IHÉS LECTURES 2014

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Contents

1. Lecture 1. Supersymmetric gauge theories. 07.10.2014 2
1.1. Field theories 2
1.1.1. Gauge theory 2
1.1.2. Yang-Mills theory 2
1.2. Global symmetries 3
1.3. Poincare symmetry 3
1.4. SuperPoincare symmetry 4
1.4.1. The spinor modules of $Spin(\mathsf{V})$ 4
1.4.2. Lie superalgebras 4
1.4.3. Poincare superalgebras 4
1.4.4. Clifford algebra 5
1.4.5. Invariant bilinear form on S 5
1.4.6. Minimal supersymmetry 6
1.5. Minimal super Yang-Mills: $\mathcal{N}=1$ SYM 7
1.6. $\mathcal{N}=1$ Super-Yang-Mills action functional 8
1.6.1. Minkowski flat space-time 8
1.6.2. Euclidean space-time and complexification 9
1.6.3. Off-shell closure of the supersymmetry algebra 9
1.7. Topologically twisted theories on $X_d = X_{d-2}\times C_2$ 11
1.8. Topological constraints on existence of $Spin$ and $Spin^c$ structures on 4,6 and 8 dimensional manifolds 13
1.9. The ring of observables of $\mathcal{N}=1$ on $X_{d-2}\times C_2$ if C_2 is $\cdot, S^1, \mathcal{E}_2$ 13
1.9.1. Equivariant T-twist of $\mathcal{N}=1$ theory 14
2. Lecture 2. The 4d $\mathcal{N}=2$ gauge theory. 14.20.2014 15
2.1. Geometrical structures 15
2.1.1. Complex 15
2.1.2. Kahler 15
2.1.3. HyperKahler 16
2.1.4. Special Kahler 17
2.1.5. Integral special Kahler 18
2.2. The $\mathcal{N}=1$ supersymmetry algebra in $d=4$ in supespace 19
2.2.1. Chiral fields 19
2.2.2. D-term: Kahler kinetic term 20
2.2.3. F-term: holomorphic superpotential 20
2.2.4. Gauge fields 20
2.3. The $\mathcal{N}=2$ supersymmetry algebra in $d=4$ as reduction of $\mathcal{N}=1$ superPoincare in $d=6$ 21

2.4. $\mathcal{N} = 2$ abelian vector mutliplet action 22
2.4.1. 4d $\mathcal{N} = 2$ hypermultiplet 23
2.4.2. 6d $\mathcal{N} = 1$ reduced to 5d 23
2.5. 6d $\mathcal{N} = 1$ reduced to 4d on T^2 24
2.6. Lagrangian $\mathcal{N} = 2$ gauge theories 24
2.7. Lagrangian $\mathcal{N} = 2$ quiver gauge theories 26
3. Lecture 3. Electro-magnetic duality, Seiberg-Witten integrable system. 21.10.2014 26
3.1. Electro-magnetic duality 26
3.1.1. Seiberg-Witten integrable system 28
3.2. $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^3 \times \mathbb{R}$ 30
3.3. Spectral curves 31
3.4. Some classes of algebraic integrable systems 31
3.4.1. Hitchin system 31
3.4.2. Group like version of Hitchin system 32
3.4.3. ADE quivers and monopoles 33
3.4.4. The elliptic group Hitchin system 33
3.4.5. Affine ADE quivers and instantons on elliptic fibration 34
4. Lecture 4. AGT and NS quantizations. 28.10.2014 34
5. Literature references 34

This is work in progress draft of the lecture notes for the 2014 IHES course. Please, all your comments and remarks are very welcomed. Just send an email. Thanks! Vasily

1. Lecture 1. Supersymmetric gauge theories. 07.10.2014

1.1. **Field theories.**

1.1.1. *Gauge theory.* Let G be a Lie group and let X be a real smooth oriented space-time manifold of dimension d. A *G-gauge theory* on X is a field theory in which the space of fields is a space of principal smooth G-bundles P on X with connections. In a given trivialization (choice of gauge) of a principal G-bundle, a connection $\nabla_A = d + A$ is represented by $\mathfrak{g}$-valued one-form on X, the section $A \in \Gamma(X, T^*_X \otimes \mathfrak{g})$, where $\mathfrak{g} = Lie(G)$.

A matter field ϕ in a G-gauge theory is defined by a choice of a vector bundle $\mathcal{E}_S$ over X and a vector bundle $\mathcal{E}_R$ over X associated to the principal G-bundle by a representation $\rho : G \to R$. Then matter field ϕ is a section $\phi \in \Gamma(X, \mathcal{E}_S \otimes_X \mathcal{E}_R)$. Most often in physical constructions the bundle $\mathcal{E}_S$ with a fixed connection comes from a (super)gravity background in which we consider the gauge theory. The matter field ϕ could be chosen to have an even (bosonic, commuting) or odd (fermionic, anticommuting) statistics. For example, the matter bundle $\mathcal{E}_S$ could be a trivial bundle $X \times W$ where W is a fixed vector space, a bundle of p-forms on X, a *Spin* or *Spin*^{c} bundle over X whenever X is a *Spin* or *Spin*^{c} manifold, etc.

1.1.2. *Yang-Mills theory.* The Yang-Mills theory is a gauge theory on X equipped with Yang-Mills functional S_{YM} on the space of gauge fields defined by a choice of the Hodge star operator on degree 2 forms $\star : \Omega^2(X) \to \Omega^{d-2}(X)$ (where $\Omega^p(X)$ denotes the space of

p-forms on X), a G-invariant symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g} = Lie(G)$, and a parameter called Yang-Mills coupling constant g_{YM}

$$S_{YM} = \frac{1}{g_{YM}^2} \int_X \langle F_A \wedge \star F_A \rangle \tag{1.1}$$

Remark 1. The above form of the action functional assumed that G is a compact simple Lie group. If G is compact reductive then there is an independent coupling constant g_{YM} for irreducible factor in G.

Remark 2. A choice of a non-degenerate metric $g \in \Gamma(X, Sym^2(T^*X))$ on X is sufficient to construct S_{YM} functional since g defines the Hodge star operator $\star_g$, but it is not necessary. For example, in dimension 2 the Hodge star $\star$ on 2-forms is defined by a choice of volume form on X (therefore 2d Yang-Mills is invariant under area-preserving diffeomorphisms of X), while in dimension 4 the Hodge star on 2-forms depends only on the conformal class of metric and is invariant under local Weyl transformation $g \to e^{2\Omega} g$, hence 4d Yang-Mills is classically a conformal theory. The quantization breaks the conformal invariance of the pure YM theory (without matter fields) in 4d, but not the area-preserving diffeomorphism invariance in 2d.

Remark 3. When X is a space-time with of Minkowski signature $(d-1, 1)$, the assumptions of unitarity and energy positivity of quantum theory require G to be a compact Lie group, the invariant form $\langle , \rangle$ on $\mathfrak{g}$ to be a positive definite metric on $\mathfrak{g}$ and g_{YM} to be a real number, so that after Wick rotation (time $t \mapsto \sqrt{-1}t$) the S_{YM} is a positive definite functional for a theory on a Euclidean manifold X.

Remark 4. Yang-Mills functional is often complemented by topological characteristic classes of the G-bundle. Let G be a simple compact Lie group. The most familiar is the second Chern class for a 4d gauge theories

$$S_{YM} = \frac{1}{g_{YM}^2} \int_X \langle F_A \wedge \star F_A \rangle + \frac{i\theta}{8\pi^2} \int_X \langle F_A \wedge F_A \rangle \tag{1.2}$$ {eq:S_{YM}}

If $\langle \rangle$ on $\mathfrak{g}$ is normalized such that the long coroot length squared is 2, then

$$k = -\frac{1}{8\pi^2} \int \langle F_A \wedge F_A \rangle \tag{1.3}$$

is integral and is called instanton charge. The instanton charge is positive for self-dual fields $F_A = - \star F_A$, e.g. $F_A^+ = 0$.

1.2. **Global symmetries.**

1.3. **Poincare symmetry.** Suppose that X is a d-dimensional affine space $X \simeq \mathsf{V} = \mathbb{R}^d$ with a constant flat metric g, the $SO_g(\mathsf{V})$ be the Lie group of special orthogonal transformations of V relative to the metric g, and the *Poincare group* $ISO_g(\mathsf{V}) = SO_g(\mathsf{V}) \ltimes \mathsf{V}$ be the extension of $SO_g(\mathsf{V})$ by the Lie translation (abelian) group V of the affine space V (we are using the same letter V for the affine space and its vector space, hopefully, without confusion)

$$\mathsf{V} \to ISO_g(\mathsf{V}) \to SO_g(\mathsf{V}) \tag{1.4}$$

In particular, the translation group $\mathsf{V} \subset ISO_g(\mathsf{V})$ is normal subgroup of Poincare group and the orthogonal group $SO_g(\mathsf{V}) = ISO_g(\mathsf{V})/\mathsf{V}$ is the quotient. Obviously, the YM functional S_{YM} on $X = (\mathsf{V}, g)$ is invariant under $ISO_g(\mathsf{V})$.

1.4. **SuperPoincare symmetry.** The notion of supersymmetry refers to the Lie supergroups that are symmetries of $\mathbb{Z}_2$-graded spaces also known as supermanifolds.

The Lie supegroups are easier to analyze in terms of their Lie superalgebras over the ground field $\mathbf{k} = \mathbb{C}$ and then specialize the global and the real structure.

1.4.1. *The spinor modules of* $Spin(\mathsf{V})$. Let $\mathsf{V} \simeq \mathbb{C}^d$ a complex vector space with a symmetric bilinear form g. Let $Spin(\mathsf{V})$ be the $\mathbb{Z}_2$ extension of $SO(\mathsf{V})$

$$\mathbb{Z}_2 \to Spin(\mathsf{V}) \to SO(\mathsf{V}) \tag{1.5}$$

and let S be the complex Dirac spinor module of $Spin(\mathsf{V})$ of $\dim_{\mathbb{C}} S = 2^{\lfloor \frac{d}{2} \rfloor}$. If d is odd then S is irreducible $Spin(\mathsf{V})$ module, if d is even then S decomposes $S = S^+ \oplus S^-$ to the direct sum of irreducible $Spin(\mathsf{V})$-modules called Weyl spinors of positive and negative chirality with $\dim_{\mathbb{C}} S^+ = \dim_{\mathbb{C}} S^- = \frac{1}{2} \dim_{\mathbb{C}} S = 2^{\frac{d-2}{2}}$.

In terms of Dynkin weights, if $d = 2k+1$ then $\mathfrak{so}(\mathsf{V}) = B_k$ and S is the irreducible highest weight B_k-module with the highest weight ω_k associated to the last node (the short root) in the B_k Dynkin diagram

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If $d = 2k$ then $\mathfrak{so}(\mathsf{V}) = D_k$ and $S^{\pm}$ are the irreducible highest weight D_k modules with highest weights ω_{k-1}-th and the ω_k corresponding the last 2 nodes of the D_k Dynkin graph where it splits

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- - -o--o--o--o
           |
           o
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1.4.2. *Lie superalgebras.* A Lie superalgebra is a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, the $\mathfrak{p}_0$ is called *the bosonic* subalgegra of the superalgebra $\mathfrak{p}$, and $\mathfrak{p}_1$ is called *the fermionic* extension. A superalgebra ($\mathbb{Z}_2$ graded algebra) means that there is an extra sign $(-1)^{|x||y|}$ in all relations every time the position of the two elements x and y is exchanged where $|x|$ denotes the $\mathbb{Z}_2$ grade of an element x. In particular, the Lie algebra bracket $[,] : \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$ decomposes as

$$[,] : \mathfrak{p}_0 \otimes \mathfrak{p}_0 \to \mathfrak{p}_0, \qquad [,]; \mathfrak{p}_0 \otimes \mathfrak{p}_1 \to \mathfrak{p}_1, \qquad [,] : \mathfrak{p}_1 \otimes \mathfrak{p}_1 \to \mathfrak{p}_0 \tag{1.6}$$

The super antisymmetry of the Lie superbracket $[,]$ implies that the fermionic-fermionic bracket $[,] : \mathfrak{p}_1 \otimes \mathfrak{p}_1 \to \mathfrak{p}_0$ is symmetric. The Jacobi identity on $\mathfrak{p}$ implies that $\mathfrak{p}_0$ is a Lie algebra itself, the $\mathfrak{p}_1$ is a $\mathfrak{p}_0$-module, and that symmetric fermionic-fermionic bracket $[,] : \mathfrak{p}_1 \otimes \mathfrak{p}_1 \to \mathfrak{p}_0$ is $\mathfrak{p}_0$-equivariant.

1.4.3. *Poincare superalgebras.* A Poincare Lie superalgebra $\mathfrak{p}$ is a fermionic(odd) extension of Poincare Lie algebra $\mathfrak{p}_0 = \mathfrak{iso}(\mathsf{V})$ by a $Spin(\mathsf{V})$ module $\mathfrak{p}_1 = \mathsf{S}$. If S is a $Spin(\mathsf{V})$ module then is automatically $ISO(\mathsf{V})$ module on which translation subgroup $\mathsf{V} \subset ISO(\mathsf{V})$ acts trivially. To complement definition of Lie superalgebra we need to specify a $Spin(\mathsf{V})$-equivariant symmetric map

$$[,] : \mathsf{S} \otimes \mathsf{S} \to \mathsf{V} \tag{1.7}$$ {eq:bracket

Such symmetric map induces super anti-symmeric Lie bracket, and the resulting Lie superalgebra $\mathfrak{siso}(\mathsf{V}|\mathsf{S}) := \mathfrak{iso}(\mathsf{V}) \oplus_s \mathsf{S}$ is called superPoincare algebra of the superspace $(\mathsf{V}|\mathsf{S})$.

Remark 5. Caution on the notations. The $Spin(\mathsf{V})$ module S in a superPoincare algebra $\mathfrak{siso}(\mathsf{V}|\mathsf{S})$ is not always just a single copy of Dirac or Weyl spinor modules S or $S^{\pm}$, but may contain several copies of S, $S^{\pm}$ in varios combinations under condition that symmetric map (1.7) exists.

Remark 6. Dualizing the map (1.7) and using the metric g on V to identify $\mathsf{V} \to \mathsf{V}^{\vee}$ we find can define the action by vectors from V on elements from S as $\cdot$ (γ-map)

$$\gamma, \cdot : \mathsf{V} \otimes \mathsf{S} \to \mathsf{S}^{\vee} \tag{1.8}$$ {eq:cdot}

This is not a exactly Clifford action, because V sends S to its dual space $\mathsf{S}^{\vee}$ rather than to itself. However, since $Spin(V)$ acts on S we have map

$$\Lambda^{\text{even}}\mathsf{V} \otimes \mathsf{S} \to \mathsf{S} \tag{1.9}$$

1.4.4. *Clifford algebra.* For a vector space with bilinear form (V, g), the Clifford algebra $Cl(\mathsf{V})$ is a free tensor algebra over V modulo relation $v \cdot v = g(v,v) \cdot 1$. If γ_μ for $\mu = 1 \dots d$, where $d = \dim \mathsf{V}$, denotes a basis in V and $g_{\mu\nu}$ is the matrix of g, then γ_μ generate $Cl(\mathsf{V})$ modulo relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \tag{1.10}$$

If the vector space V is a complex vector space of $\dim_{\mathbb{C}} \mathsf{V} = d$, then $\mathrm{C}l(\mathsf{V})$ is represented by $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$ matrices, and Dirac spinor module S is a module for $Cl(\mathsf{V})$. The action by $\mathsf{V} \subset Cl(\mathsf{V})$ on S is called Clifford action

$$(v \in \mathsf{V}, \psi \in S) \mapsto v \cdot \psi \tag{1.11}$$

and in terms of γ-matrices is written as

$$(v^\mu, \psi) \mapsto v^\mu \gamma_\mu \psi \tag{1.12}$$

This action of V on S naturally extends to the action of r-th external power $\Lambda^r(\mathsf{V})$ on S. Let $\gamma_{\underline{r}}$ be the basis in $\Lambda^r(\mathsf{V})$ where $\underline{r}$ is a multi-index notation for an r-tuple of distinct indices from $(1, 2, \dots, d)$, and $\gamma_{\mu_1 \dots \mu_r} := \gamma_{\{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_r\}}$ where $\{,\}$ denotes complete antisymmetrization. Then $v^{\underline{r}} \in \Lambda^r \mathsf{V}$ acts on S as

$$(v^{\underline{r}}, \psi) \mapsto v^{\underline{r}} \gamma_{\underline{r}} \psi \tag{1.13}$$

Notice that in even dimension $\gamma_{\underline{r}} : S^{\pm} \to S^{\mp}$ if r is odd and $\gamma_{\underline{r}} : S^{\pm} \to S^{\pm}$ if r is even.

1.4.5. *Invariant bilinear form on S.* Lemma. (An exercise is to prove it using an explicit recursive constrction of γ-matrices). Let $\mathsf{V} \simeq \mathbb{C}^d$. The Clifford algebra module S (Dirac spinor module) can be always equipped with an invariant bilinear form

$$(,)_{\mathrm{II}} : S \otimes S \to \mathbb{C}. \tag{1.14}$$

which is symmetric if $d \in \{0, 1, 6, 7\} \mod 8$ and antisymmetric otherwise, e.g. for $d \in \{2, 3, 4, 5\}$, and if d is even, with another bilinear form

$$(,)_{\mathrm{I}} : S \otimes S \to \mathbb{C}. \tag{1.15}$$

which is symmetric if $d \in \{0, 2\} \mod 8$ and antisymmetric for $d \in 4, 6 \mod 8$.

If we compose the bilinear form $(,)$ with the Clifford action $\Lambda^r(\mathsf{V})$ on S we obtain maps

$$(,)^{(r)}_{\mathrm{I,II}} : S \otimes S \mapsto \Lambda^r(\mathsf{V}^*) \tag{1.16}$$ {eq:rmap}

$d \mod 8$	0	1	2	**3**	**4**	5	**6**	7	8	9	**10**	11	12
S	S	S	S^+	S	S	$S\otimes\mathbb{C}^2$	$S^+\otimes\mathbb{C}^2$	$S\otimes\mathbb{C}^2$	S	S	S^+	S	S
$\mathcal{N}$	1	1	(0,1)	1	1	1	(0,1)	1	1	1	(0,1)	1	1
$\dim\mathsf{S}$	1	1	1	$\underline{2}$	$\underline{4}$	8	$\underline{8}$	16	16	16	$\underline{16}$	32	64
$\mathsf{S}_{\mathsf{V}=\mathbb{R}^{d-1,1}}$				$2\mathbb{R}$	$2\mathbb{C}$		$2\mathbb{H}$				$2\mathbb{O}$		

which are bilinear in spinors and valued in exterior forms on V. In the basis $\gamma_{\underline{r}}$ in $\Lambda^r(\mathsf{V})$ the above map is written as

$$(\psi,\chi)\mapsto(\psi\gamma_{\underline{r}}\chi) \tag{1.17}$$

Verify that $(,)^{(r)}_{\mathrm{II}}$ is antisymmetric in its arguments for $r\in\{2,3\}+\lfloor\frac{d}{2}\rfloor \mod 4$ and symmetric for $r\in\{0,1\}+\lfloor\frac{d-2}{2}\rfloor \mod 4$, and that $(,)^{(r)}_{\mathrm{I}}$ is antisymmetric for $r\in\{1,2\}+\frac{d-2}{2}$ mod 4 and symmetric for $r\in\{0,3\}+\frac{d-2}{2} \mod 4$.

In addition, verify that for even d the map (1.16) projects non-trivially on

$$S^{\pm}\otimes S^{\pm}\to\Lambda^r(\mathsf{V}^*) \tag{1.18}$$ {eq:S-same}

for $\frac{d}{2}+r$ even and on

$$S^{\pm}\otimes S^{\mp}\to\Lambda^r(\mathsf{V}^*) \tag{1.19}$$ {eq:S-oppos

for $\frac{d}{2}+r$ odd.

Using the metric on V we can identify $\mathsf{V}\to\mathsf{V}^*$ and hence at $r=1$ the definition (1.16) provides a map $S\otimes S\to\mathsf{V}$ that can be used in the construction of superPoincare Lie algebra *if this map is symmetric.*

1.4.6. *Minimal supersymmetry.* Now we examine the various cases of $d \mod 8$ and define minimal superPoincare algebra.

1. For $d\in\{0,1,2,3\} \mod 8$ the map $(,)^{(1)}_{\mathrm{II}}$ is symmetric and therefore we can define $[,]:S\otimes S\to\mathsf{V}$ using $(,)^{(1)}_{\mathrm{II}}$.

2. For $d\in\{2,4\} \mod 8$ the map $(,)^{(1)}_{\mathrm{II}}$ is symmetric and we can define $[,]:S\otimes S\to\mathsf{V}$ using $(,)^{(1)}_{\mathrm{I}}$.

3. For $d\in\{5,6,7\} \mod 8$ both maps $(,)^{(1)}_{\mathrm{I,II}}$ are antisymmetric and cannot be used to define odd-odd bracket $[,]:S\otimes S\to\mathsf{V}$. This can be fixed by taking $\mathsf{S}=S\otimes W$ where W is even-dimensional fixed vector space equipped witn a non-degenerate symplectic form and extending $(,)$ on S by tensoring $(,)$ on S with the symplectic form on W. The elements of $\mathsf{S}=S\otimes W$ are called symplectic spinors, and the symmetry $Sp(W)$ is called *symplectic R-symmetry.*

What is the minimal irreducible S in each of these cases? We need to examine additionally only the cases of even dimension $d\in\{0,2,4,6\} \mod 8$ where we can possibly project S to the chiral submodule S^+. The (1.18) implies that such projection possible if $d=\{2,6\}$ mod 8. Then $\mathsf{S}=S^+$ for $d=2$ and $\mathsf{S}=S^+\otimes W$ for $d=6$. In these cases the superPoincare algebra is called *chiral*.

We conclude with the table of the minimal modules S to define superPoincare algebra over complex field $\mathbb{C}$.

The table is redundantly extends (the first two lines are mod 8) to the dimension up to $d=12$ to capture all important dimensions of the superstring theory, M-theory and F-theory. The notation $\mathcal{N}$ reflects to the number of copies of the minimal extension module S used to construct superPoincare Lie algebra. In the dimensions $d=2,6,10,\ldots$ the minimal superPoincare Lie algebra is chiral, hence in these dimension the notation $(\mathcal{N}_-, \mathcal{N}_+)$ is used to denote the number of copies of the corresponding minimal extension modules $\mathsf{S}^{\pm}$. (The notation also applies also to the minimal superPoincare algebra in dimension 6 where $\mathsf{S} = S^+ \otimes \mathbb{C}^2$ involves two copies of Weyl spinors S^+ and is called $\mathcal{N} = (0,1)$). The $\dim \mathsf{S}$, i.e. the fermionic dimension of $\mathfrak{siso}(\mathsf{V}|\mathsf{S})$, is called the *number of supercharges* of superPoincare algebra.

1.5. **Minimal super Yang-Mills: $\mathcal{N} = 1$ SYM.** For a given $\dim \mathsf{S} = 2^p$ notice the maximal dimension d in which such S is possible. For $p = 1, 2, 3, 4$ the list of dimensions goes as $d = 3, 4, 6, 10$ and up to $p = 4$ satisfies

$$d - 2 = 2^{p-1} \tag{1.20}$$

The superPoincare algebra in dimensions $d = 3, 4, 6, 10$ has the following list of related very special properties

1. [Brink, Schwarz, Scherk 1977]

In $d = 3, 4, 6, 10$ the Yang-Mills functional has minimal supersymmetric extension by the fermionic fields $\psi \in \Gamma(X, \mathsf{S})$. This supersymmetric gauge theory is called $\mathcal{N} = 1$ SYM. The equivalent condition on superPoincare algebra bracket (1.7)(1.8) is the following 3-cycle rule. The map

$$\begin{aligned} \mathsf{S} \otimes \mathsf{S} \otimes \mathsf{S} &\to \mathsf{S}^\vee \\ (\psi_1, \psi_2, \psi_3) &\mapsto [\psi_1, \psi_2] \cdot \psi_3, \end{aligned} \tag{1.21}$$

where we used the map $[,]$ (1.7) to get a vector from two spinors and then map $\cdot$ (1.8) to act by the vector $[\psi_1, \psi_2]$ on ψ_3, vanishes after 3-cyclic symmetrization

$$\sum_{123 \to 231 \to 312} [\psi_1, \psi_2] \cdot \psi_3 = 0 \tag{1.22}$$ {eq:3cycli

Remark 7. The above 3-cyclic identity is not the Jacobi identity of the superPoincare Lie algebra. The map (1.8) is the bracket of superPoincare Lie algebra. The bracket operation of the superPoincare Lie algebra between translations in V and elements of S is zero.

2. [Baez-Huerta 2010] Let $H^\bullet(\mathfrak{p}, \mathbf{k})$ be Chevalley-Eilenberg cohomology of minimal superPoincare Lie algebra $\mathfrak{p} = \mathsf{siso}(\mathsf{V}|\mathsf{S})$ with values in the trivial representation $\mathbf{k}$ (ground field, e.g. $\mathbf{k} = \mathbb{C}$ or $\mathbf{k} = \mathbb{R}$). In dimensions $3, 4, 6, 10$ the $H^3(\mathfrak{p}, \mathbf{k}) = \mathbf{k}$ and $H^3(\mathfrak{p}, \mathbf{k}) = 0$ otherwise. In dimensions $d = 3, 4, 6, 10$ the non-trivial 3-cocycle in $H^3(\mathfrak{p}, \mathbf{k})$ can be used to extend the minimal superPoincare Lie algebra $\mathfrak{p}$ to a 2-Lie superalgebra in a sense of L_∞-algebras. Namely, using symmetric bracket $[,] : \mathsf{S} \otimes \mathsf{S} \to \mathsf{V}$ and metric $\langle , \rangle : \mathsf{V} \otimes \mathsf{V} \to \mathbf{k}$ on V, define the map $\alpha : \mathsf{S} \otimes \mathsf{S} \otimes \mathsf{V} \to \mathbf{k}$ by

$$(\psi, \chi, v) \mapsto \langle [\psi, \chi], v \rangle \tag{1.23}$$

Then α is a non-trivial 3-cocycle in $d = 3, 4, 6, 10$ generating $H^3(\mathfrak{p}, \mathbf{k}) = \mathbf{k}$.

3. [Kugo, Townsend 1983] In $d = 3, 4, 6, 10$ in Minkowski signature $\mathsf{V} = \mathbb{R}^{d-1,1}$ the minimal superPoincare algebra $siso(\mathsf{V}|\mathsf{S})$ over the ground field $\mathbf{k} = \mathbb{R}$ relates to the list of real normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (isomorphic to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ as vector spaces) and equivalently to the list of parallizable spheres S^0, S^1, S^3, S^7 (isomorphic to the elements of the real normed division algebras of norm 1).

Let $\mathbb{K}$ be one of $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ algebras. The module S is identified with the two copies of $\mathbb{K}$

$$\mathsf{S}_\psi = \mathbb{K}_{\psi_1} \oplus \mathbb{K}_{\psi_2} \tag{1.24}$$

and the vector space $\mathsf{V} \simeq \mathbb{R}^{d-1,1}$ is decomposed into $\mathbb{R}^{d-1,1} \simeq \mathbb{R}^{0,1}_t \oplus \mathbb{R}^{1,0}_x \oplus \mathbb{R}^{d-2,0}_y$ where $\mathbb{R}^{d-2,0}_y \simeq \mathbb{K}_y$ and identified with Hermitian 2×2 matrices valued in $\mathbb{K}$

$$\mathsf{V} = \mathrm{Mat}^{\mathrm{H}}_{2\times 2}(\mathbb{K}) \tag{1.25}$$

by the following rule

$$\begin{aligned} &\hat{}: \mathsf{V} \to \mathrm{Mat}^{\mathrm{H}}_{2\times 2}(\mathbb{K}), \\ &\hat{}: v \mapsto \hat{v} \qquad (t,x,y) \mapsto \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \end{aligned} \tag{1.26}$$ {eq:normed-

The Minkowski norm of $v \in \mathsf{V}$ corresponds to the minus determinant of $\hat{v}$

$$-v^2 = t^2 - x^2 - y^2 = \det(\hat{v}) \tag{1.27}$$

The operation (1.8) that takes $\mathsf{V} \otimes \mathsf{S} \to \mathsf{S}^\vee$ is just matrix multiplication

$$\cdot : \mathsf{V} \otimes \mathsf{S} \to \mathsf{S}^\vee, \qquad v, \psi \mapsto \hat{v} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1.28}$$

However, notice that the image lands in the dual space $\mathsf{S}^\vee$. Define the action of V on the dual space $\mathsf{S}^\vee$ by the rule

$$\cdot : \mathsf{V}^\vee \otimes \mathsf{S} \to \mathsf{S}, \qquad v, \psi \mapsto \check{v} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1.29}$$

where $\check{v}$ is

$$\begin{aligned} &\check{}: \mathsf{V} \to \mathrm{Mat}^{\mathrm{H}}_{2\times 2}(\mathbb{K}), \\ &\check{}: v \mapsto \check{v} \qquad (t,x,y) \mapsto \begin{pmatrix} -t+x & y \\ y^* & -t-x \end{pmatrix} \end{aligned} \tag{1.30}$$

It is easy to check that

$$\check{v}\hat{v}\psi = -v^2\psi \tag{1.31}$$

and hence $\mathsf{S} \oplus \mathsf{S}^\vee$ is the Clifford algebra module with the action of v represented by

$$\begin{pmatrix} 0 & \check{v} \\ \hat{v} & 0 \end{pmatrix} \tag{1.32}$$

1.6. $\mathcal{N} = 1$ Super-Yang-Mills action functional.

1.6.1. *Minkowski flat space-time.* Let $\mathfrak{siso}(\mathsf{V}|\mathsf{S})$ be the minimal superPoincare Lie algebra in one of the special dimensions $d = 3, 4, 6, 10$ over the real ground field $\mathbb{R}$. Let $X = \mathsf{V} = \mathbb{R}^{d-1,1}$ be the the flat Minkowski space-time. Let $\mathcal{E}_{\mathsf{S}}$ denote the product bundle $X \times \mathsf{S}$. Let ψ be fermionic field $\psi \in \Gamma(X, \mathcal{E}_{\mathsf{S}} \otimes_X \mathcal{E}_{\mathfrak{g}})$, called gluino superpartner of gauge connection A, and the other notations be as in 1.1.2. Then

$$S_{SYM} = \frac{1}{g^2_{YM}} \int_X \langle F_A \wedge \star F_A \rangle + \langle \psi \not{D} \psi \rangle \, \mu_X \tag{1.33}$$ {eq:SYM}

where the Dirac operator $\not{D} : \Gamma(X, \mathcal{E}_{\mathsf{S}}) \to \Gamma(X, \mathcal{E}_{\mathsf{S}^\vee})$ is a composition of the covariant derivative $D_A\psi$ and the γ-map (1.8), and μ_X is the volume form. In components $\langle \psi \not{D} \psi \rangle = (\psi \gamma^\mu D_\mu \psi)$ where $(\psi \gamma^\mu \psi)$ denotes the symmetric map $\mathsf{S} \otimes \mathsf{S} \to \mathsf{V}$ evaluated on ψ, ψ. Modulo equations of motion the SYM action functional is invariant under the following supersymmetry variation of the fields, called *vector multiplet representation*

$$\begin{aligned} \delta_\epsilon A &= \frac{1}{2}(\psi \gamma_\mu \epsilon) \\ \delta_\epsilon \lambda &= -\frac{1}{4} F_{\mu\nu} \gamma^{\mu\nu} \epsilon \end{aligned} \tag{1.34}$$

{eq:N1mult]

Our conventions are that fermionic(odd) elements of the $\mathsf{S} \subset \mathfrak{siso}$ are presented as $\epsilon^\alpha Q_\alpha$ where fermionic(odd) Q_α elements form a basis in S and bosonic(even) spinors ϵ^α are coordinates of an element $\epsilon^\alpha Q_\alpha \in \mathsf{S}$

1.6.2. *Euclidean space-time and complexification.* It is not possible to impose the real structure on the minimal complex superPoincare superalgebra $\mathfrak{siso}(\mathsf{V}|\mathsf{S})$ in special dimensions $3, 4, 6, 10$ such that $\mathsf{V} \simeq \mathbb{R}^d$ would have Euclidean metric. An easy way to see this is to notice that the vector v in the image of the symmetric odd-odd bracket $v = [\epsilon, \epsilon]$ is always null-like

$$v = [\epsilon, \epsilon] \quad \Rightarrow \quad v^2 = 0 \tag{1.35}$$

This follows immediately from the algebraic property (1.22). In Euclidean space the only null-like vector is $v = 0$, and therefore a non-trivial bracket $[\epsilon, \epsilon]$ does not exists in real Euclidean superspace.

Therefore, the $\mathcal{N} = 1$ SYM gauge theory in Euclidean space-time algebraically has to be defined over the ground field $\mathbb{C}$, while the action functional and other structures (like representation of the superPoincare algebra on the space of fields) are locally analytic (i.e. holomorphic away from singularities). The path integral of the quantum theory over bosonic(even) fields is understood as a choice of an integration half-dimensional contour in the space of complex fields, over which a top holomorphic form is integrated. Since the action functional and the supersymmetry algebra are holomorphic, the result does not depend precisely on the integration contour, but only on its homotopy class. The integration over the fermionic fields is an algebraic operation: the evaluation of the top form. Therefore it is not necessary to impose a real structure on fermions in the quantum theory.

The standard integration contour for the bosonic fields in Euclidean gauge theory is chosen in such a way that the gauge group G of the gauge theory is a compact form of the complexified gauge group $G_{\mathbb{C}}$, and the action functional on the space of bosonic fields is positive definite.

1.6.3. *Off-shell closure of the supersymmetry algebra.* By the definition of the superPoincare algebra the operator δ_ϵ^2 is the translation operator

$$\delta_\epsilon^2 = \mathcal{L}_v \tag{1.36}$$

where $\mathcal{L}_v$ is the vector field

$$v = [\epsilon, \epsilon], \qquad v^\mu = (\epsilon \gamma^\mu \epsilon). \tag{1.37}$$

The transformation rules (1.34) of the full space $\mathcal{A}$ of fields (A, ψ) is not a representation in a regular sense of the superPoincare algebra, because the algebraic relations such as $\delta_v^2 = \mathcal{L}_v$ hold in (1.34) only modulo equations of motion of the Super-Yang-Mills functional (in a flat space time $X \simeq \mathsf{V}$). In other words, the transformation rules (1.34) is a representation of the superPoincare algebra in the critical (*on-shell*) locus $\mathcal{A}_{\text{crit}} \subset \mathcal{A}$ defined by $dS = 0$.

In quantum theory it is useful (for example for localization) to have a representation of the supersymmetry algebra on the full space of fields $\mathcal{A}$ over which the path integral is computed. Often this can be done by introducing the auxiliary fields which appear in the action as free quadratic terms. This procedure is called *off-shell* closure of the representation of the supersymmetry algebra.

The *off-shell* closure of the *vector multiplet* representation of $\mathcal{N} = 1$ superPoincare algebra is easy to describe in the cases of dimensions $3, 4$ and 6 related to $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ but is much more complicated in dimension 10 related to $\mathbb{O}$ because of the non-associativity of $\mathbb{O}$. What is easy to describe is the *off-shell* closure of the $(1|1)$ supersymmetry subalgebra of the $\mathcal{N} = 1$ superPoincare algebra generated by a fixed element δ_ϵ for $\epsilon \in \mathsf{S}$.

Let us fix an element $\epsilon \in \mathsf{S}$ such that $v = [\epsilon, \epsilon] \neq 0$. Then the off-shell closure is convenient describe by the following exact sequence of the vector spaces

$$0 \to \mathsf{K} \otimes \mathfrak{g} \to \mathsf{S} \otimes \mathfrak{g} \overset{[\epsilon,-]}{\to} \mathsf{V} \otimes \mathfrak{g} \overset{([\epsilon,\epsilon],-)}{\to} \mathfrak{g} \to 0 \tag{1.38}$$ {eq:complex

The arrow from spinors to vectors S to V is an odd-odd bracket with ϵ. The last arrow from vectors to scalars is a convolution with the non-zero vector $v = [\epsilon, \epsilon]$. The vector space $\mathsf{K} \subset \mathsf{S}$ is the kernel of the map $[\epsilon, -]$ and $\mathsf{K} \to \mathsf{S}$ is the inclusion. The space K is called the space of auxiliary fields.

Not to mention the dimension of $\mathfrak{g}$, the dimensions of the vector spaces in the above complex are as follows

$$\begin{aligned} d = 3 &\qquad 0 \to 0 \to 2 \to 3 \to 1 \to 0 \\ d = 4 &\qquad 0 \to 1 \to 4 \to 4 \to 1 \to 0 \\ d = 6 &\qquad 0 \to 3 \to 8 \to 6 \to 1 \to 0 \\ d = 10 &\qquad 0 \to 7 \to 16 \to 10 \to 1 \to 0 \end{aligned} \tag{1.39}$$

This means that the off-shell closure of $\mathcal{N} = 1$ vector multiplet requires respectively $0, 1, 3, 7$ auxilary fields in $d = 3, 4, 6, 10$ (recall the list S^0, S^1, S^3, S^7 of parallizable spheres).

The space $\mathsf{K} = \ker_{[\epsilon,-]}$ is easy to describe explicitly in dimension $3, 4, 6$.

In dimension 3 we have $\mathsf{K} = 0$.

In dimension 4 let $\gamma^* = -\gamma_1\gamma_2\gamma_3\gamma_4$ be the chirality operator, and $\mathsf{S} = S \simeq \mathbb{C}^4$ be the space of the 4d Dirac spinors equipped with invariant bilinear form $(,)$ such that $(-, \gamma^\mu -)$ is symmetric map $\mathsf{S} \otimes \mathsf{S} \to \mathsf{V}$. It is easy to check that $(-, \gamma^* \gamma^\mu -)$ is antisymmetric map. Hence $[\gamma^* \epsilon, \epsilon] = 0$, and therefore $\mathsf{K} = \mathbb{C}\gamma^*\epsilon$. The generator γ^* is actually a generator of the automorphism of the $d = 4$ minimal superPoincare algebra called $U(1)$ R-symmetry. (In complexified description an element $t \in GL(1)$ of R-symmetry acts on $\mathsf{S} = S^+ \oplus S^-$ by $(S^+, S^-) \to (tS^+, t^{-1}S^-)$).

In dimension 6 recall that $\mathsf{S} = S^+ \otimes W$ where $W \simeq \mathbb{C}^2 \simeq \mathbb{C} \oplus \mathbb{C}^\vee$ is acted by the symplectic group $Sp(W) \simeq Sp(2, \mathbb{C})$. Let $\sigma_I, I = 1, 2, 3$ be the generators of $\mathfrak{sp}(2, \mathbb{C})$ (the traceless 2×2 matrices). Then, again is easy to check that $[\sigma_I \epsilon, \epsilon] = 0$, and therefore $\mathsf{K} = \oplus_{I=1}^3 \mathbb{C}\sigma_I\epsilon$. The generators σ_I for $I = 1 \dots 3$ are actually generators of the $Sp(W)$ R-symmetry automorphism group of the minimal $d = 6$ superPoincare algebra.

In dimension 10 there is no R-symmetry. The shortest geometrical description of the space K is just $\mathsf{K} = \ker_{[\epsilon,-]}$.

In all of these cases $d = 3, 4, 6, 10$ the space $\mathsf{K} = \ker_{[\epsilon,-]}$ can be also computed as the quotient of the $\mathbb{C}$-cone over the orbit of ϵ by the stabilizer of v subgroup of $Spin(\mathsf{V})$

$$K = (\mathbb{C}\mathrm{Stab}_{Spin(\mathsf{V})}(v) \cdot \epsilon)/(\mathbb{C}\epsilon) \tag{1.40}$$

In terms of the normed division algebra presentation (1.26), using a $Spin((V))$ transformation assume that an element ϵ is of the form $(k,0) \in (\mathbb{K} \oplus \mathbb{K})$. Then the vector space of auxiliary fields K is spanned by $(jk,0)$ where j is one of $(0,1,3,7)$ standard imaginary units in $(\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O})$ respectively for $d = 3,4,6,10$. (If $k \in \mathbb{K}$ is of unit norm, then k is a point in a sphere S^0, S^1, S^3, S^7. The line spanned by k is the normal bundle to the sphere. The vector space spanned by elements jk, where j runs over imaginary units of $\mathbb{K}$, is the tangent bundle to the sphere. Now we see how the $\mathcal{N} = 1$ SYM in $d = 3,4,6,10$ connects with the list of parallizable spheres.)

1.7. **Topologically twisted theories on $X_d = X_{d-2} \times C_2$.** Here we explain how to put the vector multiplet of $\mathcal{N} = 1$ SYM for $d = 3,4,6,10$ to a product space $X_d = X_{d-2} \times C_2$ where X_{d-2} is a any $d-2$-dimensional smooth real manifold with an arbitrary Riemannian metric from a certain topological class that we will describe below for $d = (3,4,6,10)$, and C_2 is a flat Euclidean two-dimensional space such as $\mathbb{R}^2, S^1 \times \mathbb{R}^1, S^1 \times S^1$. If $X_{d-2} = \mathbb{R}^{d-2}$ and $C_2 = S^1 \times S^1$ in the limit of vanishing size of C_2 the field theory from X_d reduces to the field theory on X_{d-2} (dimensional reduction).

From the supersymmetric theories $\mathcal{N} = 1, \mathcal{N} = 1, \mathcal{N} = (0,1)$ or $\mathcal{N} = (0,1)$ in dimensions $3,4,6$ or 10 respectively we obtain theories with supersymmetry $\mathcal{N} = 2, \mathcal{N} = (2,2), \mathcal{N} = 2$ or $\mathcal{N} = 1$ in dimensions $1,2,4$ or 8 that have $2,4,8$ or 16 supercharges. These theories in dimension 1,2,4,8 can be deformed in a suitable way to preserve at least one supercharge δ_ϵ for any Riemannian metric on X_{d-2} subject only to certain topological constraints on X_{d-2}.

The splitting of tangent bundle $T_{X_d} = TX_{d-2} \oplus T_{C_2}$ defines the subgroup $Spin(d-2) \times Spin(2) \subset Spin(d)$. Then $Spin(d)$ module S splits to the $\pm i$ eigen spaces of $Spin(2) \simeq U(1)$

$$\mathsf{S} = \mathsf{S}_{+\frac{1}{2}} \oplus \mathsf{S}_{-\frac{1}{2}} \tag{1.41}$$

(This is the same splitting $\mathsf{S} \simeq \mathbb{K} \oplus \mathbb{K}$ that we used in the division algebra presentation (1.26) if take (t,x) to be local coordinates in C_2.)

Consider a spinor $\epsilon \in \mathsf{S}_+ \subset \mathsf{S}$. It is clear that the vector $[\epsilon,\epsilon]$ sits inside T_{C_2}: since ϵ has charge $+\frac{1}{2}$ under $Spin(T_{C_2})$ the vector $[\epsilon,\epsilon]$ has charge $+1$ and therefore has to be a vector inside T_{C_2}.

Pick any no-where vanishing section ϵ of the $\mathsf{S}_{+\frac{1}{2}}$ bundle over X and define the supersymmetry of the vector multiplet using δ_ϵ. (We will review topological constraints on X_{d-2} due to the existence of no-where vanishing spinor in S_+). Normalize the section ϵ in such a way that we have vector field $v = [\epsilon,\epsilon]$ with

$$v^{\bar{z}} = 1 \tag{1.42}$$

where $(z,\bar{z})$ are standard flat complex coordinates on C_2 with the standard Euclidean metric.

Using (1.34) and the complex (1.38) we define the δ_ϵ complex on $X_{d-2} \times C_2$.

$$\delta A_{\bar{z}} = 0, \quad \begin{matrix} \delta A_\mu = \psi_\mu \\ \delta\psi_\mu = F_{\bar{z}\mu} \end{matrix}, \quad \begin{matrix} \delta A_z = \psi_z \\ \delta\psi_\mu = F_{\bar{z}z} \end{matrix}, \quad \begin{matrix} \delta\chi_I = K_I \\ \delta K_I = \nabla_{\bar{z}} K_I \end{matrix} \tag{1.43}$$

Here A_μ for $\mu = 1 \dots d-2$ are components of the connection 1-form along X_{d-2}. The fields (ψ_μ, ψ_z) come from the *image* of the map $[\epsilon, -]$ in the complex (1.38), all together these fields is a section of rank $d-1$ bundle. The bosonic fields K_I and fermionic fields χ_I, for $I = 1 \dots d-3$ are the sections of the auxiliary fields bundle K of rank $d-3$ over X that we have defined above in terms of the *kernel* of $[\epsilon, -]$, or using division algebra, or topologically as the pullback of the tangent bundle to the S^{d-3}-sphere bundle of non-vanishing elements of rank $d-2$ bundle S_+. The exact sequence of smooth complex vector bundles (1.38) splits in the term S of rank $2(d-2)$ into the rank $(d-1)$ bundle with sections (ψ_μ, ψ_z) and rank $(d-3)$ bundle with sections (K_I).

It is clear that

$$\delta_\epsilon^2 = \nabla_{\bar z} \tag{1.44}$$

The action functional is constructed using the auxiliary fields (K_I) as Lagrange multipliers for a certain first order elliptic PDE on X_{d-2} (usually called BPS equations): $F \in \mathrm{Stab}_\epsilon \subset \Lambda^2 T^*_{X_{d-2}}$, explicitly

$$F_I = 0, \qquad F_I = \nu_I F_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{1.45}$$

so that

$$\begin{aligned}\mathcal{L}_{SYM}[A,\psi,K] &= \delta_\epsilon\left((\chi_I, \frac{1}{2}K_I + iF_I) + (\psi_\mu, F_{z\mu}) + (F_{z\bar z}, \psi_z) + \dots\right)\\ &= \frac{1}{2}K_I K_I + iK_I F_I + (F_{\bar z\mu}, F_{z\mu}) + (F_{z\bar z}, F_{\bar z z}) + \dots\end{aligned} \tag{1.46}$$

By the usual Atiyah-Bott-Witten argument the theory localizes to the space of maps from C_2 to the moduli space

$$\mathfrak{M}^G_{X_{d-2},\epsilon} = \{\nabla_A : (F_A)_I = 0\}/\mathrm{Aut}(G\text{-bundle}) \tag{1.47}$$

After integrating out K_I we find

$$\mathcal{L}_{SYM}[A,\psi] = \frac{1}{2}(F_I, F_I) + (F_{\bar z\mu}, F_{z\mu}) + (F_{z\bar z}, F_{\bar z z}) + \dots \tag{1.48}$$

Remark 8. For $d = 10$ if $T_{X_8} \simeq \mathbb{O} \simeq S^+(T_{X_8})$ (which requires X to be the *Spin*(7) manifold) the parallizability of S^7 allows us to define the off-shell closure of 8-dimensional supersymmetry subalgebra generated by S_+. The *Spin*(10) symmetry of the original $d = 10$ Poincare superalgebra is broken after dimensional reduction to $Spin(2) \times Spin(8)$, and furthermore, after taking X_{d-2} be a curved *Spin*(7) manifold to $Spin(2) \times Spin(7)$. This construction first was described by [**?**]. If X is a compact *Spin*(7)-manifold, the resulting chiral theory on C_2 is $(0,8)$ sigma-model with *octonionic* target space (that localizes to the moduli space of *Spin*(7)-8d-instantons on X_8: $\nu_I F_{\mu\nu}\gamma^{\mu\nu}\epsilon = 0$ where $\{\nu_I\}$ spans $\mathsf{K} \subset \mathsf{S}_+$)

$$\{\delta_{\epsilon_i}, \delta_{\epsilon_j}\} = 2\delta_{ij}\nabla_{\bar z} \qquad i,j = 1,\dots,8 \tag{1.49}$$

The BPS equations $F_I = 0$ can be rewritten as

$$F - \star(F \wedge \Omega_4) = 0, \qquad \Omega_4 = (\epsilon\gamma_{\underline{4}}\epsilon) \tag{1.50}$$

Then

$$(F_I, F_I) = \frac{1}{2}(F - \star(F\wedge\Omega_4)) \wedge (\star F - F \wedge \Omega_4) = (F \wedge \star F - F \wedge F \wedge \Omega_4) \tag{1.51}$$

If $d\Omega_4 = 0$ then the above action coincides with the standard Yang-Mills functional up to a shift by the product of Ω_4 and the second Chern class $\Omega_4 \wedge \langle F \wedge F\rangle$.

For $d = 6$ if X_4 is *Spin*(4)-manifold with a no-where vanishing section of S^+ (A nowhere vanishing section of $\mathsf{S}_+ = S^+ \times (\mathbb{C} \oplus \mathbb{C})$ because of the symplectic pairing of $(\epsilon^1, \epsilon^2) \in S^+ \times (\mathbb{C} \oplus \mathbb{C})$ implies that both $\epsilon^1 \in S^+$ and $\epsilon^2 \in S^+$ nowhere vanish) implies that

$Spin(4)$ breaks to $SU(2)$, so that $S^+(X_4)$ is a trivial bundle; equivalently the bundle of self-dual two-forms $\Lambda^{2+}(T^*_{X_4})$ is a trivial bundle. The resulting chiral theory on C_2 is $(0,4)$ supersymmetric sigma-model with *quaternionic* target space (that localizes to moduli space of selt-dual connections on X_4) The equations $F_I = 0$ are equivalent to $F^+ = 0$ where F^+ is the self-dual part of the curvature two-form, and

$$(F_I, F_I) = \frac{1}{2}(F + \star F) \wedge (F + \star F) = F \wedge \star F + F \wedge F \tag{1.52}$$

For $d = 4$, the theory reduced to $X_2 \times C_2$ is a chiral $(0,2)$ supersymmetric sigma-model on C_2 with *complex* target space (that localizes to the space of 2d flat connections on X_2). . Indeed, the equation $F_I = 0$ is simply $F = 0$ and

$$(F_I, F_I) = F \wedge \star F \tag{1.53}$$

The equations are empty.

For $d = 3$ the theory reduced to $X_1 \times C_2$ is a chiral $(0,1)$ supersymmetric sigma-model on C_2 with *real* target space (the moduli space of connections of all connections on X_1, for $X_1 = S^1$ this is the gauge group G itself)

Remark 9. When the construction is generalized to a generic non-zero null vector field v on a generic X_d (with certain topological assumptions), we can define the v-orthogonal subspace $\mathsf{V}_{\perp v} \subset \mathsf{V}$ of rank $d-1$. Let $\mathsf{v} = \mathbb{C}v$ be the line subspace of V spanned by v. Since $v^2 = 0$ we have the inclusion $\mathsf{v} \subset \mathsf{V}_{\perp \mathsf{v}}$. Notice that $\mathsf{V}_{\perp v} \cup \mathsf{v} = \mathsf{V}_{\perp \mathsf{v}} \subset \mathsf{V}$ which is a proper subset of V. The $d-2$ space V_{d-2} that played the role of $T_{X_{d-2}}$ in the above construction can be defined as the quotient $\mathsf{V}_{\perp v}/\mathsf{v}$

$$\mathsf{v} \to \mathsf{V}_{\perp \mathsf{v}} \to \mathsf{V}_{d-2} \tag{1.54}$$

The splitting of $\mathsf{V}_{\perp \mathsf{v}}$ into v and V_{d-2} is not defined without an additional data.

1.8. Topological constraints on existence of *Spin* and $Spin^c$ structures on 4,6 and 8 dimensional manifolds.

1.9. The ring of observables of $\mathcal{N} = 1$ on $X_{d-2} \times C_2$ if C_2 is $\cdot, S^1, \mathcal{E}_2$. The ring of observables $\mathcal{H}_{\delta_\epsilon}$ of the twisted $\mathcal{N} = 1$ theory on $X_{d-2} \times C_2$ is the cohomology of δ_ϵ.

1. If C_2 is a point then we find the ordinary de Rham cohomology as observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times pt) = H_{\mathrm{deRham}}(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.55}$$

2. If C_2 is a circle S^1 then we find the K-theory, that is trigonometric cohomology as observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times S^1) = H_{S^1-\mathrm{deRham}}((\mathfrak{M}^G_{X_{d-2},\epsilon})^{S^1}) = \mathrm{K}(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.56}$$

The partition function itself is the Atiyah-Singer index of the (twisted) Dirac operator on $\mathfrak{M}^G_{X_{d-2},\epsilon}$.

3. If $C_2 = \mathcal{E}_\tau$ is an elliptic curve we find elliptic cohomology as the observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times S^1) = \mathrm{Ell}^\tau(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.57}$$

the partition function itself is Witten elliptic genus.

1.9.1. *Equivariant T-twist of $\mathcal{N}=1$ theory.* Now suppose that we take the previous construction with $X = X_{d-2} \times C_2$ and assume that a lie group T acts on X_{d-2}. then δ_ϵ can be promoted to the equivariant differential in the cartan model of the T equivariant cohomologies. the

1. If C_2 is a point then we find the T-equivariant de Rham cohomology as observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times pt; T) = H_{T-\mathrm{deRham}}(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.58}$$

2. If C_2 is a circle S^1 then we find the T-equivariant K-theory, that is T-equivariant trigonometric cohomology as observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times S^1; T) = H_{S^1-\mathrm{deRham}}((\mathfrak{M}^G_{X_{d-2},\epsilon})^{S^1}) = \mathrm{K}_T(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.59}$$

3. If $C_2 = \mathcal{E}_\tau$ is an elliptic curve we find T-equivariant elliptic cohomology as the observable ring

$$\mathcal{H}_{\delta_\epsilon}(X_{d-2} \times S^1) = \mathrm{Ell}^\tau_T(\mathfrak{M}^G_{X_{d-2},\epsilon}) \tag{1.60}$$

The partition function itself is an element of the (rational, trigonometric, or elliptic) T-equivariant cohomology of a point under the pushforward π_* map of 1 for

$$\pi : \mathfrak{M}^G_{X_{d-2},\epsilon} \to pt \tag{1.61}$$

$$Z = \pi_* 1 \qquad Z = \int_{\mathfrak{M}^G_{X_{d-2},\epsilon}} 1 \tag{1.62}$$ {eq:Z}

An element of the T-equivariant (rational|trigonometric|elliptic) cohomology $H_T^{\mathrm{rat|tri|ell}}(pt)$ of a point is an invariant function on the Lie algebra $\mathfrak{t}$ of T in rational case, on the group T in the trigonometric case, on the moduli space of T-bundles on elliptic curve $\mathcal{E}_\tau$ in the elliptic case (strictly speaking in the elliptic case an element of $H_T^{\mathrm{ell}}(pt)$is a section of a certain line bundle on the moduli space of T-bundles on $\mathcal{E}_\tau$, such section can be identified with a character of highest weight integrable module for Kac-Moody affine Lie algebra $\hat{\mathfrak{t}}$ and the degree of line bundle with the level).

Let $X_4 = \mathbb{R}^4$ be a flat Euclidean space with a fixed origin $0 \in \mathbb{R}^4$ and let $\mathfrak{M}^{G,k}_{X_4}$ be the moduli space of self-dual connections on $\mathbb{R}^4$ with a fixed framing at infinity and second Chern class $c_2 = k$. The group $SO(4)$ acts on $\mathbb{R}^4$ by isometrical rotation around the point 0, and the group G acts on $\mathfrak{M}^{G,k}_{X_4}$ by the chage of framing at infinity. Let $T = SO(4) \times G$. The moduli space $\mathfrak{M}^{G,k}_{X_4}$ is hyperKahler space of real dimension $4kh^\vee_G$, and let $\mathfrak{M}^G_{X_4} = \amalg_{k \in \mathbb{Z}_+} \mathfrak{M}^{G,k}_{X_4}$.

The q graded element Z of T-equivariant cohomology of a point (rational|trigonometric|elliptic) obtained by the pushforward of 1 under $\mathfrak{M}^G_{X_4} \to pt$

$$Z = \int_{\mathfrak{M}^G_{X_4}} q^{c_2} = q^k \int_{\mathfrak{M}^{G,k}_{X_4}} 1 \tag{1.63}$$

is called Nekrasov partition function of pure $N=2$ SYM. By definition, Z is a function of q and an element of (Lie algebra of T| group T| moduli space of T-bundles on $\mathcal{E}_\tau$) respectively in the (rational| trigonometric| elliptic) case of T-equivariant cohomologies.

2. Lecture 2. The 4d $\mathcal{N}=2$ gauge theory. 14.20.2014

2.1. **Geometrical structures.** The Kahler geometry on the target space is the common feature of the supersymmetric theory with 4 supercharges such as 4d $\mathcal{N}=1$ or related by dimensional reduction 2d $(2,2)$ supersymmetric models.

In the theories with 8 supercharges such as 6d $\mathcal{N}=\infty$ or its dimensional reduction to 4d $\mathcal{N}=2$ the geometry of the target space is more constrained. The geometry of 4d vector multiplet is *integral special Kahler* and the geometry of 4d hypermultiplet is *hyperKahler*.

2.1.1. *Complex.* A complex structure on $V=\mathbb{R}^{2n}$ is $I\in\mathrm{End}(V)$ such that $I^2=-1$. An eigen subspace $V^{(1,0)}\subset V\otimes\mathbb{C}$ with eigenvalue $+i$ of I is called holomorphic, and with eigenvalue $-i$ is called antiholomorphic. An almost complex structure I on a smooth $2n$-dimensional manifold M is a complex structure $I\in\mathrm{End}(T_M)$. An almost complex structure I is integrable if Lie bracket for any two $(1,0)$ vector fields is $(1,0)$ vector field.

2.1.2. *Kahler.* A metric g (symmetric positive definite bilinear form on T_M) is compatible with complex structure I if the bilinear form

$$\omega_I(u,v)=g(Iu,v) \tag{2.1}$$

is symplectic structure, i.e. ω_I is antisymmetric, non-degenerate and closed. This implies that ω_I is $(1,1)$ two-form for complex structure I called *real Kahler symplectic form*. For any Kahler metric there always locally exists function, called *Kahler potential* $K(a,\bar a)$ such that in local holomorphic coordinates $\{a^i\}$

$$g_{i\bar j}=\partial_i\partial_{\bar j}K(a,\bar a),\qquad g=g_{i\bar j}(da^i\otimes da^{\bar j}+da^{\bar j}\otimes da^i) \tag{2.2}$$

and

$$\omega=\imath\partial\bar\partial K(a,\bar a),\qquad \omega_{i\bar j}=\imath\partial_i\partial_{\bar j}K(a,\bar a),\qquad \omega=\omega_{i\bar j}da^i\wedge da^{\bar j} \tag{2.3}$$

For example $K=\frac{1}{2}|a|^2$ gives the standard metric on $\mathbb{C}$. The holonomy group of Kahler manifold is $U(n)$.

If compact group G acts on Kahler space M in Hamiltonian way with respect to symplectic structure ω_I and the $\mathfrak{g}^*$ valued function μ_I called *moment map* (that means for any $\xi\in\mathfrak{g}$ it holds that $(\xi,d\mu_I)=i_\xi\omega$ where i_ξ is contraction with the vector field ξ on M generated by G action), and $\zeta\in\mathfrak{g}^*$ is coadjoint-invariant element, the Kahler quotient $M//G$ is

$$M//G=\{\mu^{-1}(\zeta)\}/G \tag{2.4}$$

The space $M//G$ is again Kahler space of real dimension $\dim_{\mathbb{R}}M-2\dim_{\mathbb{R}}G$. As a complex manifold $M//G$ is in fact the complex quotient $M^s/G^{\mathbb{C}}$ of an open set $M^s\subset M$ of stable points on M by complexified group $G^{\mathbb{C}}$. A point is stable if its $G^{\mathbb{C}}$ orbit intersects $\mu^{-1}(\zeta)$.

For example $\mathbb{CP}^1$ is the Kahler quotient of $\mathbb{C}^2$ with moment map $\mu=\frac{1}{2}(|z_1|^2+|z_2|^2)$ and $\zeta=1$ for $U(1)$ action $z_i\to tz_i$ where t is defining character of $U(1)$, and equivalently $\mathbb{CP}^1$ is the complex quotien of $\mathbb{C}^2\setminus\{0,0\}$ by $\mathbb{C}^\times$ action $z_i\to\lambda z_i$ for non-zero complex numbers $\lambda\in\mathbb{C}^\times$. The point $\{0,0\}\in\mathbb{C}^2$ is not stable because its orbit under $\mathbb{C}^\times$ action does not intersect the locus $\mu^{-1}(1)=\{(z_1,z_2):\frac{1}{2}(|z_1|^2+|z_2|^2)=1\}$, so this point is discarded in the complex quotient definition. For Kahler quotient by an abelian group $U(1)^r$ the set of parameters $\zeta\in\mathbb{R}^r\in\mathrm{Lie}(U(1)^r)$ is called FI parameters in physics literature.

For example, the moduli space of G-flat connections $M^G_{X_2}$ on a smooth orientable 2d manifold X_2 (this moduli space is associated to the theory coming from the reduction of

$d = 4$ $\mathcal{N} = 1$ to 2d as described in the first section) is symplectic quotient of the (infinite-dimensional) affine space $\mathcal{A}$ of connection 1-forms A on a G-principal bundle P on X_2 by the (infinite-dimensional) group $\mathcal{G}$ of automorphisms of the bundle (gauge group action). The symplectic structure ω on the space of connection $\mathcal{A}$ is defined by the formula

$$\omega(\delta A, \delta A') = \int_\Sigma \langle \delta A \wedge \delta A' \rangle \tag{2.5}$$

{eq:symplec

(where $\langle , \rangle$ is invariant bilinear form on $\mathfrak{g}$). Notice that definition of ω requires only orientation of X_2, not a complex structure or a metric.

The space $\mathrm{Lie}(\mathcal{G})^*$, dual to the Lie algebra of $\mathcal{G}$, is the space of linear forms on sections of $\phi \in \Gamma(X_2, \mathrm{ad}P)$. We can identify it with the space $F \in \Gamma(X_2, \Lambda^2(T^*_{X_2}) \otimes \mathrm{ad}P)$ of adjoint valued two-forms. The value of F on ϕ is given by

$$\int_{X_2} \langle F, \phi \rangle \tag{2.6}$$

The moment map function

$$\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^* \tag{2.7}$$

for the symplectic structure (2.5) is simply the curvature two-form of the connection

$$\mu = F_A \tag{2.8}$$

{eq:moment}

Indeed, it satisfies the definition of the moment map

$$\int_\Sigma \langle \delta F_A, \phi \rangle = \int_\Sigma \langle D_A \delta A, \phi \rangle = - \int_\Sigma \langle D_A \phi \wedge \delta A \rangle \tag{2.9}$$

since vector field on the space of connections $\mathcal{A}$ generated by the element ϕ of Lie algebra $\mathcal{G}$ is $\delta A = -D_A \phi$.

2.1.3. *HyperKahler.* A hyperKahler structure on a Riemannian manifold M of real dimension $4n$ is a triplet (I, J, K) of complex structures such that $IJ = K$ and the Riemannian metric g that is Kahler with respect to each of complex structures I, J, K. HyperKahler structure implies existence of S^2-worth of complex structures: $I_\zeta = \zeta_I I + \zeta_J J + \zeta_K K$ for $\zeta \in S^2 \subset \mathbb{R}^3 | \zeta_I^2 + \zeta_J^2 + \zeta_K^2 = 1$. The holonomy group of hyperKahler manifold is $USp(2n) = U(2n) \cap Sp(2n, \mathbb{C})$.

In addition, to each complex structures I, J, K there is an associated $(2, 0)$ holomorphic two-form

$$\Omega_I = \omega_J + \imath \omega_K \qquad \Omega_J = \omega_K + \imath \omega_I \qquad \Omega_K = \omega_I + \imath \omega_J \tag{2.10}$$

If compact group G acts on hyperKahler space M in Hamiltonian way with respect to symplectic structures $(\omega_I, \omega_J, \omega_K)$ and the $\mathfrak{g}^*$ valued function (μ_I, μ_J, μ_K) called *moment map*, and $\zeta \in \mathfrak{g}^* \otimes \mathbb{R}^3$ is coadjoint-invariant element, the hyperKahler quotient $M//G$ is

$$M////G = \{\mu^{-1}(\zeta)\}/G \tag{2.11}$$

The space $M////G$ is again hyperKahler space of real dimension $\dim_{\mathbb{R}} M - 4 \dim_{\mathbb{R}} G$. For example, the moduli space of self-dual connection on hyperKahler manifold X_4 is the hyperKahler quotient of the space of all connections by the action of the (infiinite-dimensional) gauge group. The hyperKahler moment map functional is F_A^+. (The moment map functional is valued in $\Gamma(X_4, \mathbb{R}^3 \otimes \mathrm{ad}(\mathfrak{g}))$ where $\mathbb{R}^3 \simeq \Lambda^2(T^*_{X_2})$.)

Now break the hyperKahler symmetry by taking $\mathbb{R}^3 \simeq \mathbb{R} \oplus \mathbb{C}$ and define *complex moment map* $\mu_{\mathbb{C}} : M \to \mathfrak{g}^* \times \mathbb{C}$ real moment map $\mu_{\mathbb{C}} : M \to \mathfrak{g}^* \times \mathbb{R}$ by

$$\mu_{\mathbb{C}} = \mu_J + \imath \mu_K, \qquad \mu_{\mathbb{R}} = \mu_I \tag{2.12}$$

The complex moment map satisfies

$$d\mu_I^{\mathbb{C}}(\xi) = i_\xi \Omega_I \tag{2.13}$$

for $\xi \in \mathfrak{g}$ and where Ω_I is a holomorphic $(2,0)$-form. Consequently, $i_\xi \Omega_I$ is $(1,0)$ form, therefore the complex moment map $\mu_I^{\mathbb{C}}$ is *I-holomorphic* function on M. Let $\zeta_{\mathbb{R}} = \zeta_I, \zeta_{\mathbb{C}} = \zeta_J + \imath \zeta_K$. Then the hyperKahler quotient is

$$M////G = \{\mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})\}/G \tag{2.14}$$

which is symplectic quotient of the space $\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$, and consequently, as a complex manifold its isomorphic to

$$M////G = (\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}}))^s / G^{\mathbb{C}} \tag{2.15}$$

which is precisely the complexified definition of the symplectic quotient: instead of real symplectic structure we are using $(2,0)$ holomorphic symplectic structure and the action by compact group is replaced by holomorphic action of the complex group $G^{\mathbb{C}}$.

2.1.4. *Special Kahler.* A coordinate free equivalent definition [Freed, 1997] is that special Kahler manifold M is a Kahler manifold equipped with a real flat torsionfree symplectic connection ∇ on T_M such that

$$d_\nabla I = 0 \tag{2.16}$$

In the above equation $d_\nabla : \Gamma(M, T_M^* \otimes T_M) \to \Gamma(M, \Lambda^2 T_M^* \otimes T_M)$ is the exterior differential defined by the connection ∇. It acts on complex structure I viewed as a 1-form on M valued in vectors.

Equivalently, a special Kahler manifold (of real dimension $2n$) can be defined as a Kahler manifold in which locally exist a set of holomorphic coordinates $\{a^i\}_{i=1\ldots n}$ and a locally holomorphic function $F(a)$, called *prepotential* such that Kahler potential is

$$K = \frac{1}{2} \operatorname{Im}(a^{\bar{i}} \partial_i F) \tag{2.17}$$

Consequently, the Kahler metric is

$$g_{i\bar{j}} = \frac{1}{2} \operatorname{Im}(\partial_i \partial_j F) \tag{2.18}$$

and the real symplectic form is

$$\omega = \frac{\imath}{2} \partial \bar{\partial} \operatorname{Im}(a^{\bar{i}} \partial_i F), \qquad \omega_{i\bar{j}} = \frac{1}{2} \imath \operatorname{Im}(\partial_i \partial_j F) \tag{2.19}$$ {eq:omega}

The second derivative $\partial_i \partial_j F$ is called *period matrix* τ_{ij}

$$\tau_{ij} = \partial_i \partial_j F \tag{2.20}$$

for the reasons explained shortly below. The τ_{ij} is a symmetric $n \times n$ matrix whose imaginary part needs to be positive definite (that is a condition permissible $F(a)$) so that the Kahler metric $g_{i\bar{j}} = \operatorname{Im} \tau_{ij}$ is a Riemannian metric. The simplest example is $M = \mathbb{C}$ with

$$F = \frac{1}{2} \tau a^2 \tag{2.21}$$

which gives

$$\omega = \frac{\imath}{2} \operatorname{Im} \tau da \wedge d\bar{a} = \operatorname{Im} \tau (dx \wedge dy) \tag{2.22}$$

Given a function $F(a)$ of special holomorphic coordinates $\{a^i\}$ define another set of special holomorphic coordinates $\{b_i\}$ called *dual* coordinates

$$b_i = \partial_i F \equiv \frac{\partial F}{\partial a^i} \tag{2.23}$$ {eq:bi}

The symplectic form ω is actually flat in terms of the real coordinates $\operatorname{Re} a^i, -\operatorname{Re} b_i$. Indeed, first notice that because of (2.23) it holds that

$$da^i \wedge db_i = 0, \qquad da^{\bar{i}} \wedge db_{\bar{i}} = 0 \tag{2.24}$$

and from (2.19) we find

$$\omega = -\frac{1}{4}(da^{\bar{i}} \wedge db_i + da^i \wedge db_{\bar{i}}) = -\frac{1}{4}((da^i + da^{\bar{i}}) \wedge (db_i + db_{\bar{i}})) = d(\operatorname{Re} a^i) \wedge d(-\operatorname{Re} b^i) \tag{2.25}$$

Therefore, besides the holomorphic coordinates $\{a^i\}$ special Kahler manifold is equipped with Darboux flat coordinates $\{\operatorname{Re} a^i, -\operatorname{Re} b_i\}$ for the symplectic structure ω.

How ambiguous is the choice of special coordinates that we are using to describe the special Kahler geometry? Instead of the coordinate system $\{a^i\}$ consider the double set $\{a^i\} \cup \{b_i\}$ and notice that there is a pointwise linear relation between the differentials

$$db_i = d(\partial_i F) = \partial_{ij} F da^j = \tau_{ij} da^j \tag{2.26}$$

The flat symplectic structure $\omega = -\operatorname{Re} da^i \wedge \operatorname{Re} db_i$ is invariant under symplectic transformation in $Sp(2n, \mathbb{R})$. Let the constant matrix from $Sp(2n, \mathbb{R})$ that acts on $(da^1, \dots da^n, db_1, \dots, db_n)$ be denoted as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}) \tag{2.27}$$

where A, B, C, D are $n \times n$ real matrices. The tilded holomorphic double set $\{\tilde{a}^i\} \cup \{\tilde{b}_i\}$ is

$$\begin{pmatrix} d\tilde{b}^i \\ d\tilde{a}_i \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} db^i \\ da_i \end{pmatrix} \tag{2.28}$$ {eq:tilde}

It is clear that

$$d\tilde{b}_i = \tilde{\tau}_{ij} d\tilde{a}^j, \qquad \tilde{\tau} = (A\tau + B)(C\tau + D)^{-1} \tag{2.29}$$

In tilded coordinate system $\{\tilde{a}^i\} \cup \{\tilde{b}_i\}$ the tilded prepotential $\tilde{F}$ is defined by integrating the matrix of its second derivatives τ_{ij} (which is of course symmetric from (2.28)). The constant $Sp(2n, \mathbb{R})$ transformation to $\{\tilde{a}^i\} \cup \{\tilde{b}_i\}$ keeps the symplectic structure ω and the complex structure, and hence the metric invariant.

2.1.5. *Integral special Kahler.* An integral special Kahler manifold is a special Kahler manifold, say of real dimension $2n$ with an additional data: a bundle of rank $2n$ integral lattice $\Lambda \subset T_M$ with respect to which the symplectic form ω is integral. All constructions in the previous subsection hold literally except that now the ambiguity in the special coordinate system is restricted to a subgroup of $\Gamma \subset Sp(2n, \mathbb{Z}) \subset Sp(2n, \mathbb{R})$ that preserves the lattice Λ. If integral symplectic form is principal (e.g. there is a basis α^i, β_j on lattice such that $\omega(\alpha^i, \alpha^j) = 0, \omega(\alpha^i, \beta_j) = \delta^i_j, \omega(\beta_i, \beta_j) = 0$) then $\Gamma = Sp(2n, \mathbb{Z})$.

A full lattice $\Lambda \subset T_M$ induces full dual lattice $\Lambda^* \subset T^*_M$.

Now since τ_{ij} is symmetric matrix with positive definite imaginary part, we can think about it as period matrix of rank n abelian variety attached to each point of special Kahler manifold M. The defining data of integral special Kahler manifold M is equivalent to the fibration $A \to P \to M$ where at each point $u \in M$ the fiber $A_u = (T^*_M)_u/\Lambda$ is a polarized abelian variety with period matrix τ_{ij}. The lattice Λ is the lattice of 1-cycles $H_1(A, \mathbb{Z})$, and the ambiguity $\Gamma \subset Sp(2n, \mathbb{Z})$ precisely corresponds to the ambiguity in the choice of basis of Λ.

2.2. **The $\mathcal{N} = 1$ supersymmetry algebra in $d = 4$ in supespace.** The $\mathcal{N} = 1$ supersymmetry has a particular simple realization by fields on the super space $(\mathsf{V}|\mathsf{S})$. Let index α labels basis in S, the index μ labels basis in V, and let $\gamma^\mu_{\alpha\beta}$ be the symmetric map $\mathsf{S} \otimes \mathsf{S} \to \mathsf{V}$ that defines the odd-odd bracket of the superPoincare Lie algebra. The we have anti-commutator relations[1]

$$\{Q_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu \tag{2.30}$$

We can represent the superPoincare generators Q_α by the differential operators on the superspace $(\mathsf{V}, \mathsf{S})_{\langle x,\theta\rangle}$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu \tag{2.31}$$

A function $F(x, \theta)$ on a superspace (V, S) is called superfield. The operator $(1 + \epsilon^\alpha Q_\alpha)$ acts on a function $F(x, \theta)$ as differential operator

$$(1 + \epsilon^\alpha Q_\alpha)F(x, \theta) = F(x + \epsilon\gamma\theta, \theta + \epsilon) \tag{2.32}$$

2.2.1. *Chiral fields.* To define interesting representation of $\mathcal{N} = 1$ superPoincare algebra it is conventient to introduce conjugated operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu \tag{2.33}$$

whose defining property is that D_α and Q_β anti-commute

$$\{D_\alpha, Q_\beta\} = 0 \tag{2.34}$$

Recall that in $d = 4$ the spin-module $\mathsf{S} = S = S^+ + S^-$ where S are Dirac spinors and $S^\pm$ are the $\pm$-chiral spinors, as well we have splitting $Q = Q^+ + Q^-$ and $D = D^+ + D^-$. Given operators D_α anti-commuting with Q_β we can define a subspace in the space of superfields by the condition

$$D^+_\alpha \Phi^-(x, \theta) = 0 \qquad \text{:chiral constraint:} \tag{2.35}$$ {eq:chirali

Notice that

$$D^+(x + \theta^+\gamma\theta^-) = 0, \qquad D^+\theta^- = 0 \tag{2.36}$$

Define

$$y^- \equiv x + \theta^+\gamma\theta^- \tag{2.37}$$

and notice that

$$D^+y^- = 0. \tag{2.38}$$

Then a general solution of the chirality constraint (2.35) can be represented as $\Phi(y, \theta^-)$:

$$D^+\Phi^-(y^-, \theta^-) = 0 \tag{2.39}$$

The field $\Phi^-(y^-, \theta^-)$ is called chiral field.

[1] In this presentation we consider superPoincare algebra over $\mathbb{C}$ and not consequently we are not keeping track of conventional $\imath$-factors related to the choice of real structure.

2.2.2. *D-term: Kahler kinetic term.* From a generic superfield $F(x,\theta)$ a supersymmetric action can be constructed by taking

$$\int [dx] \int [d\theta] F(x,\theta) \qquad\qquad : D - term : \tag{2.40}$$ {eq:D-term}

where $\int [d\theta]$ is the Berezian integral: by definition $\int [d\theta] F(x,\theta)$ picks up the top form of $F(x,\theta)$. Indeed, supersymmetric variation vanishes

$$\int [dx] \int [d\theta] Q_\alpha F(x,\theta) = 0 \tag{2.41}$$

because the term with $\frac{\partial}{\partial\theta} F(x,\theta)$ has zero top form (by θ-degree counting), and the term with $\gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu$ is the total derivative in the x-space that vanishes after integration over x. The non-chiral actions as above are called *D-term* and are mostly used in sigma-models for construction of the kinetic term for the scalar fields using the Kahler potential $K(z,\bar{z})$ for the Kahler metric on the target space

$$\int [dx] \int [d\theta] K(\Phi^-, \Phi^+) \qquad\qquad \text{:Kahler D-term:} \tag{2.42}$$

where Φ^+ is *anti-chiral* field defined in the opposite way (swtiching $+$ with $-$) to Φ^-.

2.2.3. *F-term: holomorphic superpotential.* As well, we can notice that the chiral action defined using chiral field (2.35)

$$\int [dx] \int [d\theta^-] \Phi(y,\theta^-), \qquad\qquad : F - term : \tag{2.43}$$ {eq:F-term}

is also supersymmetric invariant. Indeed,

$$Q^+ y^-_\mu = 2\theta^- \gamma_\mu, \qquad Q^- y^-_\mu = 0 \tag{2.44}$$

hence

$$Q^+ \Phi(y^-,\theta^-) = 2\theta^- \gamma^\mu \partial_\mu \Phi(y^-,\theta^-), \qquad Q^- \Phi(y^-,\theta^-) = \partial_{\theta^-} \Phi(y^-,\theta^-) \tag{2.45}$$

so the Q-variation of (2.43) is total derivative in (x,θ) space and thus vanishes after integration over $[dx]$ and $[d\theta]$. The chiral action of the form (2.43) is called *F-term* and is usually generated by arbitrary holomorphic function, called *superpotential*, $W(z)$

$$\int [dx] \int [d\theta^-] W(\Phi(y^-,\theta^-)) : \qquad\qquad \text{:superpotential F-term:} \tag{2.46}$$

2.2.4. *Gauge fields.* We consider the $U(1)$ gauge field to simplify presentation. The supersymmetrization of the gauge connection is a superfield $V(x,\theta^+,\theta^-)$. The gauge transformation is

$$V \to V + \Phi^- + \Phi^+ \tag{2.47}$$ {eq:superga

where Φ^-, Φ^+ is chiral and antichiral fields. The supersymmetrization of the field strength is

$$\begin{aligned} W^-_\alpha &= (D^+)^2 D^-_\alpha V \\ W^+_\alpha &= (D^-)^2 D^+_\alpha V \end{aligned} \tag{2.48}$$

The superfield W^-_α is chiral because $(D^-)^3 = 0$ (indeed $\dim S^- = 2$) and the superfield W^+_α is antichiral because $(D^+)^3 = 0$ (indeed $\dim S^+ = 2$). And the superfields $W^\pm$ are obviously invariant under supergauge transformation (2.47). The superfield $W^{\pm}$ is called gauge superfield strength with the expansion

$$W^- = \psi^- + \theta^- K + \not{F}^- \theta^- + (\theta\theta)\not{D}\psi^+ \tag{2.49}$$

where in all coefficient field $(\psi^-, K^-, F^-, \psi^+)$ we have omitted arguments (y^-), the opeartor $\not{D} = \gamma^\mu D_\mu$ is Dirac differential operator, the $\not{F} = F_{\mu\nu}\gamma^{\mu\nu}$ is the Clifford action by the 2-form $F_{\mu\nu}$, the F^- denotes the anti-selfdual part of the curvature, the *gluino field* ψ is the fermionic superpartner of the gauge connection whose curvature is F, and K is the *auxiliary* scalar field[2]. In the same way

$$W^+ = \psi^+ + \theta^+ K + \not{F}^+ \theta^+ + (\theta\theta)\not{D}\psi^- \tag{2.50}$$

The gauge field action functionals $S^{\pm}_{SYM}$ are defined by complexified coupling constants τ^- and τ^+ as follows

$$\begin{aligned} S^-_{SYM} &= \frac{\imath}{4\pi}\tau^- \int [dx] \int [d\theta^-](W^-)^2 = \frac{\imath}{4\pi}\tau^- \int [dx](F^- \wedge F^- - \frac{1}{2}K^2 - \frac{1}{2}\psi\not{D}\psi) \\ S^+_{SYM} &= \frac{\imath}{4\pi}\tau^+ \int [dx] \int [d\theta^+](W^+)^2 = \frac{\imath}{4\pi}\tau^+ \int [dx](F^+ \wedge F^+ + \frac{1}{2}K^2 + \frac{1}{2}\psi\not{D}\psi) \end{aligned} \tag{2.51}$$

If a real structure for fields is chosen in Euclidean signature, then the standard convention is to denote $\tau^- = \tau$ and $\tau^+ = \bar{\tau}$ where τ is complexified coupling constant

$$\tau = \frac{4\pi\imath}{g^2_{YM}} + \frac{\theta}{2\pi} \tag{2.52}$$

then

$$S_{SYM} = S^-_{SYM} + S^+_{SYM} \tag{2.53}$$

is the standard SYM action (1.33).

2.3. **The $\mathcal{N} = 2$ supersymmetry algebra in $d = 4$ as reduction of $\mathcal{N} = 1$ superPoincare in $d = 6$.** Recall from the supersymmetry table 1.4.6 that $\mathcal{N} = 1$ superPoincare in $d = 6$ uses spinor module $\mathsf{S}_{(6)} = S^+_{(6)} \otimes \mathbb{C}^2$. Consider a decomposition of the 6d space-time into 4d and 2d spaces: $\mathsf{V}_6 = \mathsf{V}_4 \oplus \mathsf{V}_2$, and respectively the subgroup $Spin(4) \times Spin(2) \subset Spin(6)$. The chiral Weyl spinors S^+_6 (irreducible $Spin(6)$-module of dimension 4) transform as Dirac spinors $S_4 = S^+_4 + S^-_4$ (reducible $Spin(4)$-module of dimension 4) under $Spin(4) \subset Spin(6)$, so

$$S^+_6 \simeq S_4 \qquad \text{as } Spin(4) \subset Spin(6) \text{ modules} \tag{2.54}$$

The $\mathcal{N} = 1$ $d = 6$ uses spinors in $\mathsf{S}_6 = S^+_{(6)} \otimes \mathbb{C}^2$, that is the two copies of $Spin(6)$ Weyl spinors S^+_6. Under the reduction to $Spin(4) \subset Spin(6)$ we obtain two copies of $Spin(4)$ Dirac spinors: $S_4 \otimes \mathbb{C}^2$. The minimal supersymmetry $\mathcal{N} = 1$ in $d = 4$ uses one copy of Dirac spinors S_4 (see the table 1.4.6). Therefore, with respect to the decomposition $\mathsf{V}_6 = \mathsf{V}_4 \oplus \mathsf{V}_2$ the minimal 6d superPoincare algebra $\mathfrak{iso}(\mathsf{V}_6|\mathsf{S}_6)$ reduces to the 4d extended superPoincare algebra $\mathfrak{iso}(\mathsf{V}_4|\mathsf{S}_4 \otimes \mathbb{C}^2)$. Recall that the $\mathbb{C}^2$ factor in S_6 is $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}^\vee$ with canonical symplectic form and that we had to introduce this factor to have a symmetric odd-odd bracket in the superPoincare Lie algebra. The automorphism group $Sp(\mathbb{C}^2) \simeq SL(2, \mathbb{C})$ is symplectic group, called *R-symmetry*. Its compact subgroup $SU(2)$ remains after the

[2]Sometimes this auxiliary field K is denoted in supersymmetric literature as D-field, but we are already overusing symbol D

appropriate choice of the real structure for the superPoincare Lie algebra is made. This $SU(2) \subset SL(2,\mathbb{C})$ acting on the internal space $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}^\vee$ is called the $SU(2)_R$*-symmetry* group of the 4d $\mathcal{N}=2$ supersymmetry: it acts naturally on the $\mathsf{S}_4 \otimes \mathbb{C}^2$ spinors of the 4d $\mathcal{N}=2$ superPoincare $\mathfrak{iso}(\mathsf{V}_4|\mathsf{S}_4 \otimes \mathbb{C}^2)$.

Another symmetry that commutes with superPoincare $\mathfrak{iso}(\mathsf{V}_4|\mathsf{S}_4 \otimes \mathbb{C}^2)$ comes from the rotation group $Spin(\mathsf{V}_2) \simeq SO(2)$ of the 2-plane V_2 on which we reduce the 6d $\mathcal{N}=1$ superPoincare to get 4d $\mathcal{N}=2$ superPoincare. This $SO(2) \simeq U(1)_R$ is called $U(1)_R$ R-symmetry group.

Finally, the translations in V_2 also commute with $\mathfrak{iso}(\mathsf{V}_4|\mathsf{S}_4 \otimes \mathbb{C}^2)$, these translation generators in V_2 are scalars from the 4d perspective and are called central charges.

If real structure is chosen such that $\mathsf{V}_2 \simeq \mathbb{R}^2$ as Euclidean 2-plane, it is customarily to denote $\mathsf{V}_2 \simeq \mathbb{R}^2 \simeq \mathbb{C}$ and then the central charge $Z \in \mathsf{V}_2$ is complex-valued scalar operator commuting with the rest of 4d $\mathcal{N}=2$ superPoincare (hence the name "*central charge*").

Expicitly, all generators of $\mathcal{N}=2$ super Lie algebra can be listed as follows

$$\begin{array}{|c|c|c|c|c} \mathsf{V}_4 & \Lambda^2\mathsf{V}_4 & \mathsf{S}^+ \otimes \mathbb{C}^2 & \mathsf{S}^- \otimes \mathbb{C}^2 & \mathsf{V}_2 \\ P_\mu & M_{\mu\nu} & Q^+_{\alpha i} & Q^-_{\dot\alpha i} & Z \\ (\frac12,\frac12,0,0) & (1,0,0,0)\oplus(0,1,0,0) & (\frac12,0,\frac12,-\frac12) & (0,\frac12,\frac12,\frac12) & (0,0,0,1) \end{array} \tag{2.55}$$

where the last line describes the representation with respect to $SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)_R$ where $SU(2)_- \times SU(2)_+ \simeq Spin(4)$

2.4. $\mathcal{N}=2$ **abelian vector mutliplet action.** The $U(1)^r$ abelian $\mathcal{N}=2$ supersymmetric vectormultiplet is composed of $\mathcal{N}=1$ supergauge field W_α and the $\mathcal{N}=1$ chiral field Φ. The $\mathcal{N}=2$ abelian Lagrangian contains only kinetic terms

$$S_{SYM,\mathcal{N}=2} = \frac{\imath}{4\pi}\left(\int[dx]\int[d\theta^-]\mathcal{F}_{ij}(\Phi)(W^{i-}W^{j-}) + \int[dx]\int[d\theta](\Phi^{i+}e^V\mathcal{F}_i(\Phi^-)) + \text{:anti-chiral terms:}\right) \tag{2.56}$$ {eq:N2actic

Here $i,j = 1\dots r$ and Φ^{i-} and W^{i-}_α denote respectively chiral scalar field and chiral gauge field strength. The first term is the chiral type *F-term* gauge field strength action, the second terim is non-chiral *D-term* Kahler potential type action for scalars. The Kahler potential is

$$K(a^{\bar i}, a^i) = \frac{1}{2}\,\mathrm{Im}(a^{\bar i}\partial_i\mathcal{F}(a)) \tag{2.57}$$ {eq:KahlerP

where $\mathcal{F}$ is the prepotential of the special Kahler geometry on r-complex dimensional target space $\mathcal{U}$ with local special coordinates a^i. The coordinates a^i are the scalar components of the chiral superfield Φ^i. The $r \times r$ matrix

$$\tau_{ij}(a) = \frac{\partial^2\mathcal{F}}{\partial a^i \partial a^j} \tag{2.58}$$

is the dynamical (a-dependent) complexified coupling constant.

The $\mathcal{N}=2$ supersymmetry mixes up fields from the $\mathcal{N}=1$ vector W and $\mathcal{N}=1$ chiral Phi. For example, the spinors in W and the spinors in Φ form together the $SU(2)_R$ doublet. The action (2.56) ensures $\mathcal{N}=2$ supersymmetry: a single holomorphic function $\mathcal{F}(a)$ determines both the coupling constant $\tau_{ij}(a)$ for the gauge field strength and the Kahler metric with the Kahler potential (2.57) on the target space for the scalars, and the same kinetic term proportional to $\tau_{ij}(a)$ for the spinors in W and Φ.

The special Kahler manifold $\mathcal{U}$ of complex dimension r is the target space of tne non-linear $U(1)^r$ $d = 4$ $\mathcal{N} = 2$ abelian sigma model $\mathrm{Maps}(\mathbb{R}^4, \mathcal{U})$.

2.4.1. *4d $\mathcal{N} = 2$ hypermultiplet.* The 4d $\mathcal{N} = 2$ is made of a pair of 4d $\mathcal{N} = 1$ chiral multiplets. The target space for the scalars of the hypermultiplet must be hyperKahler manifold to ensure $\mathcal{N} = 2$ supersymmetyr. If hypermultiplet transforms in representation R of gauge group G, the representation R has to be quaternionic representation of G of real dimension $4n$. Starting from any complex G-representation ρ of real dimension $2n$ we can construct quaternionic representation $R = \rho \oplus \rho^*$. The associated supermultiplet (with scalars in $\rho \oplus \rho^*$) is called *full-hypermultiplet of* ρ or equivalently *half-hypermultiplet* of $R = \rho \oplus \rho^*$. Then the minimal supermultiplet associated to an abstract quaternionic representation R (wtih $4n$ real scalars for $\dim_{\mathbb{R}} R = 4n$) is called *half-hypermultiplet of* R:

$$\text{half-hyper}(\rho \oplus \rho^*) \simeq \text{full-hyper}(\rho) \tag{2.59}$$

and then for any complex representation ρ of G we have conventions

$$\dim_{\mathbb{R}}(\text{hyper}(\rho)) = 2 \dim_{\mathbb{R}} \rho = 4 \dim_{\mathbb{C}} \rho \tag{2.60}$$

where $\dim_{\mathbb{R}}(\text{full-hyper}(\rho))$ denotes the real dimension of the hyperKahler target space for real scalars in the hypermultiplet. In most literature hyper $\equiv$ full-hyper. If ρ is an irreducible complex representation of G, the hypermultiplet associated to N_F copies of ρ, that is to representation $\rho \otimes \mathbb{C}^{N_F}$ is called to have N_F *flavors.* The $U(N_F)$ action on $\mathbb{C}^{N_F}$ commutes with R-action and is called *flavor symmetry* group $U(N_F)$.

More generally, for any compact Lie group G (gauge group) and any compact Lie group F (flavor group), an a quaternionic representation R of $G \times F$ gives rise to half-hyper hypermultipet associated of R with flavor symmetry group F. The hypermultiplet masses are the background complex scalars of vector multiplet associated to the flavor symmetry group F.

Hence, a mass parameter m of half-hypermultiplet in quaternionic $G \times F$-module R is an element of the complexified Lie algebra $\mathfrak{f}_{\mathbb{C}}$ of F

$$m \in \mathrm{Lie}(F_{\mathbb{C}}) \tag{2.61}$$

The theory is invariant under symmetry F, hence by F-adjoint transformation the mass parameter can be taken to be in the complexified Cartan algebra of F

$$m \in \mathrm{Lie}(T_{F_{\mathbb{C}}}) \tag{2.62}$$ {eq:mass}

For example, for a theory with N_F flavors we have $F = U(N_F)$ and $\mathrm{Lie}(T_{F_{\mathbb{C}}}) \simeq \mathbb{C}^{N_f}$ so that the mass parameter of such hypermultiplet is N_f-tuple of complex numbers.

2.4.2. *6d $\mathcal{N} = 1$ reduced to 5d.* For the 5d reduction $\mathsf{V}_6 = \mathsf{V}_5 \oplus \mathsf{V}_1$ of 6d $\mathcal{N} = 1$ the structure does not change much. There is no 5d $U(1)_R$ (since for the 4d reduction the $U(1)_R$ came from the orthogonal group of the 2-plane V_2, but for the 5d reduction the orthogonal group of V_1 is trivial). The $SU(2)_R$ is still there in the same way acting on spinors in $S_5 \otimes \mathbb{C}^2$ where S_5 denotes irreducible spinor representation of $Spin(5)$. The central charge $Z \in \mathsf{V}_1$ is a *real* scalar Z if the real structure is chosen such that $\mathsf{V}_1 \simeq \mathbb{R}$.

The target space for the real scalars of the vector multiplet is *special real manifold* parametrized by real cubic prepotential, see e.g. [**?**].

The target space for a half-hyper in a quaternionic $G \times F$-representation R is again hyperKahler manifold.

The mass parameter of half-hyper in a quaternionic representation R of $G \times F$ is an element of the *real* Lie algebra of compact Lie group F

$$m \in \mathrm{Lie}(F) \tag{2.63}$$

2.5. **6d $\mathcal{N} = 1$ reduced to 4d on T^2.** For a reduction of 6d $\mathcal{N} = 1$ gauge theory on a 2-torus $\mathcal{E}_{\tau^\vee}$ (elliptic curve) of finite size the background F-gauge connection along $\mathcal{E}_\tau$ determines the mass-type couplings from the perspective of 4d theory obtained by KK reduction along $\mathcal{E}_{\tau^\vee}$. Consequently, in such theory the mass parameter m is a point in a moduli space of F-flat bundles on $\mathcal{E}_\tau$, or, equivalently, coarse moduli space of semi-stable holomorphic $F^{\mathbb{C}}$ principal bundles on $\mathcal{E}_\tau$

$$m \in \mathrm{Bun}_{F^{\mathbb{C}}}(\mathcal{E}_\tau) \tag{2.64}$$

The limit of $T^2 = S^1 \times S^1$ when one S^1 shrinks to zero corresponds to the 5d theory reduced on S^1 (of finite size), and equivalently to the nodal singularity of the elliptic curve $\mathcal{E}_\tau$ at $\tau \to \imath\infty$. In this case (we neglect discrete quotient by the Weyl group)

$$m \in T_{F^{\mathbb{C}}} \tag{2.65}$$

Finally, if the other circle is shrinked to zero, this correspond to the degeneration of elliptic curve $\mathcal{E}_\tau$ to the rational curve with cusp and this case as shown in (2.62) $m \in Lie(T_{F^{\mathbb{C}}})$.

This picture agrees with the degeneration hierarchy

$$\mathrm{Bun}_{F^{\mathbb{C}}}(\mathcal{E}_\tau) \rightsquigarrow T_{F^{\mathbb{C}}} \rightsquigarrow Lie(T_{F^{\mathbb{C}}}) \tag{2.66}$$

as

$$\text{smooth elliptic curve} \rightsquigarrow \text{nodal curve} \rightsquigarrow \text{cusp curve} \tag{2.67}$$

and

$$\text{6d on } T^2 \rightsquigarrow \text{5d on } S^1 \rightsquigarrow \text{4d} \tag{2.68}$$

and as in 1.9)

$$F\text{-equiv de Rham cohomology} \rightsquigarrow F\text{-equiv K-theory} \rightsquigarrow F\text{-equiv elliptic cohomology} \tag{2.69}$$

2.6. **Lagrangian $\mathcal{N} = 2$ gauge theories.** Here we summarize the data that describes the Lagrangian of 4d $\mathcal{N} = 2$ gauge theory. The representation theoretical data is the choice of

1. Compact Lie group G: the gauge group.
2. Compact Lie group F: the flavor group.
3. Quaternionic $G \times F$ representation R.

with the condition that

4. The (G, F, R)-theory is UV complete.

The triplet (G, F, R) needs to satisfy the UV-complete condition of the non-positive β-function: absence of Landau pole at the large energies, so that the QFT is asymptotically free or conformal in the limit of UV = small distances = large momenta. In other words, under renormalization group flow the theory must be weakly coupled as small distances. Such renormalizable theory is called UV-complete.

Let $\mathfrak{g} = Lie(G)$ be the reducible Lie algebra of compact Lie group G, and let

$$\mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_i \tag{2.70}$$

be the decomposition of $\mathfrak{g}$ into irreducible Lie algebras $\mathfrak{g}_i$ and let $\langle\rangle_i$ be adjoint invariant Killing form on $\mathfrak{g}_i$.

For any representation ρ_i of irreducible Lie algebra $\mathfrak{g}_i$ let $C_2(\rho_i)$ be the eigenvalue of the second Casimir operator[3] of $\mathfrak{g}_i$, and let $c_2(\rho_i)$ be the coefficient that relates bilinear form $\operatorname{tr}_{\rho_i} xy$ with $\langle xy\rangle$

$$\operatorname{tr}_{\rho_i} xy = -c_2(\rho_i)\,\langle xy\rangle\,, \qquad x, y \in \mathfrak{g}_i \tag{2.71}$$

The numbers C_{ρ_i} and $c_2(\rho_i)$ are related

$$c_2(\rho_i) = \frac{\dim \rho_i}{\dim \mathfrak{g}} C_2(\rho_i) \tag{2.72}$$

For example, if $\mathfrak{g}_i$ is simple and the Killing form is chosen such that the length of the long root squared is 2, then $c_2(\mathfrak{g}) = 2h^\vee$ where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For $\mathfrak{g}_i = SU(N)$ and ρ_i fundamental representation the $c_2(\rho_i) = 1$.

Now let $\rho_i : \mathfrak{g}_i \to \operatorname{End}(\mathbb{H}^n)$ be the quaternionic half-hypermultiplet representation $\rho_i \simeq R$ induced from the inclusion $\mathfrak{g}_i \subset \mathfrak{g}\oplus\mathfrak{f}$ and represented over quaternions $\mathbb{H}^n$ (for $\mathcal{N} = 2$ theory with $4n$ real scalars in hypermultiplets). For each i define

$$\beta_i = c_2(\mathfrak{g}_i) - c_2(\rho_i|\mathbb{H}) \tag{2.73}$$

Our conventions are such that for $\rho_i = \tilde\rho_i \oplus \tilde\rho_i^*$ where $\tilde\rho_i$ is complex representation of $G \times F$ the half-hyper of $\rho_i = \tilde\rho_i \oplus \tilde\rho_i^*$ is hyper of $\tilde\rho_i$, and $c_2(\rho_i|\mathbb{H}) = c_2(\tilde\rho_i|\mathbb{C})$. For example, the case $\tilde\rho = \mathbb{C}^N$ for $\mathfrak{g} = SU(N)$ is 1 fundamental hyper, that is the $SU(N)$ theory with $N_F = 1$, so for complex hypers defined by complex rep $\tilde\rho_i$ we have equivalently

$$\beta_i = c_2(\mathfrak{g}_i) - c_2(\tilde\rho_i|\mathbb{C}) \tag{2.74}$$

In the standard physical conventions the β-function has opposite sign to the quantity we have called β, we want $\beta_i \geq 0$ for a good UV-complete theory.

Finally, the condition of UV-completeness is

4.' For each irreducible factor $\mathfrak{g}_i$ for $\mathfrak{g} = \oplus_{i\in I}\mathfrak{g}_i$ it should hold that

$$c_2(\mathfrak{g}_i) - c_2(\rho_i|\mathbb{H}) \geq 0 \tag{2.75}$$ {eq:c2}

where $c_2(\mathfrak{g}_i)$ denotes c_2 in the adjoint representation. Since abelian factors $\mathfrak{g}_i$ have $c_2 = 0$ it follows that R should be trivial representation for all abelian factors $\mathfrak{g}_i$, and consequently they decouple (except for possibly global topological effects).

For example, if $G = SU(N)$ and $F = U(N_F)$ we consider N_F fundamental flavors for $SU(N)$ gauge group, we compute $\beta_i = 2N - N_F$ and obtain the UV-completeness bound $N_F \leq 2N$.

The continious parameters of (G, F, R) theory are

1. Complex coupling constant $\tau = (\tau_i)$ for each irreducible $\mathfrak{g}_i$ term in $\mathfrak{g} = \oplus_i\mathfrak{g}_i$ with $\operatorname{Im}(\tau_i) > 0$.

2. Mass parameter $m \in Lie(T_{F_\mathbb{C}})$.

Summary

The Lagrangian $\mathcal{N} = 2$ theory is defined by a compact Lie gauge group G, a compact Lie flavor group F and a quaternionic representation R of $G \times F$ that satisfies inequalities (2.75). The complex parameters are the complex coupling constant $\tau = (\tau_i)$, where i labels irreducible terms in $\mathfrak{g} = \oplus_i\mathfrak{g}_i$, and complex mass $m \in Lie(T_{F^\mathbb{C}})$.

[3] For a Lie algebra $\mathfrak{g}$, a basis (t_α) in $\mathfrak{g}$ and Killing form $\langle\rangle$ on $\mathfrak{g}$, the second Casimir operator is $C_2 = -(\langle t_\alpha t_\beta\rangle)^{-1} t_\alpha t_\beta$

2.7. **Lagrangian $\mathcal{N}=2$ quiver gauge theories.** Let Γ be a directed graph called quiver with the set of nodes I. To each node $i \in I$ we associate a vector space $N_i = \mathbb{C}^{N_i}$ (We use notation N_i for $\mathbb{C}^{N_i}$ as well as for $\dim_{\mathbb{C}} N^i$, but it should be clear from the context) and consider the $\mathcal{N}=2$ gauge theory with gauge group $G = \times SU(N_i)$ and full hypermultiplet in representation

$$R = \oplus_{i\to j} Hom(N_i, N_j) \tag{2.76}$$

where the sum is over all arrows in quiver Γ. This is called minimal quiver gauge theory. We can extend the minimal quiver by matter in fundamental representation: connect each node of the quiver with two extra nodes nodes $(M_i, \tilde{M}_i)$ called respectively M_i fundamental and $\tilde{M}_i$ anti-fundamental matter multiplets for i-th node. Then we take full hyper defined by complex rep of G as follows

$$R = \oplus_{i\to j} Hom(N_i, N_j) \oplus_i Hom(N_i, M_i) \oplus_i Hom(\tilde{M}_i, N_i) \tag{2.77}$$

The flavor symmetry F is

$$F = \otimes_{i\to j} U(1) \otimes_i U(M_i) \otimes_i U(\tilde{M}_i) \tag{2.78}$$

Let C_{ij} be the Cartan matrix of the graph Γ that has 2 on the diagonals and minus number of arrows, regardless of their orientation, between i-th and j-th node. The UV-completeness β-function inequlities leads to the bound

$$C_{ij} N_j \ge M_i + \tilde{M}_i \qquad \forall i \tag{2.79}$$

The solution of these bounds is equivalent to the classification of simply-laced finite and affine generalized Cartan matrices. The result is that the quiver Γ is the Dynkin diagram of ADE or affine ADE. Moreover, if Γ is affine ADE, then $M_i = \tilde{M}_i = 0$ and $N_i = a_i^\vee N$ where $a_i^\vee$ are dual Dynkin labels on Γ and N is positive integer ($N > 1$ for $\Gamma = \hat{A}_r$ when all $a_i^\vee = 1$ in order for the theory with the gauge group $\times SU(N_i)$ to be non-empty).

3. Lecture 3. Electro-magnetic duality, Seiberg-Witten integrable system. 21.10.2014

3.1. **Electro-magnetic duality.** Electro-magnetic duality that exchanges the electric and magnetic fields as well as electric and magnetic charges is the basic symmetry of abelian Maxwell equations. We consider the electro-magnetic duality for $U(1)^r$ theory in the presence of θ-term couplings $\operatorname{tr} F \wedge F$. Let F^i for $i = 1, \dots, r$ label the 2-forms of $U(1)^r$ field strengh that take value in the Lie algebra $\mathbb{R}^r$ of $U(1)^r$ and let $\star$ be the Hodge dual operator on the Euclidean space-time X. The abelian Yang-Mills action is now (we inducing the metric on the Lie algebra by $-\operatorname{tr}$ in the fundamental representation)

$$S_{YM} = \frac{\imath}{4\pi} \int_X F^i \wedge (\tau'_{ij} F^j - \imath \tau''_{ij} \star F^j) \tag{3.1}$$ {eq:YMr}

where τ'_{ij} and positive definite τ''_{ij} are real symmetric matrices that combine into the complex symmetric matrix

$$\tau_{ij} = \tau'_{ij} + \imath \tau''_{ij} \tag{3.2}$$

For $r = 1$ and

$$\tau = \frac{4\pi\imath}{g_{YM}^2} + \frac{\theta}{2\pi} \tag{3.3}$$

the above action reduces to (1.2). The field configuration in the presence of electric charge $n \in \mathbb{Z}^r$ that moves on contour $\gamma \subset X$ is defined by extremizing the action (3.1) in the presence of the source term generated by electric charge

$$S_e = -\imath \int_\gamma n_i A^i \tag{3.4}$$

For abelian theory $F^i = dA^i$ and extremization of $S = S_{YM} + S_e$ is elementary

$$\int_X \frac{\imath}{2\pi} \delta A^i \wedge d(\tau'_{ij} F^j - \imath \tau''_{ij} \star F^j) = \imath n_i \int \delta A^i \tag{3.5}$$

implies

$$\frac{1}{2\pi} d(\tau'_{ij} F^j - \imath \tau''_{ij} \star F^j) = n_i \tilde{\gamma} \tag{3.6}$$

where $\tilde{\gamma}$ is the delta-function like three-form that is Hodge dual to γ, that is $\int_\gamma A = \int_X \gamma^\vee$. Let S^2 be a two-sphere canonically linked γ. Then we find Gauss flux associated to electric charge $n = \{n_i\}$.

$$\frac{1}{2\pi} \int_{S^2} (\tau'_{ij} F^j - \imath \tau''_{ij} \star F^j) = n_i \tag{3.7}$$ {eq:el}

A magnetic charge on a contour γ creates the flux of F^i, by definition

$$\frac{1}{2\pi} \int_{S^2} F^i = m^i \tag{3.8}$$ {eq:m}

Electric-magnetic duality (EM duality), by definition, is a linear integral transformation for the lattice $(n, m) \in \mathbb{Z}^{2r}$ of electro-magnetic charges that preserves the canonical symplectic structure ω on the space of charges

$$\omega((n, m); (\tilde{n}, \tilde{m})) = n_i \tilde{m}^i - \tilde{n}_i m^i \tag{3.9}$$

All electro-magnetic duality transformations form infinite discrete group $\Gamma = Sp(2n, \mathbb{Z})$. Let an element $g \in \Gamma$ be denoted as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.10}$$

where A, B, C, D are $r \times r$ matrices. Let $T_{ij} = \tau'_{ij} - \imath \tau''_{ij} \star$ be a matrix of linear operator that acts on $\Gamma(X, \Lambda^2 T^*_X \otimes \mathbb{R}^r)$. Then electric and magnetic flux (3.7) equations are

$$\begin{aligned} \frac{1}{2\pi} \int_X T_{ij} F^j &= n_i \\ \frac{1}{2\pi} \int_X 2\pi F^i &= m^i \end{aligned} \tag{3.11}$$

After we apply electro-magnetic duality $g \in \Gamma$ to the charge lattice we find new charges $(\tilde{n}, \tilde{m})$ to be given by

$$\begin{pmatrix} \tilde{n} \\ \tilde{m} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \tag{3.12}$$

And we want to satisfy the equations

$$\begin{aligned} \frac{1}{2\pi}\int_X \tilde{T}_{ij}\tilde{F}^j &= \tilde{n}_i \\ \frac{1}{2\pi}\int_X \tilde{F}^i &= \tilde{m}^i \end{aligned} \tag{3.13}$$

on the linearly transformed field strength $(\tilde{F}^i)$. Then we find

$$\tilde{T} = (AT+B)(CT+D)^{-1} \tag{3.14}$$ {eq:tildeT}

and

$$\tilde{F} = (CT+D)F \tag{3.15}$$

Since algebraically the operator $-\imath\star$ in the definition of $T = \tau'_{ij} - \imath\tau''_{ij}\star$ satisfies $(-\imath\star)^2 = -1$ it is equivalent to $\sqrt{-1}$ in the formula (3.14). That is, define $\tilde{T} = \tilde{\tau}'_{ij} - \imath\tilde{\tau}''_{ij}\star$ and $r\times r$ complex matrix $\tilde{\tau} = \tilde{\tau}' + \imath\tilde{\tau}''$, then

$$\tilde{\tau} = (A\tau+B)(C\tau+D)^{-1} \tag{3.16}$$

exactly like integral symplectic transformation for change of special coordinates in special Kahler manifold (2.28).

The main message learned as a consequence of electric-magnetic duality but in fact it is a principally polarized abelian variety!

3.1.1. *Seiberg-Witten integrable system.* The deformations of a given vacua of $\mathcal{N}=2$ theory correspond either to massless vectormultiplet or massless hypermultiplet excitations. The deformations associated with massless vector multiplets give rise to the space $\mathcal{U}$ called *Coulomb branch of the moduli space of vacua*, for the reason that the low-energy theory of excitations around such vacua if $U(1)^r$ abelian theory in which charged particles interact by the Coulomb law (equivalent to (3.7)(3.8)). Algebraically, $\mathcal{U}$ is the spectrum of the 1/2-BPS chiral ring operators of the vector-multiplet type, like $\langle P(\phi)\rangle$ where ϕ is the complex G-adjoint valued scalar of G-gauge vector-multiplet and P is an adjoint invariant polynomial on $\mathfrak{g}_{\mathbb{C}}$.

For example, in the simplest case of $G = SU(2)$ theory, the only independent 1/2-BPS chiral ring operator is $u = \operatorname{tr}\phi^2$. The variable u is a coordinate on the 1-dimensionl affine space $\mathcal{U}$: the moduli space of vacua of the 4d $\mathcal{N}=2$ theory.

From the analysis of $\mathcal{N}=2$ supersymmetry for the abelian $U(1)^r$ theory we have found that such theory is a non-linear $\mathcal{N}=2$ sigma-model of maps from the 4d space-time to the target space $\mathcal{U}$. The target space $\mathcal{U}$ must be equipped with special Kahler structure to satisfy $\mathcal{N}=2$ supersymmetry, and moreover, this special Kahler structure must be integral for so that electric-magnetic duality holds.

The special Kahler structure on $\mathcal{U}$ implies that there are local special coordinates a^i and holomorphic function $F(a)$, called prepotential, that determines symmetric $r\times r$ matrix $\tau_{ij}(a) = \partial_{ij}F(a)$ of coupling constants for $U(1)^r$ theory, moreover, the $\operatorname{Im}(\tau_{ij})$ is positive definite: it determines the positive-definite action functional for the gauge fields and the positive definite Kahler metric on $\mathcal{U}$.

Therefore, at a given point $u\in\mathcal{U}$ the τ_{ij} can be interpreted as a period matrix of rank r principally polarized abelian variety. (A rank r polarized abelian variety A can be viewed as r-dimensional complex torus $\mathbb{C}^r/\Lambda$, where Λ is a full rank $2r$ lattice, that has symmetric period matrix τ_{ij} with positive definite $\operatorname{Im}(\tau_{ij})$. The condition on τ_{ij} implies that one can well-define θ-functions on A which are quasi-periodic, that is holomorphic sections of a holomorphic line bundle on A, and then use these θ-functions to map A into a projective space. Hence A is a projective algebraic variety and an abelian group (complex torus), hence A is abelian variety. A principal polarization means a symplectic form on $H_1(A,\mathbb{Z})$ for which exists basis of 1-cycles α^i, β_i in canonical intersection $\alpha^i \cap \alpha^j = 0, \beta_i \cap \beta_j = 0, \alpha^i \cap \beta_j = \delta^i_j$)

The electric-magnetic duality implies that $\tau_{ij}(a)$ is defined only up to $Sp(\mathbb{Z}^{2r})$ where $\mathbb{Z}^{2r}$ is the lattice of electric-magnetic charges (n_i, m^i).

Hence, the defining data of $\mathcal{N} = 2$ low-energy theory for $U(1)^r$ vector multiplet can be recast in the form of the fibration $\mathcal{P}$ of rank r abelian varieties A over the moduli space of vacua $\mathcal{U}$:

$$A \to \mathcal{P} \to \mathcal{U} \tag{3.17}$$

where $\mathcal{P}$ of complex dimension $2r$ denotes the total space of the fibration.

The lattice of electric-magnetic charges is identified with $H_1(A,\mathbb{Z}) \simeq \mathbb{Z}^{2r}$.

The central charge Z evaluated on a state with electric-magnetic charge (n,m) is

$$Z = n_i a^i + m^i b_i \tag{3.18}$$

The central charge is a locally holomorphic funtion on $\mathcal{U}$. Since lattice of charges (n,m) is identified with $H_1(A,\mathbb{Z})$ we can represent the central charge function Z of a given (n,m) state by

$$Z(n,m) = \int_{n_i\alpha^i + m^i\beta_i} \lambda \tag{3.19}$$

where λ is a meromorphic 1-form. Hence

$$a^i = \int_{\alpha_i} \lambda, \qquad b_i = \int_{\beta^i} \lambda \tag{3.20}$$

Moreover, since on the base $\mathcal{U}$ it holds that

$$db_i = \tau_{ij}(a) da^j \tag{3.21}$$

we find that the α_i and β^i periods of $d\lambda$ from the equations

$$da^i = \int_{\alpha_i} d\lambda, \quad db_i = \int_{\beta^i} d\lambda \tag{3.22}$$

should be related by the period matrix τ_{ij} of the abelian variety A at a given base point in $\mathcal{U}$.

Therefore, the 2-form

$$\Omega = d\lambda \tag{3.23}$$

is a closed holomorphic non-degenerate $(2,0)$ form on the total space $\mathcal{P}$ that turns $\mathcal{P}$ into a holomorphic symplectic manifold. Moreover, the fibration $\mathcal{P} \to \mathcal{U}$ is Lagrangian, which means that Ω evaluates to zero when restricted to the fiber directions.

We see that the defining data of $\mathcal{N} = 2$ abelian $U(1)^r$ sigma-model with target $\mathcal{U}$ are exactly equivalent to the structure of algebraic completely integrable system

$$\mathcal{P} \to \mathcal{U} \tag{3.24}$$

The ring of Poisson commuting functions on $\mathcal{P}$ is the ring $\mathcal{O}(\mathcal{U})$ of holomorphic functions on the base $\mathcal{U}$: these functions are holomorphic Hamiltonians of completely integrable system. In the $\mathcal{N}=2$ gauge theory language the ring $\mathcal{O}(\mathcal{U})$ is the chiral ring of 1/2-BPS operators of the vectormultiplet type.[4]

For example, for $SU(2)$ theory the base $\mathcal{U}\simeq\mathbb{C}$ is one-dimensional complex space, and the only independe hamiltonian function on the complex phase space $\mathcal{P}\to\mathcal{U}$ is u. Without proof, let us quote the result for algebraic integrable system associated to $G=SU(2), F=1, R=0$ theory, that is the minimal $SU(2)$ 4d $\mathcal{N}=2$ theory with no hypers. Let $\mathfrak{q}=\exp(2\pi\imath\tau_{YM})$ be the exponentiated coupling constant.[5]

The base $\mathcal{U}\simeq\mathbb{C}$ with complex coordinate u. The fibers are rank 1-abelian varieties, that is simply elliptic curves. At a point $u\in\mathcal{U}$ the elliptic curve Σ_u is defined by algebraic equation in $\mathbb{C}\times\mathbb{C}^{\times}$ complex surface with coordinates (x,y) by the equation

$$\Sigma_u: \quad y+\frac{\mathfrak{q}}{y}=T(x;u) \tag{3.25}$$ {eq:Sigma_u

where

$$T(x;u)=x^2-u \tag{3.26}$$

The symplectic structure on $\mathbb{C}\times\mathbb{C}^{\times}$ surface is $dx\wedge dy/y$ and the meromorphic Liouville-Seiberg-Witten form λ on a curve Σ_u is

$$\lambda = x\frac{dy}{y} \tag{3.27}$$

In this example the total space $\mathcal{P}$ can be identified with $\mathbb{C}\times\mathbb{C}^{\times}$ space which we used to describe the spectral curve Σ_u by algebraic equation (3.25). Denote $y=-e^{\phi}$ and $x=p$. Then the symplectic form on $\mathcal{P}$ is $\Omega=dp\wedge d\phi$, while the equation of the spectral curve Σ_u gives Hamiltonian function u

$$u=p^2+e^{\phi}+\mathfrak{q}e^{-\phi} \tag{3.28}$$

This is an example of famous Toda integrable system, here is the simplest non-relativistic closed 2-particle Toda Hamiltonian reduced by the center of mass motion.

Remark 10. In domain of real Hamiltonian mechanics, in dimension 1 any real Hamiltonian function is trivially integrable, because there is no integrable conditions to check. However, in complex dimension 1 on the phase space $dp\wedge d\phi$ not every holomorphic Hamiltonian function u produces an algebraic integrable system. It is a non-empty condition that the complex curve for a fixed value of Hamiltonian function is an abelian variety, i.e. an elliptic curve.

3.2. **$\mathcal{N}=2$ gauge theory on $\mathbb{R}^3\times\mathbb{R}$.** The special coordinates a^i on the base $\mathcal{U}$ of the fibration $A\to\mathcal{P}\to\mathcal{U}$ are holomorphic action variables. Is there gauge theory interpretation of the angle variables parametrizing the holomorphic Arnold-Liouville toric fibers A? The answer is yes, and to see the angle variables in gauge theory we need to put the $\mathcal{N}=2$ theory on $\mathbb{R}^3\times S^1$ periodic space-time. Each $U(1)$-abelian vector multiplet after compactification on S^1 produces two additional periodic scalars: one scalar comes from the holonomy of the gauge field around the compactification circle S^1, the other periodic sclar is the 3d dual photon.

[4]The 1/2-BPS operators of hypermultiplet type parametrize Higg branch of the moduil space of vacua that we do not consider in these notes.

[5]Of course the theory is asymptotically free, therefore the coupling constant $\mathfrak{q}$ is dimension-full parameter defined at a certain scale.

Therefore, the holomorphic symplectic phase space $\mathcal{P}$ is the moduli space of vacua of the $\mathcal{N}=2$ QFT on $\mathbb{R}^3\times S^1$ background. In fact, after KK reduction such theory can be viewed as 3d theory (with infinite tower of particles coming from KK modes), and such 3d theory has 8 supercharges. The relevant massless multiplet of 3d theory with 8 supercharges is a hypermultiplet. The original complex scalar of the 4d $\mathcal{N}=2$ vector multiplet combines with two other real periodic scalars that came from the reduction on S^1 into quaternionic scalar parametrizing the hyperKahler moduli space of vacua. Hence, the symplectic space $\mathcal{P}$ carries not only holomorphic symplectic structure, but full hyperKahler metric. For a finite radius of the compactification circle S^1 the metric is highly non-trivial receiving non-perturbative contributions from the massive states wrapping S^1. Only in the limit of S^1 of infinite radius we can neglect the non-perturbative contributions and approximate the metric on $\mathcal{P}$ by the naive metric, flat in the fiber directions, defined by the canonical flat along fibers metric on the total space $T^*\mathcal{U}/\Lambda$. The space $\mathcal{P}=T^*\mathcal{U}/\Lambda$ defined on a special Kahler manifold carries a canonical hyperKahler metric, but this metric is not the exact metric of the moduli space of vacua $\mathcal{P}_R$ of the $\mathcal{N}=2$ theory reduced on S^1 of finite radius R.

Even the exact metric depends non-trivially on the radius R of the S^1 compactification, one of the hyperKahler complex structures, customarily denoted by I, is independed of the radius R. As an I-complex holomorphic symplectic manifold the space $\mathcal{P}_R$ does not depend on R. This can be shown by the analysis of the supersymmetry: I-holomorphic quantities are protected from R-corrections.

3.3. **Spectral curves.** In practice, often Seiberg-Witten integrable system is described not by a family of abelian varieties fibered over the base $\mathcal{U}$, but by a familty of curves Σ fibered over the same base $\mathcal{U}$ with the idea that the abelian variety is the Jacobian of spectral curve (or more generally Prym subspace of Jacobian for some correspondence symmetry on the curve)

$$A_u = Jac(\Sigma_u) \tag{3.29}$$

Let S be a holomorphic symplectic surface and (Σ, L) be a pair: a holomorphic curve $\Sigma_u\subset S$ of a fixed homology class and a line bundle on L. The moduli space of such pairs (Σ, L) is equivalent to the moduli space of rank 1 sheaves on S of a fixed cohomology class determined by the homology class of Σ. It is clear that the holomorhic symplectic structure on S induces the holomorphic symplectic structure on the moduli space of sheaves on S. Thus the moduli space $\mathcal{P}=\{(\Sigma,L)\}$ for $\Sigma\subset S$ for a holomorphic surface S carries holomorphic symplectic structure as well as the projection $\mathcal{P}\to\mathcal{U}$ to the moduli space $\mathcal{U}$ of holomorphic curves $\Sigma\subset S$ of a fixed homology class. The fiber of the projection map $\mathcal{P}\to\mathcal{U}$ is the moduli space of line bundles on Σ, that is $A_u=Jac(\Sigma_u)$.

3.4. **Some classes of algebraic integrable systems.**

3.4.1. *Hitchin system.* One well-known class of algebraic integrable system is associated to the symplectic surface $S=K_C$ which is the total space of the canonical bundle K (holomorphic cotangent bundle) to a holomorphic curve C, possibly with marked points. On C we consider a moduli space of Higgs bundles. Let $G^{\mathbb{C}}$ be a complex (algebraic) reductive group. A Higgs bundle is a pair

$$\text{Higgs bundle} = (\text{stable principal } G^{\mathbb{C}}\text{-bundle on } C, \text{holomorphic section } \phi\in\Gamma(K\otimes \mathrm{ad}_{G^{\mathbb{C}}})) \tag{3.30}$$

The moduli space

$$\mathfrak{M}^{Hit}_{G^{\mathbb{C}}}(C) = \{G^{\mathbb{C}}\text{-Higgs bundles on C}\} \tag{3.31}$$

is called Hitchin moduli space. Let $\mathrm{Bun}_{G^{\mathbb{C}}}(C)$ denote the moduli space of principal $G^{\mathbb{C}}$-holomorphic bundles on C. It is clear that in a generic point

$$\mathfrak{M}^{Hit}_{G^{\mathbb{C}}}(C) \simeq T^*\mathrm{Bun}_{G^{\mathbb{C}}}(C) \tag{3.32}$$

and hence $\mathcal{P} = \mathfrak{M}^{Hit}_{G^{\mathbb{C}}}(C)$ is a holomorphic symplectic variety. Let the collection of $\{P_{d_k}\}$ be the polynomial generators (of degree d_k) of the ring of adjoint invariant polynomials on $\mathfrak{g}_{\mathbb{C}}$. Then the Higgs field ϕ induces a holomorphic section $P_{d_k}(\phi) \in \Gamma(C, K^{\otimes d_k})$. This defines projection $\mathcal{P} \to \mathcal{U}$:

$$\mathfrak{M}^{Hit}_{G^{\mathbb{C}}}(C) \to \oplus_k H^0(C, K^{\otimes d_k}) \tag{3.33}$$ {eq:Hint}

for $\mathcal{U} = \oplus_k H^0(C, K^{\otimes d_k})$. Hitchin has shown that $\dim_{\mathbb{C}} \mathcal{U} = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{P}$ and that the fibers of the projection are holomorphic symplectic Lagrangian varieties. Hence (3.33) is an algebraic integrable system. Let R be representation of $G_{\mathbb{C}}$. The curve $\Sigma \subset T^*C$ defined by the spectral equation

$$\Sigma : \det{}_R(\phi - t) = 0 \tag{3.34}$$

where t is a coordinate in the cotangent fiber of T^*C is called *spectral curve* for a Hitchin system. For $G_{\mathbb{C}} = GL(n)$ and fundamental representation R the fibration $\mathcal{P} \to \mathcal{U}$ is precisely of the type reviewed in section 3.3. The base $\mathcal{U}$ is the family of n-fold spectral covers $\Sigma \to C$ in the total space $S = T^*C$, and the fiber A_u at a given point $u \in \mathcal{U}$ is the moduli space of line bundles on Σ_u, that is $Jac(\Sigma_u)$. For a general G and R one needs to deal with correspondence on Σ and define Prym subvariety of $Jac(\Sigma_u)$, see [Donagi, Spectral Covers].

3.4.2. *Group like version of Hitchin system.* [Hurtubise, Markman 2002]

What happens if we try to compactify $\mathbb{C}$-fibers of T^*C to cylinders $\mathbb{C}^{\times}$? Namely, let C be a genus 0 or genus 1 curve with flat structure dz, which means that C is either of $\mathbb{C}, \mathbb{C}^{\times}, \mathcal{E}$ (where $\mathcal{E}$ is elliptic curve) with a fixed holomorphic differential dx.

We consider the surface $S = C \times \mathbb{C}^{\times}$. The vertical curve $\mathbb{C}^{\times}$ is equipped with $\mathbb{C}^{\times}$ invariant holomorphic differential $\frac{dy}{y}$. The surface S is holomorphic symplectic with symplectic structure

$$\Omega = dx \wedge \frac{dy}{y} \tag{3.35}$$

The compactification of S in the limiting points 0 and ∞ of $\mathbb{C}^{\times}$ has singularities for Ω along the curves C_0 and C_{∞} covering C. Define the holomorphic symplectic phase space $\mathcal{P}$

$$\mathcal{P} = \{G_{\mathbb{C}} - \text{principal bundles on } S \text{ with fixed trivialization at } C_0 \text{ and } C_1\} \tag{3.36}$$

and define the base

$$\mathcal{U} = \mathrm{Maps}(C_0, \mathrm{Bun}_{G_{\mathbb{C}}}(C^{\times})) \tag{3.37}$$

It is clear we have projection

$$\mathcal{P} \to \mathcal{U} \tag{3.38}$$

Moreover, it turns out that the fibers of this projection are holomorphic Lagrangian varieties. The space $\mathcal{P}$ can be described as a group-valued Higgs bundle

$$\mathcal{P} = \{G_{\mathbb{C}} - \text{holomorphic bundle on } C, \text{ holomorphic section } \phi \in \Gamma(C, Ad_{G_{\mathbb{C}}})\} \tag{3.39}$$

with certain asymptotic conditions at the infinite points of the base curve C.

The base $\mathcal{U}$ can characterized similar to Hitchin system. Let χ_i be the generators of the ring of adjoint invariant functions on the group $G_{\mathbb{C}}$. For example, for a simple $G_{\mathbb{C}}$ we take χ_i to be the characters of the highest weight fundamental $G_{\mathbb{C}}$ modules with fundamental highest weight ω_i. Then $\chi_i(\phi)$ is a holomorphic function on C with a certain asymptotic growth conditions at infinity points of C, let the space of such holomorphic functions be denotes by $\mathcal{O}_{d_i}(C)$. Later we will characterize the degrees d_i

$$\mathcal{U} = \oplus_i \mathcal{O}_{d_i}(C) \tag{3.40}$$

The vertical curve $C^{\times}$ can be contrated to S^1. Therefore, the group Hitchin system is equivalent to the integrable system on the moduli space of monopoles on $C \times S^1$.

The group valued Higgs field ϕ is the monodromy of the complexified connection $A + i\Phi$ where (A, Φ) are solving the BPS monopole equations $F_A = \star d_A \Phi$ on $C \times S^1$.

The definition of group Hitchin system can be extended for the ramified case when group valued Higgs field ϕ develops poles. In the language of monopoles one chooses a point in $C \times S^1$ and a coweight $\lambda^\vee : U(1) \to G$ and request that the monopole configuration develops the standard Dirac type $U(1)$ monopole singularity with $\frac{1}{2\pi}\int_{S^2} F = 1$ (where S^2 surronds the point supporting singular monopole) that embedds into G-monopole configuration by the coweight $\lambda^\vee$.

For such monopole system with singularities the group Higgs field $\phi(x)$ has poles at the points on C obtained by the projection of the points of monopole singularities on $C \times S^1$, and for a given $\lambda^\vee$-Dirac type singularity the pole of $\phi(x)$ is characterized by $\lambda^\vee$.

A local characterization of $\lambda^\vee$ singularity contribution to $\mathcal{P}$ is $\lambda^\vee$ orbit in affine $G_{\mathbb{C}}$ Grassmanian which has complex dimension $2(\rho, \lambda^\vee)$ where ρ is the Weyl vector in the weight space of $\mathfrak{g}_{\mathbb{C}}$.

For rigid boundary conditions at infinity of C we find

$$\dim \mathcal{P} = \sum_i 2(\rho, \lambda_i^\vee) \tag{3.41}$$

where the sum is over singular points i and $\lambda_i^\vee$ denotes the type of $U(1)$ Dirac singularity at given point. The above formula is a well-known dimension [Pauly] of the moduli space of monopoles with singularities.

3.4.3. *ADE quivers and monopoles.* The integrable system of $G_\Gamma = ADE$ quiver gauge theories in 4d, 5d on S^1 or 6d on $\mathcal{E}^\vee$ is the $G_\Gamma^{\mathbb{C}}$-group Hitchin system on $C = \mathbb{C}, \mathbb{C}^\times, \mathcal{E}$ respectively. Each fundamental multiplet attached to i-th node of the quiver Γ with mass m inserts Dirac singularity at point $m \in C$ and of type $\lambda_i^\vee$.

3.4.4. *The elliptic group Hitchin system.* The elliptic group Hitchin system is defined in a similar way, but now the vertical curve $\mathbb{C}^\times$ is replace by a non-degenerate elliptic curve $\mathcal{E}_v$. The phase space $\mathcal{P}$ is the moduli space of $G_{\mathbb{C}}$-bundles on $C \times \mathcal{E}_v$, and the base $\mathcal{U}$ is the space of holomorphic maps from C to $\mathrm{Bun}_{G_{\mathbb{C}}}(C)$. For elliptic group Hitchin system we do not allow singularities.

The $\mathcal{N} = 2$ gauge theory is affine ADE quiver theory which also does not have room for fundamental matter that has been associated with Dirac monopoles in the case of finite ADE quivers.

3.4.5. *Affine ADE quivers and instantons on elliptic fibration.* The integrable system of $G_\Gamma = \hat{A}\hat{D}\hat{E}$ quiver gauge theories in 4d, 5d on S^1 or 6d on $\mathcal{E}^\vee$ is the $G_\Gamma^{\mathbb{C}}$ elliptic group Hitchin system on $C = \mathbb{C}, \mathbb{C}^\times, \mathcal{E}$ respectively. The vertical elliptic curve $\mathcal{E}$ has elliptic modulus defined by $\times_{i \in I_\Gamma} \mathfrak{q}_i^{a_i^\vee}$ where $\mathfrak{q}_i$ is exponentiated coupling constant for $SU(N_i)$ gauge group of $\mathcal{N} = 2$ QFT.

4. Lecture 4. AGT and NS quantizations. 28.10.2014

[IN PROGRESS...]

5. Literature references

[IN PROGRESS...]