Rigorous derivation of a model for two-phase bubbly flows with surface tension

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Rigorous derivation

- Average model from a microscopic description
- Full model: PDE's and source terms
- Mathematical theory (Navier–Stokes type equations, homogeneisation)

Physical setting

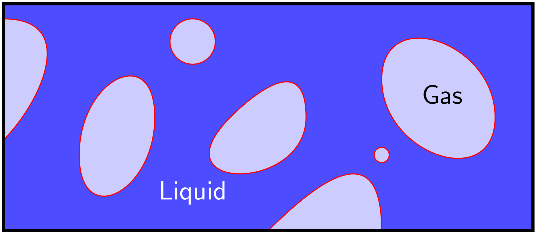
- Compressible bubbles in a surrounding compressible liquid
- Viscous flows (smooth enough solutions)
- Surface tension

To be compared with

- [Stewart, Wendroff, 84], [Zhang, Prosperetti, 94]...
- [Drew, Passman, 98], [Ishii, Hibiki, 06]...
- [Gavrilyuk, Saurel, 02], [Drui, 17], [Cordesse, 20]...
- [Serre, 91 & 01], [E, 92], [Hillairet, 07], [Bresch, Huang, 11], [Bresch, Hillairet, 15 & 19], [Hillairet, 18], [Bresch, Burtea, Lagoutière, 20]...

Averaging process for two-phase flows

Goal. Construction of an average model from a *microscopic* description



Microscopic description [Drew, Passman, 98]...

- Instantaneous local conservation laws for each separated phase
- Jump conditions through the interfaces

Averaging process [Drew, Passman, 98]...

- Introduce small (time and/or volume) scales, or random disturbances
- Average the microscopic model wrt the small scales

Averaging process for two-phase flows

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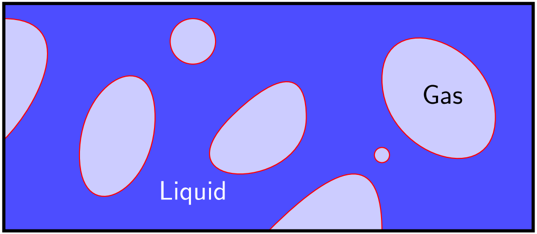

Homogenized Cauchy problem [Hillairet, Mathis, S., 21]...

- Start with some macroscopic initial data
- Deduce a family of microscopic initial data, indexed by $N \in \mathbb{N}$
- Solve the microscopic model
- Pass to the limit $N \to \infty$ to deduce macroscopic variables
- Find the macroscopic model associated with these variables

The microscopic model

Compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho_i + \mathrm{div}_x(\rho_i u_i) = 0 \\ \partial_t(\rho_i u_i) + \mathrm{div}_x(\rho_i u_i \otimes u_i) = \mathrm{div}_x \Sigma_i \\ \text{with } \Sigma_i = 2\mu_i \big(D(u_i) - \frac{1}{3}\mathrm{div}_x u_i \mathbb{I}_3 \big) + \lambda_i \mathrm{div}_x u_i - \mathrm{p}_i(\rho_i)\mathbb{I}_3 \end{cases}$$

where $i = f$ (the fluid) or $i = k$ (the k-th inclusion B_k), with $k = 1, \ldots, N$

The microscopic model

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Jump conditions

On each bubble boundary ∂B_k:

- Continuity of the velocity field $\quad u_f = u_k$
- Surface tension $\quad (\Sigma_f - \Sigma_k)n = \kappa_k n$

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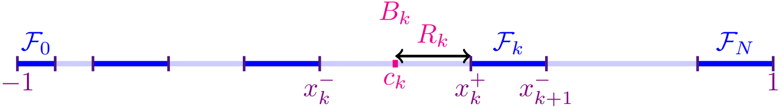
Jump conditions
On each bubble boundary ∂B_k:

- Continuity of the velocity field $u_f = u_k$
- Surface tension $(\Sigma_f - \Sigma_k)n = \kappa_k n$

Geometrical constraint
For all $k = 1, \ldots, N$, $D(u_k) - \frac{1}{3}\mathrm{div}_x u_k \mathbb{I}_3 = 0$ (ie $\mu_k \to +\infty$)
$\Longrightarrow$ the bubbles B_k remain spherical (translation, rotation, dilatation)

The one-dimensional microscopic model



On the fluid domain $\mathcal{F}(t)$:

$$\begin{cases} \partial_t \rho_f + \partial_x(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \partial_x(\rho_i (u_f)^2) = \partial_x \Sigma_i \\ \text{with } \Sigma_i = \mu_f \partial_x u_f - \mathrm{p}_f(\rho_f) \end{cases}$$

In each bubble $B_k(t) = B(c_k(t), R_k(t)) = (x_k^-, x_k^+)$ of (constant) mass m_k:

$$\begin{cases} \rho_k(t) = \dfrac{m_k}{2R_k(t)} \\ u_k(t,x) = \dot{c}_k + \dfrac{\dot{R}_k}{R_k}(x - c_k) \end{cases} \qquad \begin{cases} m_k \ddot{c}_k = \Sigma_f(t, x_k^+) - \Sigma_f(t, x_k^-) \\ \dfrac{m_k}{3} \ddot{R}_k = \Sigma_f(t, x_k^-) + \Sigma_f(t, x_k^+) - 2\Sigma_k + \dfrac{\gamma_s}{R_k} \\ \text{with } \Sigma_k = \mu_g \partial_x u_k - \mathrm{p}_g\left(\dfrac{m_k}{2R_k}\right) \end{cases}$$

Macroscopic to microscopic initial data

At the macroscopic scale, both fluids are present everywhere in the domain Ω

Macroscopic initial data

- Density of the fluid $\bar{\rho}_f^0 \in H^1(\Omega)$
- Density of the gas $\bar{\rho}_g^0 \in H^1(\Omega)$
- Mean velocity $\bar{u}^0 \in H^1(\Omega)$
- Probability distribution of the bubbles, in position x and radius r, $\bar{S}_g^0 = \bar{S}_g^0(x,r) \in L^1(\Omega \times \mathbb{R}^+)$

Moments of the probability distribution $\bar{S}_g^0$:

- Volume?

$$\bar{f}_g^0(x) = \int_{\mathbb{R}^+} \bar{S}_g^0(x,r)\mathrm{d}r$$

- Void fraction

$$\bar{\alpha}_g^0(x) = \int_{\mathbb{R}^+} (2r)\bar{S}_g^0(x,r)\mathrm{d}r$$

Macroscopic to microscopic initial data

Family of microscopic initial data to be constructed from $\bar{S}_g^0$, $\bar{\rho}_{f,g}^0$ and $\bar{u}^0$

For any bubble number $N \geqslant 1$:

1. Define a bubble distribution from $\bar{S}_g^0$: $(c_k^{(N)}, R_k^{(N)})_{k=1,\ldots,N}$
2. Define the densities

$$\begin{cases} \rho_f^{(N)}(0,x) = \bar{\rho}_f^0(x) & \text{on } \mathcal{F}^{(N)}(0) \\ \rho_k^{(N)}(0,x) = m_k^{(N)}/(2R_k^{(N)}(0)) & \text{on } B_k^{(N)}(0) \\ \text{with } m_k^{(N)} = \int_{B_k^{(N)}(0)} \bar{\rho}_g^0(x)\,\mathrm{d}x \end{cases}$$

3. Define the velocities (recall $x_k^{\pm} = c_k^{(N)} \pm R_k^{(N)}$)

$$\begin{cases} u_f^{(N)}(0,x) = \bar{u}^0(x) & \text{on } \mathcal{F}^{(N)}(0) \\ u_k^{(N)}(0,x) = \dfrac{\bar{u}^0(x_k^+) - \bar{u}^0(x_k^-)}{x_k^+ - x_k^-}(x - x_k^-) + \bar{u}^0(x_k^-) & \text{on } B_k^{(N)}(0) \end{cases}$$

The macroscopic model

Well-posedness of the Cauchy problem for the microscopic initial data

Microscopic Cauchy problem [Hillairet, Mathis, S., 21]

There exists a time $T > 0$, only depending on the macroscopic initial data, such that

$$\left((c_k^{(N)}, R_k^{(N)})_{k=1,\dots,N}, \rho_f^{(N)}, u_f^{(N)}, (\rho_k^{(N)}, u_k^{(N)})_{k=1,\dots,N}\right)$$

exists and is unique...

- Scaling $(m_k)_{k\in\{1,\dots,N\}} \sim N^{-1}$, $(R_k)_{k\in\{1,\dots,N\}} \sim N^{-1}$, $\gamma_s \sim N^{-1}$
- Additional restrictions on the macroscopic initial data
  $\longrightarrow$ (small) time of existence independent of N
- Extension of microscopic phase variables to Ω
- Convergence of these extended variables when $N \to +\infty$
  $\longrightarrow$ macroscopic variables $\bar{\rho}_{f,g}$, $\bar{u}$ and $\bar{S}_g$
- Convergence inside the microscopic system of PDE's
  $\longrightarrow$ macroscopic model

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The macroscopic model

The macroscopic closed system:

$$
\begin{cases}
\bar\alpha_f + \bar\alpha_g = 1, \quad \bar\rho = \bar\alpha_f\bar\rho_f + \bar\alpha_g\bar\rho_g \\
\partial_t\bar\alpha_f + \bar u\partial_x\bar\alpha_f = \dfrac{\bar\alpha_g\bar\alpha_f}{\bar\alpha_f\mu_g + \bar\alpha_g\mu_f}\left[(\mu_g-\mu_f)\partial_x\bar u + (\mathrm{p}_f(\bar\rho_f) - \mathrm{p}_g(\bar\rho_g)) - \bar\gamma_s\dfrac{\bar f_g}{\bar\alpha_g}\right] \\
\partial_t\bar f_g + \partial_x(\bar f_g\bar u) = 0 \\
\partial_t(\bar\alpha_f\bar\rho_f) + \partial_x(\bar\alpha_f\bar\rho_f\bar u) = 0 \\
\partial_t(\bar\alpha_g\bar\rho_g) + \partial_x(\bar\alpha_g\bar\rho_g\bar u) = 0 \\
\partial_t(\bar\rho\bar u) + \partial_t(\bar\rho^2\bar u) = \partial_x\bar\Sigma \\
\text{with } \bar\Sigma = \frac{\mu_g\mu_f}{\bar\alpha_f\mu_g+\bar\alpha_g\mu_f}\left[\partial_x\bar u - \left(\frac{\bar\alpha_f}{\mu_f}\mathrm{p}_f(\bar\rho_f) + \frac{\bar\alpha_g}{\mu_g}\mathrm{p}_g(\bar\rho_g)\right) - \frac{\bar\gamma_s}{\mu_g}\bar f_g\right]
\end{cases}
$$

Additional equation for the probability distribution

$$
\partial_t\bar S_g + \partial_x(\bar S_g\bar u) + \frac{1}{\mu_g}\partial_r((r(\bar\Sigma_g + \mathrm{p}_g(\bar\rho_g)) + \bar\gamma_s/2)\bar S_g) = 0
$$

To be continued...

- Two-pressure one-velocity two-phase flow model
  - Both phases are compressible and viscous
  - "Extension" of the Bresch–Hillairet models
- Bubbly flows
  - Additional description via
    - the new variable $\bar{f}_g$ ($\sim$ interfacial area? 3D?)
    - the probability distribution $\bar{S}_g$ wrt (x, r)
  - New "pressure" term due to the surface tension
  - Constant "number" of bubbles
- Comparison with other models of bubbly flows