq.
2. " u steepens $\quad \partial_x^2 u \gg \partial_x u$: parabolic eq (diff eq.)