How to solve hyperbolic equations regularised by inertia - type dispersive terms

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What does ‘inertia-type dispersive terms’ mean?
Think champagne or any ‘bubbly fluids’!

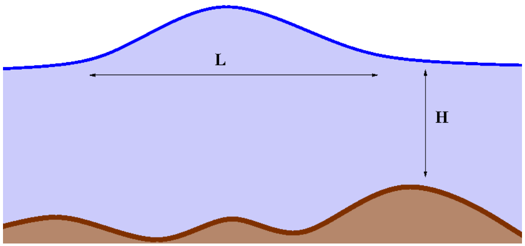
ATTENTION : this is not advertising for soda companies!

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \ \frac{\partial}{\partial t}(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p \mathbf{I}) = 0,$$

$$p = \rho \frac{\delta W(\rho, \dot{\rho})}{\delta \rho} - W(\rho, \dot{\rho}) = f(\rho, \dot{\rho}, \ddot{\rho}),$$

$$\dot{\rho} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \rho, \quad \ddot{\rho} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)^2 \rho.$$

Better to think surface gravity waves ... because mathematically it is the same!
And it will save me from being accused of favouritism towards soda companies.



$$\varepsilon = \frac{H}{L} \ll 1.$$

BBM equation

Hopf equation + 'inertia' term

$$u_t + \left(\frac{u^2}{2}\right)_x - \varepsilon^2 u_{txx} = 0.$$

Serre-Green-Naghdi equations

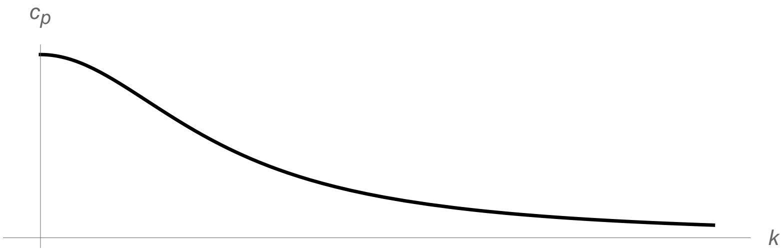
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU) = 0,$$

$$\frac{\partial}{\partial t}(hU) + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + \frac{\varepsilon^2}{3}h^2\ddot{h}\right) = 0,$$

$$\dot{h} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}, \quad \ddot{h} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\dot{h}.$$

Both models share common features

1. The phase and group velocity of linear waves are bounded for all wave numbers.



2. They admit a variational formulation.

Lagrangian ($\varepsilon = 1$)

BBM (P. Olver)

$$\mathcal{L} = -\frac{\varphi_t \varphi_x}{2} + \frac{\varphi_t \varphi_{xxx}}{2} - \frac{\varphi_x^3}{6}, \quad u = \varphi_x.$$

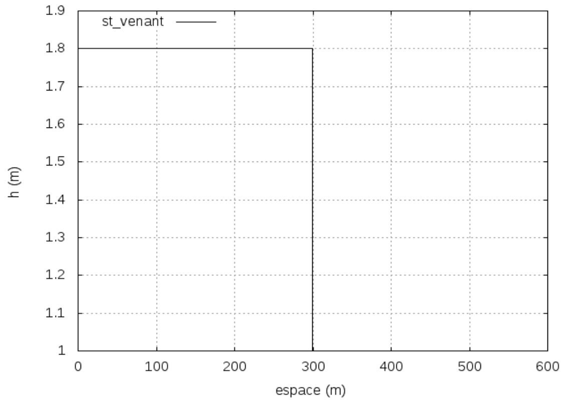
SGN (Salmon 1988, SG & Teshukov 2001)

$$\mathcal{L} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{h|\mathbf{U}|^2}{2} + \frac{h\dot{h}^2}{6} - \frac{g(h-h_\infty)^2}{2} \right) dx\, dy.$$

Constraint :

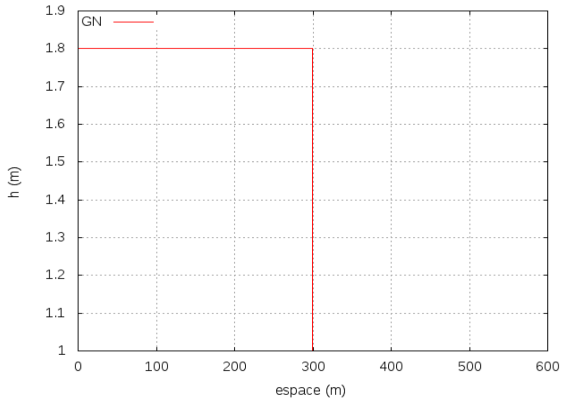
$$h_t + \mathrm{div}(h\mathbf{U}) = 0.$$

Dam-break problem, Saint-Venant equations



$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU) = 0, \ \frac{\partial}{\partial t}(hU) + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2}\right) = 0.$$

Serre-Green-Naghdi equations



$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU) = 0, \; \frac{\partial}{\partial t}(hU) + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + \frac{1}{3}h^2\ddot{h}\right) = 0.$$

What are the difficulties in solving these equations?

Serre-Green-Naghdi equations in mass Lagrangian coordinates :

$$q = \int_0^X h_0(s)ds,$$

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} - \tilde{e}(\tau, \tau_t) \right) dq,$$

$$u = x_t, \quad \frac{1}{h} = \tau = x_q, \quad \tilde{e}(\tau, \tau_t) = \frac{g}{2\tau} - \frac{h_t^2}{6}.$$

SGN equations :

$$\tau_t - u_q = 0, \;\; u_t + p_q = 0,$$

with

$$p = -\frac{\delta \tilde{e}}{\delta \tau} = -\left(\frac{\partial \tilde{e}}{\partial \tau} - \frac{\partial}{\partial t} \left(\frac{\partial \tilde{e}}{\partial \tau_t} \right) \right).$$

Inversion of an elliptic operator (O. Le Metayer, SG & S. Hank (2010))

System to solve :

$$\tau_t - u_q = 0,\ \ u_t - \left(\frac{\partial \tilde{e}}{\partial \tau} - \frac{\partial}{\partial t}\left(\frac{\partial \tilde{e}}{\partial \tau_t}\right)\right)_q = 0.$$

Or

$$\tau_t - u_q = 0,\ \ K_t - \left(\frac{\partial \tilde{e}}{\partial \tau}\right)_q = 0,$$

$$K = u + \left(\frac{\partial \tilde{e}}{\partial \tau_t}\right)_q = u - \frac{1}{3}\left(\frac{u_q}{\tau^4}\right)_q = \mathcal{A}u.$$

To find u we need to invert $\mathcal{A}$.

Shortcomings of the method

- Prohibitively expensive computations
- Difficulties in formulating “transparent” boundary conditions (C. Besse, M. Ehrhardt, P. Noble, M. Kazakova, ...)

Idea

Smooth initial data are needed for dispersive systems.
But, sometimes, one needs to solve the equations with discontinuous data : dam break problems or water hammer problem.

Hyperbolic equations are better then! Godunov type methods could be used!

Hyperbolic regularization, classical relaxation methods

The idea is not new : Cattaneo approach to solve the heat equation (relaxation method)

$$T_t = Q_x, \quad Q_t = \frac{T_x - Q}{\tau_r}, \quad 0 < \tau_r << 1$$

Further development of relaxation methods : V. Yu. Liapidevskii (1998), M. Antuono, V. Yu. Liapidevskii and M. Broccini (2008), A. A. Chesnokov and V. Yu. Liapidevskii (2020) for dispersive shallow water equations; I. Peshkov, E. Romenskii and M. Dumbser for Navier-Stokes equations, ...

New idea for dispersive Euler-Lagrange equations : to modify the 'master' Lagrangian.

Penalty method (trivial example)

'Master' Lagrangian and E-L equations :

$$L = \left(\frac{dx}{dt}\right)^2 - \frac{x^2}{2}, \quad \frac{d^2x}{dt^2} + x = 0.$$

Augmented Lagrangian :

$$\hat{L} = \left(\frac{dy}{dt}\right)^2 - \frac{x^2}{2} - \lambda\frac{(y-x)^2}{2}.$$

Euler-Lagrange equations :

$$\frac{d^2y}{dt^2} + \lambda(y-x) = 0, \quad -x + \lambda(y-x) = 0.$$

One has :

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \quad x(t,\lambda) = \omega^2 y(t,\lambda),\ \omega^2 = \frac{\lambda}{1+\lambda}.$$

Solution of the Cauchy problem : $y(0) = A,\ \dot{y}(0) = B$:

$$y(t,\lambda) = A cos\left(\omega t\right) + \frac{B}{\omega} sin\left(\omega t\right).$$

Augmented Lagrangian

- At least a one-parameter family of 'augmented' Lagrangians should be choosen. When the parameter goes to infnity, we find out our 'master' lagrangian.
- The Euler-Lagrange equations for the 'augmented' Lagrangian should be unconditionally hyperbolic.
- In the linear approximation, the Whitham type condition should be satisfed : the phase velocities of waves corresponding to the 'master' lagrangian should be interplaced between the phase velocities corresponding to the 'augmented' Lagrangian for any wave numbers.

Augmented Lagrangian for the BBM equation : SG and K. M. Shyue, 2021

$$\hat{\mathcal{L}} = -\frac{\varphi_t \varphi_x}{2} - \frac{\varphi_x^3}{6} - \frac{\psi_t \psi_x}{2c} - \varphi_x \psi_x + \psi_x \chi - \frac{\chi_t \chi_x}{2} - \frac{\chi_x^2}{2c}.$$

Here $c > 0$ is a large positive constant.

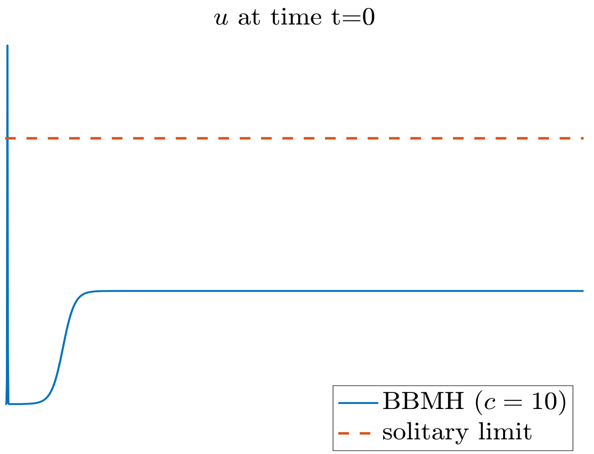
Hyperbolic BBM (BBMH) system

The Euler-Lagrange equations for $u = \varphi_x$, $v = \psi_x$ and $w = \chi_x$:

$$u_t + uu_x + v_x = 0, \quad \frac{v_t}{c} + u_x = w, \quad w_t + \frac{w_x}{c} = -v.$$

1. Can be written in terms of Riemann invariants.
2. Two eigenfields are genuinely non-linear and the third one is linearly degenerate.
3. Admits 3 conservation laws (exactly as in the case of the BBM equations (P. Olver, 1979)).

BBMH : Interaction of a solitary wave with a step



Application to the SGN equations : ‘master’ lagrangian

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} + \frac{h_t^2}{6} - \frac{g}{2\tau} \right) dq,$$

$$u = x_t, \quad \frac{1}{h} = \tau = x_q.$$

Augmented Lagrangian (N. Favrie, SG, Nonlinearity, 2017)

$$\hat{\mathcal{L}} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} + \frac{\eta_t^2}{6} - \frac{g}{2\tau} - \lambda \frac{(\eta\tau - 1)^2}{6} \right) \, dq$$

Governing equations (two types of virtual displacements) :

$$\begin{cases} \tau_t - u_q = 0, \\ u_t - \left(\dfrac{g}{\tau^3} + \dfrac{\lambda}{3}\eta^2 \right) \tau_q - \dfrac{\lambda}{3}(2\tau\eta - 1)\eta_q = 0, \\ \eta_{tt} = -\lambda \left(\eta\tau - 1 \right) \tau. \end{cases}$$

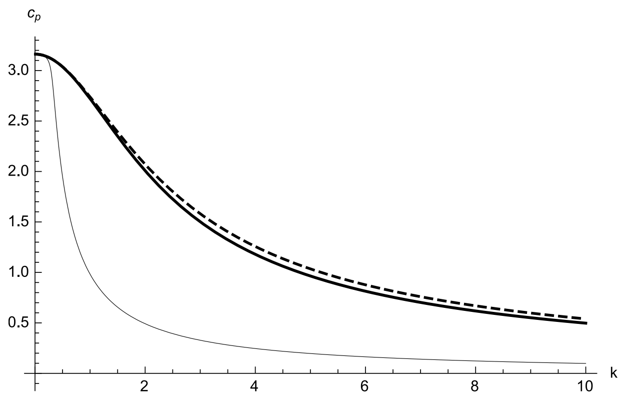
Characteristics :

$$\xi_{1,2} = 0 \text{ (lin. degenerate)}, \quad \xi_{3,4} = \pm\sqrt{\frac{g}{\tau^3} + \frac{\lambda}{3}\eta^2} \text{ (gen. nonlinear)}$$

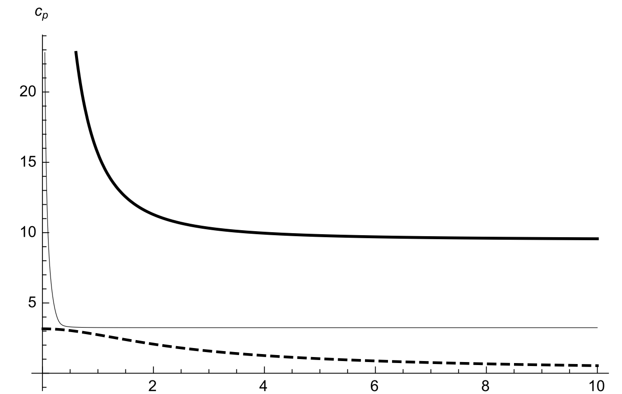
Dispersive properties

Whitham condition :

$$c_p^-(k) < c_p(k) < c_p^+(k), \quad c_p(k) = \sqrt{\frac{g}{\tau_0^3 + \frac{k^2}{3\tau_0}}}.$$



$(c_p^-)^2$



$(c_p^+)^2$

$\tau_0 = 1\ m^{-1}$, $\lambda = 1\ m^2\ s^{-2}$, $\lambda = 160\ m^2\ s^{-2}$

Rigourous justification of the method of augmented Lagrangian for the SGN equations

V. Duchene, Nonlinearity **32** (10), 2019.

Numerical results 2D (Tkachenko S.)

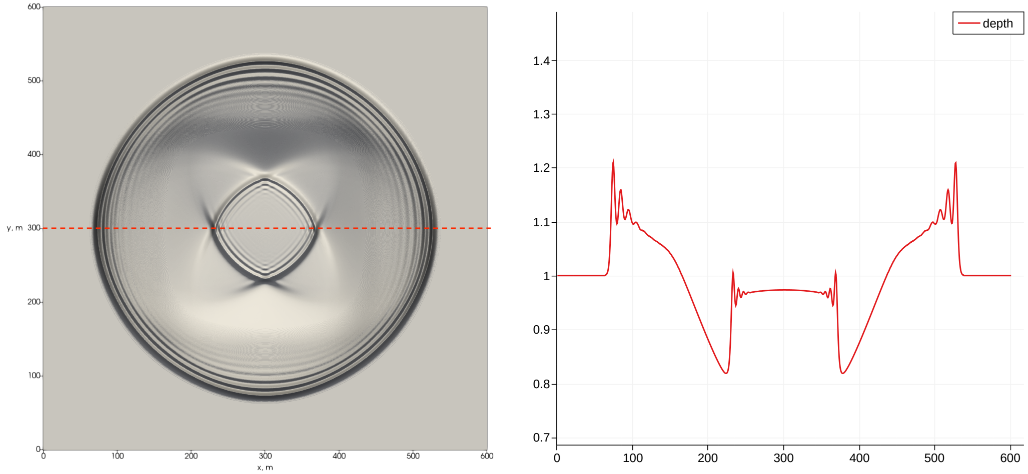
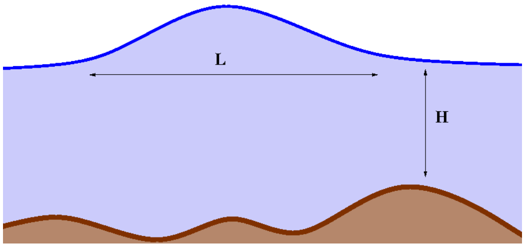


Figure 1: Numerical solution to 2D Riemann problem for the extended SGN model, $t = 20s$.

S. Busto, C. Escalante, M. Dumbser, N. Favrie and SG (JSC 2021)



$$\mathcal{L} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} L \, dx_1 dx_2,$$

$$L(\overline{\mathbf{v}}, h, \dot{h}, b, \dot{b}) = h \left(\frac{|\overline{\mathbf{v}}|^2}{2} + \frac{1}{6} \left(\dot{h} + \frac{3}{2} \dot{b} \right)^2 + \frac{1}{8} \dot{b}^2 \right) - \frac{gh}{2}(h+2b) - Ch.$$

The Euler-Lagrange equations are obtained under the incompressibility (mass conservation) constraint.

The ‘augmented’ Lagrangian is :

$$\hat{\mathcal{L}} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{L} dx_1 dx_2,$$

with

$$\hat{L}(\overline{\mathbf{v}}, h, \eta, \dot{\eta}, b, \dot{b})$$

$$= h\left(\frac{|\overline{\mathbf{v}}|^2}{2} + \frac{1}{6}\left(\dot{\eta} + \frac{3}{2}\dot{b}\right)^2 + \frac{1}{8}\dot{b}^2\right) - \frac{gh}{2}(h+2b) - \frac{\lambda h}{6}\left(\frac{\eta}{h} - 1\right)^2 - Ch.$$

Mild bottom approximation

$$\hat{L}(\overline{\mathbf{v}}, h, \eta, \dot{\eta}, b, \dot{b})$$

$$\approx h\left(\frac{|\overline{\mathbf{v}}|^2}{2} + \frac{1}{6}\left(\dot{\eta} + \frac{3}{2}\dot{b}\right)^2 \cancel{+\frac{1}{8}\dot{b}^2}\right) - \frac{gh}{2}(h+2b) - \frac{\lambda h}{6}\left(\frac{\eta}{h} - 1\right)^2 - Ch.$$

Solitary wave interaction with an island

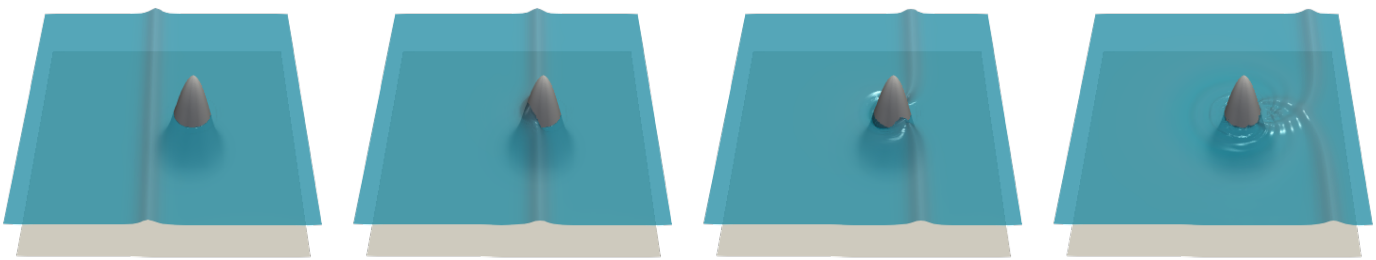


Figure 2: Interaction of a solitary wave with an island (S. Busto, C. Escalante, M. Dumbser, N. Favrie and SG (JSC 2021)).

Conclusion :

- An augmented Lagrangian method for dispersive shallow water models is proposed.
- The corresponding Euler-Lagrange equations are hyperbolic and approximate with a good precision the ‘master’ system.
- Allows to work with discontinuous initial data.