Gravitational Theories
— Introduction to Modified Gravity —

1. Introduction
2. GR and Lovelock gravity
3. PPN formalism
4. EFT of scalar tensor theory
5. Massive gravity
6. Horava-Lifshitz gravity
7. Summary

Shinji Mukohyama (YITP, Kyoto U)

INTRODUCTION

Why modified gravity?

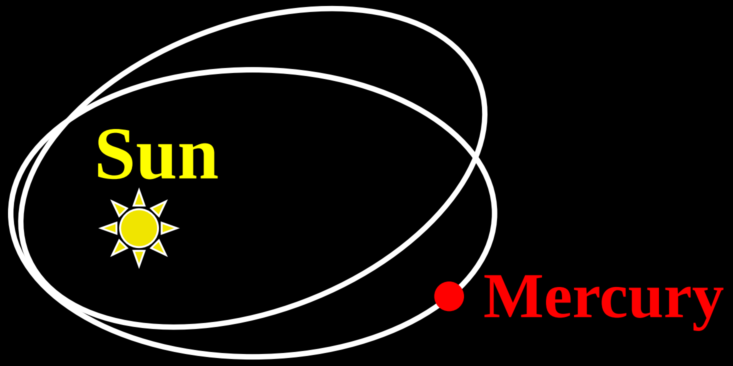
A motivation for IR modification

- Gravity at long distances
  Flattening galaxy rotation curves
  extra gravity
  Dimming supernovae
  accelerating universe

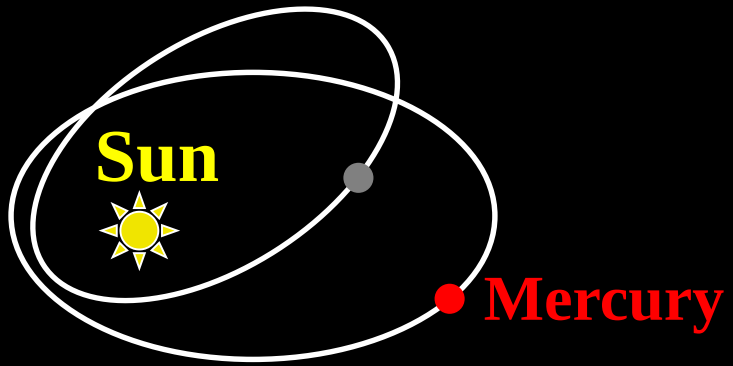
- Usual explanation: new forms of matter (DARK MATTER) and energy (DARK ENERGY).

Dark component in the solar system?

Precession of perihelion
observed in 1800's…



which people tried to explain with a "dark planet", Vulcan,

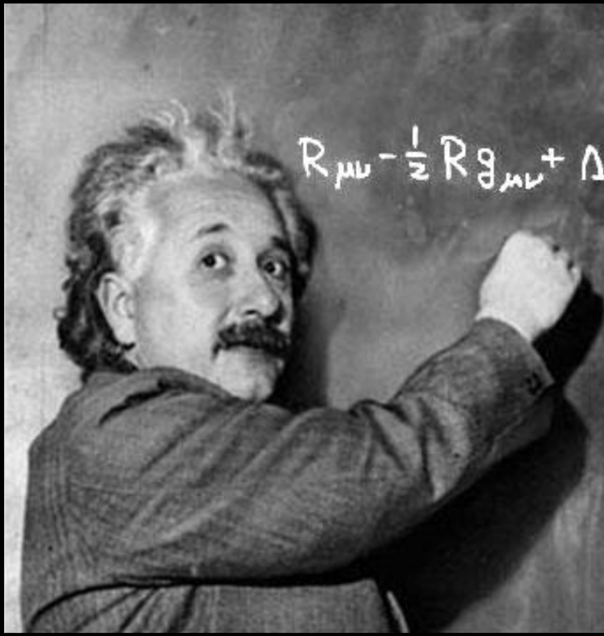


But the right answer wasn't "dark planet", it was "change gravity" from Newton to GR.

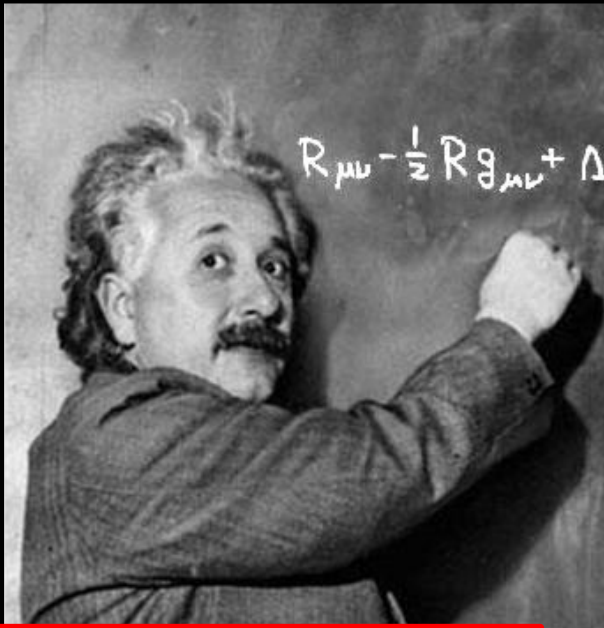
Why modified gravity?

- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.

How to unify Quantum Theory with General Relativity?



How to unify Quantum Theory with General Relativity?



Probably we need to modify GR at short distances

Why modified gravity?

- Can we address **mysteries in the universe**?
  Dark energy, dark matter, inflation, big-bang singularity, cosmic magnetic field, etc.
- Help constructing a **theory of quantum gravity**?
  Superstring, Horava-Lifshitz, etc.
- Do we really **understand GR**?
  One of the best ways to understand something may be to break (modify) it and then to reconstruct it.
- …

Three conditions for good alternative theories of gravity (my personal viewpoint)

1. Theoretically consistent
   e.g. no ghost instability
2. Experimentally viable
   solar system / table top experiments
3. Predictable
   e.g. protected by symmetry

Three conditions for good alternative theories of gravity (my personal viewpoint)

1. Theoretically consistent
   e.g. no ghost instability
2. Experimentally viable
   solar system / table top experiments
3. Predictable
   e.g. protected by symmetry

Some examples

I. Effective field theory (EFT) approach
   IR modification of gravity
   motivation: dark energy/matter

II. Massive gravity
   IR modification of gravity
   motivation: "Can graviton have mass?"

III. Horava-Lifshitz gravity
   UV modification of gravity
   motivation: quantum gravity

IV. Superstring theory
   UV modification of gravity
   motivation: quantum gravity, unified theory

Some examples

I. Effective field theory (EFT) approach
   IR modification of gravity
   motivation: dark energy/matter

II. Massive gravity
   IR modification of gravity
   motivation: “Can graviton have mass?”

III. Horava-Lifshitz gravity
   UV modification of gravity
   motivation: quantum gravity

IV. Superstring theory
   UV modification of gravity
   motivation: quantum gravity, unified theory

Implication of GW170817 on gravity theories @ late time

$$X = -\partial^\mu \phi \partial_\mu \phi$$

- $|(c_{gw} - c_\gamma)/ c_\gamma| < 10^{-15}$
- Horndeski theoy (scalar-tensor theory with 2nd-order eom):
  Among 4 free functions, $G_4(\phi,X)$ & $G_5(\phi,X)$ are strongly constrained. Still $G_2(\phi,X)$ & $G_3(\phi,X)$ are free.
  $G_3(\phi,X)$ may be constrained due to GW-DE interactions [Creminelli, Tambalo, Vernizzi, Yingcharoenrat 2019]
- Generalized Proca theory (vector-tensor theory):
  Among 6 (or more) free functions, $G_4(X)$ & $G_5(X)$ are strongly constrained. Still $G_2(X,F,Y,U)$, $G_3(X)$, $G_6(X)$, $g_5(X)$ are free.

$$X = -A^\mu A_\mu$$

- Horava-Lifshitz theory (renormalizable quantum gravity):
  The coefficient of $R^{(3)}$ is strongly constrained
  → IR fixed point with $c_{gw} = c_\gamma$? How to speed up the RG flow?
- Ghost condensation (EFT of scalar-tensor theory in Minkowski/de Sitter):
  No additional constraint
- Massive gravity (simplest modification of GR):
  Upper bound on graviton mass ≈ 10^{-22}eV
  Much weaker than the requirement from acceleration

c.f. "All" gravity theories (including general relativity):
The cosmological constant is strongly constrained ≈ 10^{-120}.

GENERAL RELATIVITY AND LOVELOCK GRAVITY

Equivalence principle and metric theories of gravity

- **Weak equivalence principle (WEP)**

  uncharged test body
  initial event in spacetime
  initial velocity

  ➡ subsequent trajectory is independent of internal structure & composition

- **Einstein's equivalence principle (EEP)**

  i) **WEP is valid**

  ii) outcome of any local nongravitational test experiment is independent of the velocity of freely falling apparatus and of the time and position in the universe

  Basically saying that **gravity ≃ acceleration**

- **EEP → metric theory**

  EEP → validity of special relativity in local free-falling frame

  → **$^{\exists}$tensor $g_{\mu\nu}$ that reduces to $\eta_{\mu\nu}$ in local free-falling frame**

  This argument does not exclude existence of other metrics.

Einstein’s theory

- **Assumptions**
  EEP (→ ∃metric $g_{\mu\upsilon}$)
  gravity is described by the metric **$g_{\mu\upsilon}$ only**
- **Invariant action** $I = \int d^4x \sqrt{-g}\, L$
  L : scalar made of **$g_{\mu\upsilon}$ & its derivatives**
  up to 1st derivatives → constant only
  up to 2nd derivatives → scalar made of $g_{\mu\upsilon}$ & $R_{\mu\upsilon\rho\sigma}$
  …
- **Ingredients in L**

$$1, R, R^2, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \nabla^\mu R \nabla_\mu R, R^3, \cdots$$

⬇ scale M

$$L = c_0 M^4 + c_1 M^2 R + c_2 R^2 + c_3 R^{\mu\nu}R_{\mu\nu} + c_4 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \cdots$$

⬇ **truncate @ terms with two derivatives**

$$L = c_0 M^4 + c_1 M^2 R = \frac{M_{Pl}^2}{2}(R - 2\Lambda) \qquad \left(M_{Pl}^2 = 2c_1 M^2, \Lambda = -\frac{c_0}{2c_1}M^2\right)$$

This is Einstein-Hilbert action!

c.f. cosmological constant problem = “Why $\left|\frac{c_0}{4c_1^2}\right| \ll 1$?”

Einstein's theory

- **Field equation**

$$I_{EH}=\frac{\mathrm{M_{Pl}^2}}{2}\int d^4x\,\sqrt{-g}(\mathrm{R}-2\Lambda)$$

$$\delta\left(\sqrt{-g}\right)=\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}\qquad (\leftarrow \delta\,(\ln\det A)=\delta(Tr\ln A)=Tr(A^{-1}\delta A))$$

$$\delta\left(\sqrt{-g}R\right)=\sqrt{-g}\{-G^{\mu\nu}\delta g_{\mu\nu}+\nabla^{\mu}\left[\nabla^{\nu}\delta g_{\mu\nu}-\nabla_{\mu}\left(g^{\rho\sigma}\delta g_{\rho\sigma}\right)\right]\}$$

$$\therefore\ \ \delta I_{EH}=\frac{\mathrm{M_{Pl}^2}}{2}\int d^4x\,\sqrt{-g}\left[-(G^{\mu\nu}+\Lambda g^{\mu\nu})\delta g_{\mu\nu}\right]$$

$$\boldsymbol{I_{tot}=I_{EH}+I_{matter}}$$

$$\delta I_{matter}=\int d^4x\,\left[\frac{\sqrt{-g}}{2}T^{\mu\nu}\delta g_{\mu\nu}+(matter\ eom)\delta(matter)\right]$$

$$(T^{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta I_{matter}}{\delta g_{\mu\nu}})$$

$\boldsymbol{\delta I_{tot}=0}$ ➡ $\boldsymbol{M_{Pl}^2(G^{\mu\nu}+\Lambda g^{\mu\nu})=T^{\mu\nu}}$

Einstein eq with $\boldsymbol{G_N=\frac{1}{8\pi M_{Pl}^2}}$

of d.o.f. in general relativity

- 10 metric components → 20-dim phase space @ each point

ADM decomposition

- Lapse N, shift N^i, 3d metric h_{ij}

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

- Einstein-Hilbert action

$$I = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, {}^{(4)}R$$
$$= \frac{M_{\rm Pl}^2}{2} \int dt d^3\vec{x} N \sqrt{h} \left[K^{ij} K_{ij} - K^2 + {}^{(3)}R \right]$$

- Extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - D_i N_j - D_j N_i)$$

of d.o.f. in general relativity

- 10 metric components → 20-dim phase space @ each point
- Einstein-Hilbert action does not contain time derivatives of N & N^i → $\pi_N = 0$ & $\pi_i = 0$

of d.o.f. in general relativity

- 10 metric components → 20-dim phase space @ each point
- Einstein-Hilbert action does not contain time derivatives of N & N^i → $\pi_N = 0$ & $\pi_i = 0$
  All constraints are independent of N & N^i → π_N & π_i "commute with" all constraints → 1st-class

1st-class vs 2nd-class

- **2nd-class constraint S**
  $\{ S , C_i \} \not\approx 0$ for $^{\exists}i$
  Reduces 1 phase space dimension
- **1st-class constraint F**
  $\{ F , C_i \} \approx 0$ for $^{\forall}i$
  Reduces 2 phase space dimensions
  Generates a symmetry
  Equivalent to a pair of 2nd-class constraints

$\{ C_i \mid i = 1,2,\ldots \}$: complete set of independent constraints

$A \approx B$ ⟺ $A = B$ when all constraints are imposed

(weak equality)

of d.o.f. in general relativity

- 10 metric components → 20-dim phase space @ each point
- Einstein-Hilbert action does not contain time derivatives of N & N^i → $\pi_N = 0$ & $\pi_i = 0$
  All constraints are independent of N & N^i → π_N & π_i "commute with" all constraints → 1st-class
- 4 generators of 4d-diffeo: 1st-class constraints
- 20 – (4+4) x 2 = 4 → 4-dim physical phase space @ each point → 2 local physical d.o.f.

# of d.o.f. in GR = 2,
corresponding to TT gravitational waves

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- Lovelock’s theorem

(i) $A^{\mu\upsilon}$ is a symmetric tensor (μ,ν = 0,1,2,3)
(ii) $A^{\mu\upsilon} = A^{\mu\upsilon}(g_{\rho\sigma}, g_{\rho\sigma,\alpha}, g_{\rho\sigma,\alpha\beta})$ (++++) (valid for (-+++) as well)
(iii) $A^{\mu\upsilon}{}_{;\nu} = 0$ (;ν represents covariant derivative)
(iv) 4-dimensions

➡ $A^{\mu\nu} = a\,G^{\mu\nu} + bg^{\mu\nu}$ (a, b: constants, $G^{\mu\nu}$: Einstein tensor)

- Motivation for assumptions (i)-(iii)

(i) $A^{\mu\nu}$ is to be EOM for $g_{\mu\upsilon}$ and thus should be symmetric.
(ii) If EOM depends on 3rd or higher derivatives of $g_{\mu\upsilon}$ then # of d.o.f. (in Lorentzian case) may increase.
(iii) If $\exists I$ s.t. $A^{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta I}{\delta g_{\mu\nu}}$ and if I is diffeo invariant then $A^{\mu\nu}{}_{;\nu} = 0$.

$$\left(\begin{array}{l} \because \; g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \;\; \delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \\ \quad 0 = \delta I = \int d^4x \frac{\delta I}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -2\int d^4x \sqrt{g}\, \xi_\nu A^{\mu\nu}{}_{;\nu} \quad \text{for}^{\forall} \xi_\nu \end{array} \right)$$

c.f. “symmetric” in (i) can be dropped (J.Math.Phys. 13, 874 (1972)) .

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- What Lovelock actually proved

  In n-dim. Lovelock proved theorems 1 & 2 below

- **Theorem 1**

(i)-(iii) ➡ $\boldsymbol{A^{\mu\nu} = \sum_{k=1}^{m-1} c_k \theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} \prod_{h=1}^{k} R_{\alpha_{4h-1}\alpha_{4h-3}\alpha_{4h-2}\alpha_{4h}} + b g^{\mu\nu}}$

$$\mathrm{m} = \begin{cases} \frac{n}{2} & (n: even) \\ \frac{1}{2}(n+1) & (n: odd) \end{cases} , \qquad c_k, b : \text{const.}$$

$\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}$ $(\mathrm{k} = 1, \cdots, m-1)$: a tensor satisfying (a)-(d) below

(a) $\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} = \theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}(g_{\rho\sigma})$

(b) symmetric in (ij) and in $(i_{2h-1}i_{2h})$ for $h = 1, \cdots, 2k$

(c) symmetric under interchange of the pair $(\mu\nu)$ with the pair $(\alpha_{2h-1}\alpha_{2h})$ for all $h = 1, \cdots, 2k$

(d) the cyclic sum involving any three of the four indices $(\mu\nu)$ $(\alpha_{2h-1}\alpha_{2h})$ for $h = 1, \cdots, 2k$ vanishes

(c) follows from (b)&(d)

$$0 = \theta^{\mu(\nu\alpha_1\alpha_2)} + \theta^{\nu(\alpha_1\alpha_2\mu)} - \theta^{\alpha_1(\alpha_2\mu\nu)} - \theta^{\alpha_2(\mu\nu\alpha_1)} = \frac{1}{3}(\theta^{\mu\nu\alpha_1\alpha_2} - \theta^{\alpha_1\alpha_2\mu\nu})$$

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

- **Theorem 2**

p : positive integer

$\psi^{\mu\nu\alpha_1\cdots\alpha_{2p}}$ is a tensor with the following properties (a)'-(d)'

(a)' : (a) with $\theta \to \psi, 4k \to 2p$

(b)'-(d)' : (b)-(d) with $2k \to p$

➡ $$(n-p)\boxed{\psi^{\mu\nu\alpha_1\cdots\alpha_{2p}}} = g^{\mu\nu}\boxed{g_{\rho\sigma}\psi^{\rho\sigma\alpha_1\cdots\alpha_{2p}}} - \frac{1}{2}\sum_{h=1}^{2p} g^{\alpha_h\nu}\boxed{g_{\rho\sigma}\psi^{\rho\sigma\alpha_1\cdots\alpha_{h-1}\mu\alpha_{h+1}\cdots\alpha_{2p}}}$$

Theorem 2 shows a way to calculate $\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}$ defined in Theorem 1 and its uniqueness (up to an overall constant factor).

- **Corollary 1**

n=2 → $A^{\mu\nu} = bg^{\mu\nu} \quad (b: const.)$

(proof of corollary 1)

m=1 for n=2 ∴ corollary 1 follows from theorem 1 Q.E.D.

- **Corollary 2**

n=3 or 4 → $A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \quad (a, b: const.)$

Corollary 2 for n=4 is what is usually known as Lovelock's theorem.

Uniqueness of GR (ref. D.Lovelock, Aquationes Mathematica 4 (1970) 127)

(proof of corollary 2)

- m=2 for n=3 or 4
- Theorem 2 with p=2

$$\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} \equiv g_{\rho\sigma}\theta^{\rho\sigma\alpha_1\alpha_2\alpha_3\alpha_4}$$

$$(n-2)\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4} = g^{\mu\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} - \frac{1}{2}\Big(g^{\alpha_1\nu}\tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4} + g^{\alpha_2\nu}\tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4} + g^{\alpha_3\nu}\tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4} + g^{\alpha_4\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}\Big)$$

with p=1

$$(n-1)\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4} = g^{\alpha_1\alpha_2}\tilde{\tilde{\theta}}^{\alpha_3\alpha_4} - \frac{1}{2}\Big(g^{\alpha_3\alpha_2}\tilde{\tilde{\theta}}^{\alpha_1\alpha_4} + g^{\alpha_4\alpha_2}\tilde{\tilde{\theta}}^{\alpha_3\alpha_1}\Big)$$

$$\tilde{\tilde{\theta}}^{\alpha_1\alpha_2} \equiv g_{\rho\sigma}\tilde{\theta}^{\rho\sigma\alpha_1\alpha_2}$$

$\tilde{\tilde{\theta}}^{\alpha_1\alpha_2}(g_{\mu\nu})$ is a symmetric tensor → $\tilde{\tilde{\theta}}^{\alpha_1\alpha_2} = \tilde{a}g^{\alpha_1\alpha_2}$ $(\tilde{a} : const.)$

[lemma A2 of D.Lovelock, Arch.Ratl.Mech.Anal. 33 (1969) 54 restricted to symmetric part]

➡ $$2(n-2)\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = 2g^{\mu\nu}\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} - \tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4}R_{\alpha_3}{}^{\nu}{}_{\alpha_2\alpha_4} - \tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4}R_{\alpha_3\alpha_1}{}^{\nu}{}_{\alpha_4} - \tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4}R^{\nu}{}_{\alpha_1\alpha_2\alpha_4} - \tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}R_{\alpha_3\alpha_1\alpha_2}{}^{\nu}$$

$$\tilde{\theta}^{\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = -\frac{3\tilde{a}}{2(n-1)}R$$

$$\tilde{\theta}^{\mu\alpha_2\alpha_3\alpha_4}R_{\alpha_3}{}^{\nu}{}_{\alpha_2\alpha_4} = \tilde{\theta}^{\alpha_1\mu\alpha_3\alpha_4}R_{\alpha_3\alpha_1}{}^{\nu}{}_{\alpha_4} = \tilde{\theta}^{\alpha_1\alpha_2\mu\alpha_4}R^{\nu}{}_{\alpha_1\alpha_2\alpha_4} = \tilde{\theta}^{\alpha_1\alpha_2\alpha_3\mu}R_{\alpha_3\alpha_1\alpha_2}{}^{\nu} = -\frac{3\tilde{a}}{2(n-1)}R^{\mu\nu}$$

➡ $$\theta^{\mu\nu\alpha_1\alpha_2\alpha_3\alpha_4}R_{\alpha_3\alpha_1\alpha_2\alpha_4} = \frac{3\tilde{a}}{(n-1)(n-2)}G^{\mu\nu}$$

- Theorem 1

➡ $$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu} \qquad a = \frac{3c_1\tilde{a}}{(n-1)(n-2)}$$

Q.E.D.

How to go beyond GR?

- Lovelock's theorem (to be more precise, corollary 2) assumes
  - 4-dim. (pseudo-)Riemannian geometry
  - the metric is the only physical field

(The theorem is at the level of eoms.)

- Modification of GR (at the level of eoms) then requires at least one of the following
  - extra dimension
  - extra dof.
  - Lorentz violation
  - non (pseudo-)Riemannian geometry

Lovelock gravity (simplest generalization of GR in higher-dim.)

(ref. D.Lovelock, J.Math.Phys. 12 (1971) 498)

- **A solution to the recursion relation in theorem 2** with p=2k, 1≦k≦m-1

$$\psi^{\mu\nu\alpha_1\cdots\alpha_{4k}} = \left(\delta^{\mu\rho_1\cdots\rho_{2k}}_{\beta\sigma_1\cdots\sigma_{2k}} g^{\beta\nu} + \delta^{\nu\rho_1\cdots\rho_{2k}}_{\beta\sigma_1\cdots\sigma_{2k}} g^{\beta\mu}\right) g^{\sigma_1\lambda_1}\cdots g^{\sigma_{2k}\lambda_{2k}} D^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\rho_1\rho_2\lambda_1\lambda_2}\cdots D^{\alpha_{4k-3}\alpha_{4k-2}\alpha_{4k-1}\alpha_{4k}}_{\rho_{2k-1}\rho_{2k}\lambda_{2k-1}\lambda_{2k}}$$

$$m = \begin{cases} \frac{n}{2} & (n : \text{even}) \\ \frac{n+1}{2} & (n : \text{odd}) \end{cases} \qquad \delta^{\alpha_1\cdots\alpha_N}_{\beta_1\cdots\beta_N} = \det \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \cdots & \delta^{\alpha_1}_{\beta_N} \\ \vdots & & \vdots \\ \delta^{\alpha_N}_{\beta_1} & \cdots & \delta^{\alpha_N}_{\beta_N} \end{vmatrix}$$

$$D^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\lambda} = \frac{1}{2}(\delta^\mu_\alpha\delta^\nu_\lambda + \delta^\mu_\lambda\delta^\nu_\alpha)(\delta^\rho_\beta\delta^\sigma_\gamma + \delta^\rho_\gamma\delta^\sigma_\beta)$$

- Since Theorem 2 implies **the uniqueness of $\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}}$** in Theorem 1,

$$\theta^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} = b_k\psi^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} \qquad (b_k : const.)$$

- It is straightforward to calculate

$$\psi^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{4k-1}\alpha_{4k}} \prod_{h=1}^{k} R_{\alpha_{4h-1}\alpha_{4h-3}\alpha_{4h-2}\alpha_{4h}} = 2\left(\frac{3}{2}\right)^k \delta^{\mu\alpha_1\cdots\alpha_{2k}}_{\rho\beta_1\cdots\beta_{2k}} g^{\rho\nu} \prod_{h=1}^{k} R_{\alpha_{2h-1}\alpha_{2h}}{}^{\beta_{2h-1}\beta_{2h}}$$

The r.h.s. is symmetric in (μν).

- In this way, **Lovelock established the following theorem**.

Lovelock gravity (simplest generalization of GR in higher-dim.)

(ref. D.Lovelock, J.Math.Phys. 12 (1971) 498)

- **Theorem 3**

  If $A^{\mu\nu}$ satisfies (i)-(iii) then $a_k, a : const.$

$$A^{\mu\nu} = \sum_{k=1}^{m-1} a_k g^{\nu\rho} \delta^{\mu\alpha_1\cdots\alpha_{2k}}_{\rho\beta_1\cdots\beta_{2k}} \prod_{h=1}^{k} R_{\alpha_{2h-1}\alpha_{2h}}{}^{\beta_{2h-1}\beta^{2h}} + bg^{\mu\nu}$$

$$m = \begin{cases} \frac{n}{2} & (n : \text{even}) \\ \frac{n+1}{2} & (n : \text{odd}) \end{cases} \qquad \delta^{\alpha_1\cdots\alpha_N}_{\beta_1\cdots\beta_N} = \det \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \cdots & \delta^{\alpha_1}_{\beta_N} \\ \vdots & & \vdots \\ \delta^{\alpha_N}_{\beta_1} & \cdots & \delta^{\alpha_N}_{\beta_N} \end{vmatrix}$$

- Lovelock then found an action whose Euler-Lagrange eq is $A^{\mu\nu} = 0$.

$$I = \int d^n x \sqrt{-g} \left[\sum_{k=1}^{m-1} 2a_k \delta^{\alpha_1\cdots\alpha_{2k}}_{\beta_1\cdots\beta_{2k}} \prod_{h=1}^{k} R_{\alpha_{2h-1}\alpha_{2h}}{}^{\beta_{2h-1}\beta^{2h}} + 2b \right]$$

This theory is called Lovelock gravity.
The first two are Einstein-Hilbert (k=1) & Gauss-Bonnet (k=2) terms.
The last (2b) is cosmological constant term.

PARAMETRIZED POST-NEWTONIAN (PPN) FORMALISM

Formalism

[ref. C. M. Will, "Theory and experiment in gravitational physics" (Cambridge)]

- **Stress-energy tensor (perfect fluid)**

$$T_{\mu\nu} = \boldsymbol{\rho}(1+\boldsymbol{\Pi})u_\mu u_\nu + \boldsymbol{P}\left(u_\mu u_\nu + g_{\mu\nu}\right)$$

$u_\mu = g_{\mu\nu}u^\nu,\ u^\mu = \left(u^0, u^0 \boldsymbol{v}^i\right)$: 4-velocity

$$\left.\begin{array}{l} \boldsymbol{v}^i \text{ : 3-velocity} \\ \boldsymbol{\rho} \text{ : rest mass density} \\ \boldsymbol{P} \text{ : isotropic pressure} \\ \boldsymbol{\Pi} \text{ : specific energy density} \end{array}\right\} \quad \text{basic variables}$$

$$u^\mu u_\mu = -1,\ \nabla_\mu(\rho u^\mu) = 0,\ \nabla_\mu T^{\mu\nu} = 0$$

- **Post-Newtonian bookkeeping**

$$\boxed{v = \mathcal{O}(\epsilon)} \qquad |\partial_t| \sim \mathcal{O}(\epsilon)|\vec{\nabla}| \qquad \frac{G_{\rm N} l^3}{L}\rho = \mathcal{O}(\epsilon^2) \qquad \frac{P}{\rho} = \mathcal{O}(\epsilon^2) \qquad \Pi = \mathcal{O}(\epsilon^2)$$

- **Newtonian metric**

$$\left[\begin{array}{l} g_{00} = -1 + 2U + \mathcal{O}(\epsilon^4) \\ g_{0i} = \mathcal{O}(\epsilon^3) \\ g_{ij} = \delta_{ij} + \mathcal{O}(\epsilon^2) \end{array}\right. \qquad \left(\begin{array}{l} U(t,\vec{x}) = G_N \int \frac{\rho'}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \\ \rho' = \rho(t,\vec{x}') \end{array}\right)$$

Formalism

- **PPN metric**

$$
\begin{cases}
g_{00} = -1 + 2U + 2(\psi - \beta U^2) + \zeta_{\mathcal{B}}\mathcal{B} + \mathcal{O}(\epsilon^6) \\
\left(\begin{aligned}
\psi &= \frac{1}{2}(2\gamma + 1 + \alpha_3 + \zeta_1 - \zeta_{\mathcal{B}} - 2\xi)\Phi_1 + (1 - 2\beta + \zeta_2 + \xi)\Phi_2 \\
&\quad + (1 + \zeta_3)\Phi_3 + (3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - \frac{1}{2}(\zeta_1 - \zeta_{\mathcal{B}} - 2\xi)\Phi_6 - \xi\Phi_W \\
\mathcal{B} &= -X_{,00} + \Phi_1 - \Phi_6
\end{aligned}\right) \\
g_{0i} = -\left[2(1+\gamma) + \frac{1}{2}\alpha_1\right] V_i - \frac{1}{2}\left[1 + \alpha_2 - \zeta_1 + \zeta_{\mathcal{B}} + 2\xi\right] X_{,0i} + \mathcal{O}(\epsilon^5) \\
g_{ij} = (1 + 2\gamma U)\delta_{ij} + \mathcal{O}(\epsilon^4)
\end{cases}
$$

- **Def. of potentials**

$$\Delta\Phi = -4\pi s \quad \Leftrightarrow \quad \Phi = \int \frac{s'}{|\vec{x} - \vec{x}'|} d^3\vec{x}$$

$$\Delta^2\Psi = -8\pi s \quad \Leftrightarrow \quad \Psi = \int s'|\vec{x} - \vec{x}'| d^3\vec{x}$$

$$\Delta U = -4\pi G_{\rm N}\rho^* \,,$$

$$\Delta^2 X = -8\pi G_{\rm N}\rho^* \,,$$

$$\Delta V_i = -4\pi G_{\rm N}\rho^*\delta_{ij}v^j \,,$$

$$\Delta\Phi_1 = -4\pi G_{\rm N}\rho^* v^2 \,,$$

$$\Delta\Phi_2 = -4\pi G_{\rm N}\rho^* U \,,$$

$$\Delta\Phi_3 = -4\pi G_{\rm N}\rho^*\Pi \,,$$

$$\rho^* \equiv \rho\sqrt{-g}u^0$$

$$\Delta\Phi_4 = -4\pi G_{\rm N} P \,,$$

$$\Delta^2\Phi_6 = 8\pi G_{\rm N}\left[\partial_i\partial_j(\rho^* v^i v^j) - \frac{1}{2}\Delta(\rho^* v^2)\right] \,,$$

$$\Delta\Phi_W = -2\delta^{ik}\delta^{jl}\partial_i\partial_j X\partial_k\partial_l U - 4\delta^{ij}\partial_i U\partial_j U + 4\pi G_{\rm N}\rho^* U$$

c.f. $\zeta_{\mathcal{B}}$ cannot be set to zero if time-diffeo is broken either explicitly or spontaneously, e.g. in Horava gravity [Lin, Mukohyama, Wang, Zhu 2013].

Formalism

- **Residual gauge freedom (in 4d-diffeo invariant theories)**
  $x^\mu \rightarrow x^\mu + \xi^\mu \; with \; \boldsymbol{\xi_0 = \lambda \partial_0 X}, \xi_i = 0, \lambda = const.$

$$\left\{\begin{array}{l} g_{00} \to g_{00} + 2\lambda(\Phi_6 + \mathcal{B} - \Phi_1) \\ g_{0i} \to g_{0i} - \lambda X_{,0i} \\ g_{ij} \to g_{ij} \end{array}\right. \qquad \left\{\begin{array}{l} \zeta_{\mathcal{B}} \to \zeta_{\mathcal{B}} + 2\lambda \\ \text{others unchanged} \end{array}\right.$$

  $\zeta_{\mathcal{B}}$ can be set to zero by time diffeo

- **10 (+1) PPN parameters**
  $\boldsymbol{\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4}$ **10 observable parameters**
  $\boldsymbol{\zeta_{\mathcal{B}}}$ **unobservable if the matter sector has 4d-diffeo invariance**
  even if the gravity sector does not
  (In 4d-diffeo invariant theories, $\zeta_{\mathcal{B}}$ is gauge freedom. In theories without time-diffeo, $\zeta_{\mathcal{B}}$ is physical but cannot be probed by matter if the matter sector is (approximately) diffeo-invariant.)

- **General relativity**
  $\gamma = 1, \beta = 1, \xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$
  ($\zeta_{\mathcal{B}}$ is gauge freedom.)

Limits on PPN parameters

[ref. C. M. Will, “Theory and experiment in gravitational physics” (Cambridge); C. M. Will, Living Rev. Relativity 17 (2014) 4]

$\gamma - 1$	2.3×10^{-5} (time delay) , 1.2×10^{-4} (light deflection)	
$\beta - 1$	8×10^{-5} (periherion shift) , 2.3×10^{-4} (Nordtvedt effect)	
ξ	10^{-3} (Earth tides)	
α_1	10^{-4} (orbital polarization)	
α_2	4×10^{-7} (spin precession)	
ζ_1	2×10^{-2} (combined PPN bound)	
ζ_3	10^{-8} (Newton’s 3rd law)	
$\hat{\xi}$	4×10^{-9} (spin precession)	Strong gravity
$\hat{\alpha}_1$	7×10^{-5} (orbital polarization)	
$\hat{\alpha}_2$	2×10^{-9} (spin precession)	
$\hat{\alpha}_3$	4×10^{-20} (pulsar acceleration)	
$\hat{\zeta}_2$	4×10^{-5} (binary acceleration)	

Scalar-tensor theory as an example

- Basic variables

metric $g_{\mu\nu}$, scalar ϕ , matter $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I_{matter}}{\delta g_{\mu\nu}}$

- Action $I = I_g\left[g_{\mu\nu}, \phi\right] + I_{matter}\left[g_{\mu\nu}, matter\right]$

$$I_g = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right]$$

- ϕ-eom

$$(3 + 2\omega)\nabla^2\phi + \frac{d\omega}{d\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 8\pi T \qquad (T \equiv T_\mu^\mu)$$

- g-eom

$$\phi R_{\mu\nu} - \left(\nabla_\mu \nabla_\nu \phi + \frac{1}{2} \nabla^2 \phi g_{\mu\nu}\right) - \frac{\omega}{\phi} \partial_\mu \phi \partial_\nu \phi = 8\pi \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

Scalar-tensor theory as an example

- PPN expansion

  $g_{\mu\nu} \to$ PPN metric with $\zeta_{\mathcal{B}} = 0$ (and thus $\zeta_1 = \tilde{\zeta}_1$)

  $\phi = \phi_0 + \phi_2 + \phi_4 + \mathcal{O}(\epsilon^6)$

  $$\phi_0 = const. = \mathcal{O}(\epsilon^0)$$
  $$\phi_2 = 2\gamma_\phi U$$
  $$\phi_4 = c_{UU}U^2 + c_W\Phi_W + c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 + c_4\Phi_4 + c_6\Phi_6 + c_{\mathcal{B}}\mathcal{B}$$

  $T_{\mu\nu}$ →perfect fluid form

- 10 PPN parameters + G_N (+ unobservable parameters)

$\boldsymbol{\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4}$	**10**	**observable**
$\boldsymbol{G_N}$	**1**	**defines the unit**
$\gamma_\phi, c_{UU}, c_W, c_1, c_2, c_3, c_4, c_6, c_{\mathcal{B}}$	**9**	**unobservable**

- Computation

  i) ϕ-eom of $\mathcal{O}(\epsilon^2)$
  $(g$-eom$)_{00}$ of $\mathcal{O}(\epsilon^2)$ } → solve w.r.t. (G_N, γ_ϕ)

  $$G_N = \frac{2(2+\omega_0)}{\phi_0(3+2\omega_0)}, \gamma_\phi = \frac{\phi_0}{2(2+\omega_0)}$$

Scalar-tensor theory as an example

- Computation continued

ii) $\delta^{ij}(g\text{-eom})_{ij}$ of $\mathcal{O}(\epsilon^2)$ → solve w.r.t. γ

$$\gamma = 1 - \frac{1}{2+\omega_0}$$

iii) $(g\text{-eom})_{0i}$ of $\mathcal{O}(\epsilon^3)$ → solve w.r.t. α_1

$$\boldsymbol{\alpha_1 = 0}$$

iv) ϕ-eom of $\mathcal{O}(\epsilon^4)$ → solve w.r.t. $(c_{UU}, c_W, c_1, c_2, c_3, c_4, c_6, c_{\mathcal{B}})$

$$c_{UU} = \frac{\phi_0[2\omega_0 - \left(\frac{d\omega}{d\phi}\right)_0 \phi_0 + 3]}{2(3+2\omega_0)(2+\omega_0)^2}, c_W = 0, c_1 = \frac{\phi_0}{2(2+\omega_0)}, c_2 = \frac{\phi_0[4\omega_0^2 + 8\omega_0 - \left(\frac{d\omega}{d\phi}\right)_0 \phi_0 + 3]}{(3+2\omega_0)(2+\omega_0)^2}$$

$$c_3 = \frac{\phi_0}{2+\omega_0}, c_4 = -\frac{3\phi_0}{2+\omega_0}, c_6 = -\frac{\phi_0}{2(2+\omega_0)}, c_{\mathcal{B}} = -\frac{\phi_0}{2(2+\omega_0)}$$

v) $(g\text{-eom})_{00}$ of $\mathcal{O}(\epsilon^4)$ → solve w.r.t. $(\beta, \xi, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$

$$\beta = 1 + \frac{\left(\frac{d\omega}{d\phi}\right)_0 \phi_0}{4(3+2\omega_0)(2+\omega_0)^2}$$

$$\boldsymbol{\xi = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0}$$

vi) Setting $G_N = \frac{2(2+\omega_0)}{\phi_0(3+2\omega_0)} = 1$ → $\beta = 1 + \frac{\left(\frac{d\omega}{d\phi}\right)_0}{(3+2\omega_0)^2(4+2\omega_0)}$

Summary of PPN formalism

- One can go beyond GR, but only to the extent that it is consistent with all experimental constraints.
- There are many theories and many experiments.
- Thanks to the PPN formalism, possible deviations from GR at the solar system scale are universally constrained by experiments.
- 10 PPN parameters + G_N : calculable from theories and constrained by solar system scale experiments.
- Table of constraints on PPN parameters.
- Calculation of PPN parameters in scalar-tensor theory as an example.
- Similar calculations can be done in your favorite theories!

EFFECTIVE FIELD THEORY OF SCALAR TENSOR THEORY

Many modified gravity theories

- 3 check points
  “What are the physical d.o.f. ?”
  “How do they interact ?”
  “What is the regime of validity ?”
- If two (or more) theories give the same answers to the 3 questions above then they are the same even if they look different.
  → **Universal description**

Scalar-tensor theories

- Metric $g_{\mu\nu}$ + scalar field ϕ
- Jordan (1955), Brans & Dicke (1961), Bergmann (1968), Wagoner (1970), …
- Most general scalar-tensor theory with 2nd order covariant EOM: Horndeski (1974)
- DHOST theories beyond Horndeski: Langlois & Noui (2016)
- All of them (and more) are universally described by an effective field theory (EFT)

EFT of inflation/DE

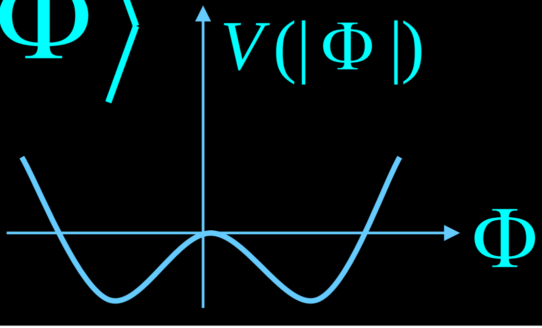
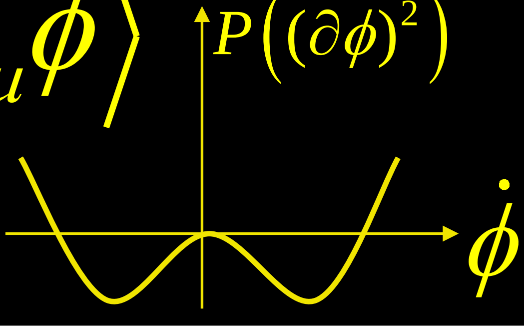
- **Time diffeo is broken by the background but spatial diffeo is preserved.**
- All terms that respect spatial diffeo must be included in the EFT action.
- Derivative & perturbative expansions
- Diffeo can be restored by introducing NG boson

EFT of inflation/DE

- **Time diffeo is broken by the background but spatial diffeo is preserved.**
- All terms that respect spatial diffeo must be included in the EFT action.
- Derivative & perturbative expansions
- Diffeo can be restored by introducing NG boson

Simplest : ghost condensation

ref. Arkani-Hamed, Cheng, Luty, Mukohyama 2004

	Higgs mechanism	**Ghost condensate** Arkani-Hamed, Cheng, Luty and Mukohyama 2004
Order parameter	$\langle \Phi \rangle$ 	$\langle \partial_\mu \phi \rangle$ 
Instability	Tachyon $-\mu^2\Phi^2$	Ghost $-\dot{\phi}^2$
Condensate	V'=0, V''>0	P'=0, P''>0
Broken symmetry	Gauge symmetry	Time translational symmetry
Force to be modified	Gauge force	Gravity
New force law	Yukawa type	Newton+Oscillation

Systematic construction of Low-energy effective theory

Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405:074,2004

Backgrounds characterized by

✧ $\langle \partial_\mu \phi \rangle \neq 0$ and timelike

✧Background metric is maximally symmetric, either Minkowski or dS.

➡

$$
\begin{aligned}
L_{eff} = L_{EH} + M^4 \Big\{ & \left(h_{00} - 2\dot{\pi}\right)^2 - \frac{\alpha_1}{M^2}\left(K + \vec{\nabla}^2 \pi\right)^2 \\
& - \frac{\alpha_2}{M^2}\left(K^{ij} + \vec{\nabla}^i \vec{\nabla}^j \pi\right)\left(K_{ij} + \vec{\nabla}_i \vec{\nabla}_j \pi\right) + \cdots \Big\}
\end{aligned}
$$

Gauge choice: $\phi(t,\vec{x}) = t.$ $\pi \equiv \delta\phi = 0$

(Unitary gauge)

Residual symmetry: $\vec{x} \to \vec{x}'(t,\vec{x})$

⇨ Write down most general action invariant under this residual symmetry.

(⇨ Action for π: undo unitary gauge!)

Start with flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Under residual ξ^i

$$\delta h_{00} = 0, \delta h_{0i} = \partial_0 \xi_i, \delta h_{ij} = \partial_i \xi_j + \partial_j \xi_i$$

Action invariant under ξ^i

$$\begin{cases} \left(h_{00}\right)^2 & \textbf{OK} \\ \cancel{\left(h_{0i}\right)^2} & \\ K^2, K^{ij}K_{ij} & \textbf{OK} \end{cases}$$

Beginning at quadratic order, since we are assuming flat space is good background.

$$K_{ij} = \frac{1}{2}\left(\partial_0 h_{ij} - \partial_j h_{0i} - \partial_i h_{0j}\right)$$

$$\Rightarrow \quad L_{eff} = L_{EH} + M^4\left\{\left(h_{00}\right)^2 - \frac{\alpha_1}{M^2}K^2 - \frac{\alpha_2}{M^2}K^{ij}K_{ij} + \cdots\right\}$$

Action invariant under ξ^i

$$\begin{cases} (h_{00})^2 & \textbf{OK} \\ \cancel{(h_{0i})^2} & \\ K^2, K^{ij}K_{ij} & \textbf{OK} \end{cases}$$

Beginning at quadratic order, since we are assuming flat space is good background.

$$K_{ij} = \frac{1}{2}\left(\partial_0 h_{ij} - \partial_j h_{0i} - \partial_i h_{0j}\right)$$

$$\Rightarrow \quad L_{eff} = L_{EH} + M^4\left\{(h_{00})^2 - \frac{\alpha_1}{M^2}K^2 - \frac{\alpha_2}{M^2}K^{ij}K_{ij} + \cdots\right\}$$

Action for π

$$\xi^0 = \pi \quad \begin{cases} h_{00} \rightarrow h_{00} - 2\partial_0\pi \\ K_{ij} \rightarrow K_{ij} + \partial_i\partial_j\pi \end{cases}$$

$$\Rightarrow \quad \boxed{\begin{aligned} L_{eff} = L_{EH} + M^4\Big\{&(h_{00} - 2\dot{\pi})^2 - \frac{\alpha_1}{M^2}\left(K + \vec{\nabla}^2\pi\right)^2 \\ &- \frac{\alpha_2}{M^2}\left(K^{ij} + \vec{\nabla}^i\vec{\nabla}^j\pi\right)\left(K_{ij} + \vec{\nabla}_i\vec{\nabla}_j\pi\right) + \cdots\Big\} \end{aligned}}$$

$$\left.\begin{aligned} E &\to rE \\ dt &\to r^{-1}dt \\ dx &\to r^{-1/2}dx \\ \pi &\to r^{1/4}\pi \end{aligned}\right\}$$

Make invariant ➡ $\int dtd^3x\left[\frac{1}{2}\dot{\pi}^2 - \frac{\alpha(\vec{\nabla}^2\pi)^2}{M^2} + \cdots\right]$

Leading nonlinear operator in infrared $\int dtd^3x\frac{\dot{\pi}(\nabla\pi)^2}{\tilde{M}^2}$

has scaling dimension 1/4. **(Barely) irrelevant**

⇨ Good low-E effective theory

Robust prediction

e.g. Ghost inflation [Arkani-hamed, Creminelli, Mukohyama, Zaldarriaga 2004]

Extension to FLRW background = EFT of inflation/dark energy

Creminelli, Luty, Nicolis, Senatore 2006
Cheung, Creminelli, Fitzpatrick, Kaplan, Senatore 2007

- Action invariant under x^i → $x^i(t,x)$
- Ingredients
  $g_{\mu\nu}, g^{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\mu,$ t & its derivatives
- 1^{st} derivative of t

$$\partial_\mu t = \delta^0_\mu \qquad n_\mu = \frac{\partial_\mu t}{\sqrt{-g^{\mu\nu}\partial_\mu t \partial_\nu t}} = \frac{\delta^0_\mu}{\sqrt{-g^{00}}}$$

$$g^{00} \qquad h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- 2^{nd} derivative of t

$$K_{\mu\nu} \equiv h^\rho_\mu \nabla_\rho n_\nu$$

Unitary gauge action

$$\boxed{I = \int d^4x\sqrt{-g}L(t,\delta^0_\mu, K_{\mu\nu}, g_{\mu\nu}, g^{\mu\nu}, \nabla_\mu, R_{\mu\nu\rho\sigma})}$$

derivative & perturbative expansions

$$\begin{aligned} I &= M_{Pl}^2\int dx^4\sqrt{-g}\left[\frac{1}{2}R + c_1(t) + c_2(t)g^{00}\right. \\ &\qquad \left. +L^{(2)}(\tilde{\delta}g^{00}, \tilde{\delta}K_{\mu\nu}, \tilde{\delta}R_{\mu\nu\rho\sigma}; t, g_{\mu\nu}, g^{\mu\nu}, \nabla_\mu)\right] \end{aligned}$$

$$\begin{aligned} L^{(2)} &= \lambda_1(t)(\tilde{\delta}g^{00})^2 + \lambda_2(t)(\tilde{\delta}g^{00})^3 + \lambda_3(t)\tilde{\delta}g^{00}\tilde{\delta}K^\mu_\mu \\ &\quad +\lambda_4(t)(\tilde{\delta}K^\mu_\mu)^2 + \lambda_5(t)\tilde{\delta}K^\mu_\nu\tilde{\delta}K^\nu_\mu + \cdots \end{aligned}$$

NG boson

- Undo unitary gauge $t \rightarrow \tilde{t} = t - \pi(\tilde{t}, \vec{x})$

$$\begin{aligned} H(t) &\rightarrow H(t+\pi), \quad \dot{H}(t) \rightarrow \dot{H}(t+\pi), \\ \lambda_i(t) &\rightarrow \lambda_i(t+\pi), \quad a(t) \rightarrow a(t+\pi), \\ \delta^0_\mu &\rightarrow (1+\dot{\pi})\delta^0_\mu + \delta^i_\mu \partial_i \pi, \end{aligned}$$

- NG boson in decoupling (subhorizon) limit

$$\boxed{\begin{aligned} I_\pi = M_{Pl}^2 \int dt d^3\vec{x}\, a^3 \Bigg\{ &-\frac{\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2}\right) \\ &-\dot{H}\left(\frac{1}{c_s^2} - 1\right)\left(\frac{c_3}{c_s^2}\dot{\pi}^3 - \dot{\pi}\frac{(\partial_i \pi)^2}{a^2}\right) + O(\pi^4, \tilde{\epsilon}^2) + L^{(2)}_{\delta K, \delta R} \Bigg\} \end{aligned}}$$

$$\frac{1}{c_s^2} = 1 - \frac{4\lambda_1}{\dot{H}}, \quad c_3 = c_s^2 - \frac{8 c_s^2 \lambda_2}{-\dot{H}}\left(\frac{1}{c_s^2} - 1\right)^{-1}$$

- Sound speed

c_s : speed of propagation for modes with $\omega \gg H$

$\omega^2 \simeq c_s^2 \frac{k^2}{a^2}$ for $\pi \sim A(t)\exp\left(-i\int \omega dt + i\vec{k}\cdot\vec{x}\right)$

Application: non-Gaussianity of inflationary perturbation $\zeta = -H\pi$

$$I_\pi = M_{Pl}^2 \int dt d^3\vec{x}\, a^3 \left\{ -\frac{\dot H}{c_s^2}\left(\dot\pi^2 - c_s^2 \frac{(\partial_i\pi)^2}{a^2}\right) \right.$$

$$\left. -\dot H\left(\frac{1}{c_s^2}-1\right)\left(\frac{c_3}{c_s^2}\boxed{\dot\pi^3} - \boxed{\dot\pi\frac{(\partial_i\pi)^2}{a^2}}\right) + O(\pi^4,\tilde\epsilon^2) + L^{(2)}_{\tilde\delta K,\tilde\delta R} \right\}$$

2 types of 3-point interactions

→ power spectrum

$$P_\zeta(\vec k) = \frac{\Delta}{k^3}, \quad \Delta = \left.\frac{H^4}{-4M_{Pl}^2 \dot H c_s}\right|_{c_s k \simeq aH}$$

→ non-Gaussianity

$$\langle \zeta_{\vec k_1}(t)\,\zeta_{\vec k_2}(t)\,\zeta_{\vec k_3}(t)\rangle = (2\pi)^3\delta^3(\vec k_1+\vec k_2+\vec k_3)B_\zeta$$

c_s^2 → size of non-Gaussianity

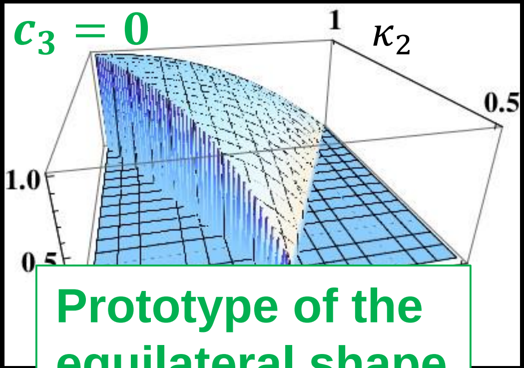
$$k^6 B_\zeta|_{k_1=k_2=k_3=k} = \frac{18}{5}\Delta^2\left(f_{NL}^{\dot\pi(\partial_i\pi)^2} + f_{NL}^{\dot\pi^3}\right)$$

$$f_{NL}^{\dot\pi(\partial_i\pi)^2} = \frac{85}{324}\left(1-\boxed{\frac{1}{c_s^2}}\right) \qquad f_{NL}^{\dot\pi^3} = \frac{5\boxed{c_3}}{81}\left(1-\boxed{\frac{1}{c_s^2}}\right)$$

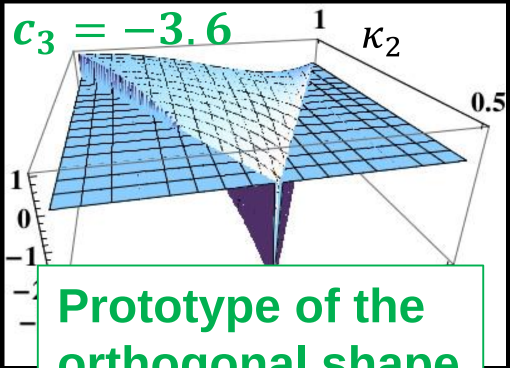
$\propto \frac{1}{c_s^2}$ **for small c_s^2**

c_3 → shape of non-Gaussianity

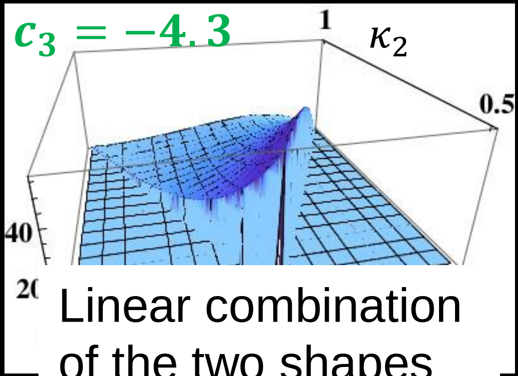
plots of $B_\zeta(k,\kappa_2 k,\kappa_3 k)/B_\zeta(k,k,k)$



Prototype of the equilateral shape



Prototype of the orthogonal shape



Linear combination of the two shapes

Summary of EFT of scalar-tensor theory

- Ghost condensation is a universal description of scalar-tensor theories around Minkowski/de Sitter background.
- Extension of ghost condensation to FLRW backgrounds results in the EFT of inflation/dark energy.
- This EFT provides a universal description of all known scalar-tensor theories, including Horndeski theory, DHOST theory and more.