IRREGULAR RIEMANN-HILBERT CORRESPONDENCE

PROGRAM FOR THE SCHOOL IN AUSSOIS, SEPTEMBER 14–18, 2020

by

Claude Sabbah

Introduction

The objects coming into play are

- on the one hand, meromorphic vector bundles $\mathscr{V}$ on a complex manifold X with poles along a divisor D with normal crossing—in other words, locally free $\mathscr{O}_X(*D)$-modules of finite rank—endowed with an *integrable* connection $\nabla : \mathscr{V} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{V}$, that we call *flat meromorphic bundles*,
- and on the other hand, local systems of $\mathbf{C}$-vector spaces on $X \smallsetminus D$, possibly endowed with a supplementary structure, called a *Stokes structure*, along D.

According to Deligne [**Del70**], the Riemann-Hilbert correspondence can be regarded as an equivalence between the category of flat meromorphic bundles with *regular singularities* and that of locally constant sheaves of finite-dimensional $\mathbf{C}$-vector spaces on $X \smallsetminus D$. Let $j : X \smallsetminus D \hookrightarrow X$ denote the inclusion. The main ingredient of this theorem is the *comparison theorem*, which states that the holomorphic *de Rham* functor DR for such a $(\mathscr{V}, \nabla)$ satisfies

$$\mathrm{DR}(\mathscr{V}, \nabla) \xrightarrow{\sim} \boldsymbol{R}j_*j^{-1}\,\mathrm{DR}(\mathscr{V}, \nabla).$$

The assumption of regular singularities is essential, and the purpose of the talks is to understand how the de Rham functor can be enriched in order to bypass this constraint.

- Monday and Tuesday: dimension one.
- Wednesday to Friday: dimension $\geqslant 2$.

Simplifying assumption for the talks. The talks will assume that the flat meromorphic bundles, when they are *good*, are *unramified*. These notions will be explained during the talks. This assumption is made in order to avoid some technicalities in the exposition. In practice, the proofs are reduced to the case when this assumption is satisfied: in local analytic geometry, one uses a local finite morphism ramified along the divisor, while in the global projective setting, one makes use of *Kawamata's lemma*.

Suggested reading. For the dimension-one setting, a good survey is given in [**Var96**].

1. Monday

The talks on Monday are devoted to the classical theory and the Riemann-Hilbert-Deligne correspondence in dimension one. The talks explain the theory with the simplifying assumption of non-ramification.

1.1. The Levelt-Turrittin theorem and the sectorial decomposition. The goal of this introductory talk is to formulate the classification problem of germs of meromorphic connections. The theorem of Levelt-Turrittin classifies formal germs. The asymptotic theorem lifts this classification in local sectors around the singularity (finer theorems specify the size of sectors, but they are not relevant for the main topic). It seems reasonable not to give proofs of the main theorems in the talk, but explain their statement. Extract them e.g. from [**Var96**].

- Statement of the theorem of Levelt-Turrittin (decomposition with coefficients in $\widehat{\mathscr{O}}$). A proof in a simple case and with parameters will be given on Wednesday.
- The sheaf $\mathscr{A}$ on the real oriented blow-up. The basic exact sequence

  $$0 \longrightarrow \mathscr{A}^{<0} \longrightarrow \mathscr{A} \longrightarrow \widehat{\mathscr{O}} \longrightarrow 0$$

  Source: [**Mal91**, Chap. 4, §1], [**SvdP03**, §7.4].
- Statement of the decomposition with coefficients in the sheaf $\mathscr{A}$. Source: [**Mal91**, App. 1, Prop. 2].

1.2. Malgrange-Sibuya theorem and Stokes torsors. Source: Sections 3.4 and 4.1–4.5 of [**BV89**, Part I].

- Explain the notion of marked meromorphic connection.
- Main result: Classification theorem of marked meromorphic connections: Theorem 4.5.1, Malgrange-Sibuya. (See also [**SvdP03**, Th. 9.5 & Cor. 9.7].)

1.3. Stokes-filtered local system. Reference: [**Sab13**]. This talk introduces the notion of Stokes-filtered local system in an abstract way without convincing justifications, which are however given just after, in the next talk.

- Explain Sections 2.b, 2.f and 3.c.
- Try to explain the notion of level structure.
- State the abelianity theorem 3.5.
- Explain Lemma 3.12 and Remark 3.14.

1.4. The Riemann-Hilbert-Deligne correspondence. Reference: [**Sab13**]. Beginning of Section 5.b (= 5.2) (do not explain Lemmas 5.1 and 5.2). Section 5.c (= 5.3) (stop before Proposition 5.10).

2. Tuesday

The talks of Tuesday are devoted to the local moduli theory in dimension one. The talks extract from the sources the part concerned with non-ramified objects.

2.1. Torsors and non-abelian cohomology. Reference: [**BV89**, Part II, Ch. 1]. Sections 1.1–1.4. Insist on Proposition 1.2.2.

2.2. Representability by affine spaces. Reference: [**BV89**, Part II, Ch. 2]. Insist on Proposition 2.3.3. Introduce the notion of "elementary" in the case the ramification index is $d = 1$. Relation with the level structure of Talk 1.3. Statement of Theorem 2.4.1, but proof only in the case of one level. Emphasize the proof that a Stokes torsor has no non-trivial automorphism (Theorem 2.4.1 (a)).

2.3. Affine structure on the set of Stokes torsors. Reference: [**BV89**, Part II, Ch. 3]. Proposition 3.2.1 and its proof, and statement of the theorems in Section 3.4.

2.4. Local moduli for marked meromorphic flat bundles. Reference: [**BV89**, Part III, Ch. 1]. Aim: Proof of Theorem 1.1.2. As usual, set the ramification index d equal to one in order to simplify the explanations.

3. Wednesday

Higher dimension, poles along a *smooth* divisor. This case is a natural generalization of the case in dimension one. It enables to softly introduce the notion of *good* formal decomposition, sheaf of Stokes torsors and Stokes-filtered local systems. References: [**Sab07**, §II.5, 6] and [**Sab13**, Chap. 10].

3.1. Notion of good formal decomposition, sectorial decomposition with parameter. Reference: [**Sab07**, §II.5, 6]. Aim: Proof of the Levelt-Turrittin theorem with parameters in a simple case (Theorem 5.7).

- Emphasize the goodness property.
- Introduce the real blow-up space in the local setting, and then in the global setting ([**Sab13**, §8.2]). It is useful for the next talks to consider the general case of a divisor with strict normal crossings, although only the case D smooth is used on Wednesday.
- Introduce the sheaf $\mathscr{A}$ and state the sectorial decomposition theorem [**Sab07**, II.5.12].
- Introduce the Stokes sheaf [**Sab07**, §II.6.a] and the sheaf $\mathscr{H}_X$ [**Sab07**, §II.6.b].
- State Theorems and II.6.1 and II.6.3.

3.2. Classification theorem of marked meromorphic flat bundles. Reference: [**Sab07**, §II.6]. Aim: Proof of Theorems II.6.1 and II.6.3.

- Proof that $\mathscr{H}_X$ is a sheaf ([**Sab07**, §II.6.b]).
- Proof that the presheaf of Stokes torsors is a sheaf (Lemma 1.6.1 and Corollary 1.6.2 in [**Tey19b**].
- Proof of Corollary II.6.4.
- Section 6.d.
- If time permits, Base change (Proposition II.6.9).

3.3. Riemann-Hilbert correspondence along a smooth divisor. Reference: [**Sab13**, Chap. 10].

- Structure of covering of the exponential factors involved in the local Turrittin-Levelt decomposition of a good meromorphic flat bundle, non-ramified along a smooth divisor D.
- Order when lifting to the real blow-up along D, and corresponding notion of Stokes-filtered local system.
- The Riemann-Hilbert functor
- Proof of full faithfulness.
- Proof of essential surjectivity.

4. Thursday

Higher dimension, poles along a divisor with normal crossings, non-ramified setting as usual. The talks will introduce the notion of Stokes-filtered local system in higher dimension. The local exponential factors (irregular values in the terminology of Mochizuki) together with the local growth order between them are globalized in a structure called "good system of irregular values" by Mochizuki [**Moc11a, Moc11b**] which has the topology of a possibly non-Hausdorff space with a local homeomorphism onto the divisor. The goal is to prove the Riemann-Hilbert correspondence generalizing that of 1.4. The last talk, together with the first one on Friday, introduces the new notion of *irregular constructible/perverse sheaf* due to Kuwagaki [**Kuw18**], based on the theorems of D'Agnolo-Kashiwara, in order to treat Riemann-Hilbert correspondence for general holonomic D-modules.

4.1. Irregular values, Stokes filtration. References: [**Moc11b**, §3], [**Sab13**, Chap. 9]. Setting: a complex manifold X and a ncd D. $\widetilde{X}(D)$ is the real oriented blow-up space considered on Wednesday (3.1).

- Notion of good system of irregular values [**Moc11b**, §2.1.1].
- Order [**Moc11b**, §3.1].
- Notion of Stokes data [**Moc11b**, §3.2] and Stokes filtration [**Moc11b**, §4.1].
- If time permits, sheaf-theoretic interpretation with the sheaf space [**Sab13**, §9.5].

4.2. Main properties of Stokes-filtered local systems. References: [**Moc11b, Sab17**].

- Statement and proof of [**Moc11b**, Th. 3.9]. See also [**Sab17**, §2.g & App.].

4.3. The Riemann-Hilbert correspondence for good meromorphic connections. References: [**Moc11a, Moc11b, Sab13**].

- Statement of the RH correspondence.
- Proof of full faithfulness [**Moc11b**, §4.2].
- Sketch of the proof of essential surjectivity [**Moc11a**, §4.3]. See also [**Sab13**, §12.5].

4.4. Irregular perverse sheaves (1). Reference: [**Kuw18**].

- Sections 1 to 4 of [**Kuw18**]. See also `http://www.cmls.polytechnique.fr/perso/sabbah/exposes/sabbah_varsovie190927.pdf` as an introduction to this work.

5. Friday

Two talks on Friday are devoted to the generalization in higher dimension of the results of Babbitt-Varadarajan seen on Tuesday. These are due to Teyssier [**Tey19a, Tey19b**]. A panoramic view on present and future research in this direction is given by Teyssier in the last talk.

5.1. Irregular perverse sheaves (2). Reference: [**Kuw18**].

- Sections 5.2 to 5.4 of [**Kuw18**].
- Section 9.
- Section 8: State the second part of Theorem 8.5 (Riemann-Hilbert correspondence). Very short hint for a proof, but do not give details on the notion of enhanced ind-sheaf.

5.2. Moduli of Stokes torsors in higher dimension (1). Reference: [**Tey19a, Tey19b**]. Relies much on the talks of Tuesday.

- Section 1 of [**Tey19a**].
- Section 1, up to 1.7 of [**Tey19b**].

5.3. Moduli of Stokes torsors in higher dimension (2). Reference: [**Tey19b**].

- Sections 1.8, 1.9 and 2 of [**Tey19b**].

5.4. J.-B. Teyssier: Panoramic view on Stokes stacks.

References

[BV89] D.G. Babbitt & V.S. Varadarajan – *Local moduli for meromorphic differential equations*, Astérisque, vol. 169-170, Société Mathématique de France, Paris, 1989, `http://www.numdam.org/item/AST_1989__169-170__1_0/`.

[Del70] P. Deligne – *Équations différentielles à points singuliers réguliers*, Lect. Notes in Math., vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[Kuw18] T. Kuwagaki – “Irregular perverse sheaves”, `arXiv:1808.02760`, 2018.

[Mal91] B. Malgrange – *Équations différentielles à coefficients polynomiaux*, Progress in Math., vol. 96, Birkhäuser, Basel, Boston, 1991.

[Moc11a] T. Mochizuki – *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque, vol. 340, Société Mathématique de France, Paris, 2011.

[Moc11b] ———, “Stokes structure of a good meromorphic flat bundle”, *Journal de l'Institut mathématique de Jussieu* **10** (2011), no. 3, p. 675–712.

[Sab07] C. Sabbah – *Isomonodromic deformations and Frobenius manifolds*, Universitext, Springer & EDP Sciences, 2007, (in French: 2002).

[Sab13] ———, *Introduction to Stokes structures*, Lect. Notes in Math., vol. 2060, Springer-Verlag, 2013, `doi:10.1007/978-3-642-31695-1` & `arXiv:0912.2762`.

[Sab17] ———, “A remark on the irregularity complex”, *J. Singul.* **16** (2017), p. 100–114, Erratum: *Ibid.*, p. 194.

[SvdP03] M. Singer & M. van der Put – *Galois Theory of Linear Differential Equations*, Grundlehren Math. Wiss., vol. 328, Springer-Verlag, 2003.

[Tey19a] J.-B. Teyssier – “Skeletons and moduli of Stokes torsors”, *Ann. Sci. École Norm. Sup. (4)* **52** (2019), no. 1, p. 337–365, `arXiv:1609.01559`.

[Tey19b] ———, “Moduli of Stokes torsors and singularities of differential equations”, `arXiv:1802.00289v3`, 2019.

[Var96] V.S. Varadarajan – “Linear meromorphic differential equations: a modern point of view”, *Bull. Amer. Math. Soc.* **33** (1996), p. 1–42, `link to the pdf file`.

C. Sabbah, CMLS, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau cedex, France • *E-mail :* `Claude.Sabbah@polytechnique.edu`
Url : `http://www.math.polytechnique.fr/perso/sabbah`