Stochastic derivation of Baer-and-Nunziato models: homogenization of two-phase hyperbolic terms and discussions on other cases

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## Main objectives

- Baer-and-Nunziato model
  - Two-phase flows: phases k and  $\bar{k}$
  - Mass conservation for each phase, out of equilibrium model

$$\begin{aligned} \partial_t \alpha_k + \mathbf{u}_I \cdot \nabla_{\mathbf{x}} \alpha_k &= \mu (P_k - P_{\bar{k}}) \\ \partial_t (\alpha_k \rho_k) + \operatorname{div}_{\mathbf{x}} (\alpha_k \rho_k \mathbf{u}_k) &= 0 \\ \partial_t (\alpha_k \rho_k \mathbf{u}_k) + \operatorname{div}_{\mathbf{x}} (\alpha_k (\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k)) &= P_I \nabla_{\mathbf{x}} \alpha_k + \lambda (\mathbf{u}_{\bar{k}} - \mathbf{u}_k) \\ \partial_t (\alpha_k \rho_k E_k) + \operatorname{div}_{\mathbf{x}} (\alpha_k (\rho_k E_k + P_k) \mathbf{u}_k) &= P_I \mathbf{u}_I \cdot \nabla_{\mathbf{x}} \alpha_k \\ &- \mu P_I (P_k - P_{\bar{k}}) \\ &+ \lambda \mathbf{u}_I \cdot (\mathbf{u}_{\bar{k}} - \mathbf{u}_k), \end{aligned}$$

- References
  - M.R. Baer & J.W. Nunziato (1986).
  - Richard Saurel & Rémi Abgrall (1999).
- Interfacial terms, relaxation coefficients.
- Objectives
  - How to compute  $\mathbf{u}_{l}$ ,  $P_{l}$ ,  $\lambda$ ,  $\mu$ ?
  - Are some mathematical properties ensured?

### Drew & Passman (2006) method

- Drew & Passman (2006), Theory of multicomponent fluids
  - Derive an exact model for one known flow topology
  - Average the variables with respect to the flow topology

### New idea

- Provide an explicit model for the flow topology
- Use a stochastic model
- Outline of the talk
  - Operivation of a two phase model when the topology is explicit
  - Ostochastic modeling of two-phase flows
  - Baer-and-Nunziato models obtained and their properties
  - Possible extensions

## Derivation of two-phase model

# Derivation of a two-phase model (1/3)

- Two phases: k and  $\bar{k}$ .
- $\chi_k$  is the indicator function of the phase k.
  - $\chi_k = 1$  in the phase k
  - $\chi_k + \chi_{\bar{k}} = 1$  everywhere
- level set function f

$$\chi_k(\mathbf{x}) = \mathbf{1}_{\{f(\mathbf{x}) \ge 0\}}.$$

• Inside each of the fluid, the Euler system holds

 $\partial_t \mathbf{U}_k + \operatorname{div}_{\mathbf{x}} \mathbf{F}_k(\mathbf{U}_k) = 0$ 

• The indicator function follows

$$\partial_t \chi_k + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} \chi_k = \mathbf{0}$$



• Advection on the interface

$$\partial_t \chi_k + \mathbf{v}_i \cdot \nabla_{\mathbf{x}} \chi_k = \mathbf{0}$$

• The following equation holds everywhere

$$\chi_k\left(\partial_t \mathbf{U}_k + \operatorname{div}_{\mathbf{x}} \mathbf{F}_k(\mathbf{U}_k)\right) = 0$$

• and can be rephrased as

$$\partial_t \left( \chi_k \hat{\mathbf{U}}_k 
ight) + \operatorname{div}_{\mathbf{x}} \left( \chi_k \hat{\mathbf{F}}_k (\hat{\mathbf{U}}_k) 
ight) = \left( \hat{\mathbf{F}}_k (\hat{\mathbf{U}}_k) - \mathbf{v}_i \hat{\mathbf{U}}_k 
ight) \nabla_{\mathbf{x}} \chi_k$$

• with 
$$\hat{\mathbf{F}} = (0, \mathbf{F})$$
 and  $\hat{\mathbf{U}} = (1, \mathbf{U})$ .

## Derivation of a two-phase model (3/3)

• Suppose that  $f(\mathbf{x}) = 0$  and  $\nabla_{\mathbf{x}} f(\mathbf{x}) \neq 0$ 



• Solve the Riemann problem  $[\mathbf{U}_{\bar{k}}, \mathbf{U}_k]$  in the direction **n** 

$$\begin{pmatrix} \hat{\mathbf{F}}_{k}(\hat{\mathbf{U}}_{k}) - \mathbf{v}_{i}\hat{\mathbf{U}}_{k} \end{pmatrix} \nabla_{\mathbf{x}}\chi_{k} = \hat{\mathbf{F}}_{\bar{k}k}^{lag} \left( \frac{\nabla_{\mathbf{x}}f(\mathbf{x})}{\|\nabla_{\mathbf{x}}f(\mathbf{x})\|} \right) \|\nabla_{\mathbf{x}}f(\mathbf{x})\|$$
  
• with  $\hat{\mathbf{F}}_{\bar{k}k}^{lag}(\mathbf{n}) = \begin{pmatrix} -u_{\bar{k}k}^{\star}(\mathbf{n}) \\ 0 \\ P_{\bar{k}k}^{\star}(\mathbf{n})\mathbf{n} \\ P_{\bar{k}k}^{\star}(\mathbf{n})u_{\bar{k}k}^{\star}(\mathbf{n}) \end{pmatrix}$ 

## Averaging

• Final system

$$\partial_t \left( \chi_k \hat{\mathbf{U}}_k \right) + \operatorname{div}_{\mathbf{x}} \left( \chi_k \hat{\mathbf{F}}_k (\hat{\mathbf{U}}_k) \right) = \hat{\mathbf{F}}_{\bar{k}k}^{lag} \left( \frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\nabla_{\mathbf{x}} f(\mathbf{x})\|} \right) \|\nabla_{\mathbf{x}} f(\mathbf{x})\|$$

• Take the average  $\mathbb{E}\left[
ight]$ 

$$\mathbb{E}\left[\partial_t\left(\chi_k\hat{\mathbf{U}}_k\right)\right] + \mathbb{E}\left[\operatorname{div}_{\mathbf{x}}\left(\chi_k\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k)\right)\right] = \mathbb{E}\left[\hat{\mathbf{F}}_{\bar{k}k}^{lag}\left(\frac{\nabla_{\mathbf{x}}f(\mathbf{x})}{\|\nabla_{\mathbf{x}}f(\mathbf{x})\|}\right)\|\nabla_{\mathbf{x}}f(\mathbf{x})\|\right]$$

### Conservative part

if  $\mathbf{U}_k$  does not depend on the averaging operator

$$\partial_t \left( \alpha_k \hat{\mathbf{U}}_k \right) + \operatorname{div}_{\mathbf{x}} \left( \alpha_k \hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) \right) = \mathbb{E} \left[ \hat{\mathbf{F}}_{\bar{k}k}^{lag} \left( \frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\nabla_{\mathbf{x}} f(\mathbf{x})\|} \right) \|\nabla_{\mathbf{x}} f(\mathbf{x})\| \right]$$

• Much more difficult for the nonconservative part

• Define an explicit model for two-phase flows.

## Stochastic model of two-phase flows



- Sign of a 2-dimensional Gaussian process
- Mean 0
- Exponential-square autocorrelation function.



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### Pointwise Gaussian

At each point,  $g_x$  follows a Gaussian law

$$g_{\mathbf{x}} \sim rac{1}{\sigma\sqrt{2\pi}} \,\mathrm{e}^{-rac{(u-m(\mathbf{x}))^2}{2\sigma^2}} \,\mathrm{d} u \qquad (\sigma=1)$$

### Two-points correlation

Any vector  $[g_x, g_y]$  is a Gaussian vector

$$\left[g_{\mathbf{x}},g_{\mathbf{y}}\right] \sim \frac{1}{2\pi \left|\det \boldsymbol{\Sigma}\right|^{1/2}} \mathrm{e}^{-\frac{1}{2}\left(\mathbf{v}-\mathbf{m}\right)^{T} \boldsymbol{\Sigma}^{-1}\left(\mathbf{v}-\mathbf{m}\right)^{T}} \mathrm{d}\mathbf{v}$$

 $R(\mathbf{x}, \mathbf{y})$  is the covariance of the points  $\mathbf{x}$  and  $\mathbf{y}$ , and

$$\mathbf{m} = \begin{pmatrix} m(\mathbf{x}) \\ m(\mathbf{y}) \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & R(\mathbf{x}, \mathbf{y}) \\ R(\mathbf{x}, \mathbf{y}) & 1 \end{pmatrix}$$

## Derivability of the model

- System involves  $abla_{\mathbf{x}}\chi_k$ , with  $\chi_k(\mathbf{x})=rac{1+\mathrm{sgn}(g_{\mathbf{x}})}{2}$ 
  - Derivability of gx?
  - Derivability of  $\nabla_{\mathbf{x}} \chi_k(\mathbf{x})$ ?

### Derivability of $g_x$

We suppose that  $g_x$  is Gaussian, and that its auto-correlation function R is  $\mathscr{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ . Then

- $\nabla_{\mathbf{x}} g_{\mathbf{x}}$  exists in the mean square sense, is Gaussian, and has  $\nabla_{\mathbf{x}} m_k(\mathbf{x})$  as mean and  $\partial_{\mathbf{x}\mathbf{y}}^2 R(\mathbf{x}, \mathbf{x})$  as variance
- The vector  $[g_x, \nabla_x g_x]$  is Gaussian, and has

$$\begin{pmatrix} 1 & \partial_{\mathbf{y}} R(\mathbf{x}, \mathbf{x})^T \\ \partial_{\mathbf{y}} R(\mathbf{x}, \mathbf{x}) & \partial_{\mathbf{xy}}^2 R(\mathbf{x}, \mathbf{x}) \end{pmatrix},$$

as variance.

### Derivability of $\chi_k$

 $\nabla_{\mathbf{x}}\chi$  is measurable if and only if  $\partial_{\mathbf{y}}R(\mathbf{x},\mathbf{x}) = 0$  and  $\partial_{\mathbf{xy}}^2R(\mathbf{x},\mathbf{x})$  is non-negative.

- Sketch of the proof
  - For a regular function f,

$$abla_{\mathbf{x}}\left(\frac{1+\operatorname{sgn}(f)}{2}\right)(\mathbf{x}) = \delta_{\{f(\mathbf{x})=0\}} \nabla_{\mathbf{x}} f(\mathbf{x}).$$

• If f is replaced by a Gaussian process, integrability of

$$\begin{split} \mathbf{x}_{d} & \frac{1}{((2\pi)^{d+1} \det \Sigma)^{1/2}} \exp\left(-\frac{\check{X}^{T} \Sigma^{-1} \check{X}}{2}\right) \, \mathrm{d}\mathbf{x}_{d}, \\ \check{X}^{T} \Sigma \check{X} &= -m_{k}(\mathbf{x})^{2} + 2\partial_{\mathbf{y}} R(\mathbf{x}, \mathbf{x}) \cdot (\mathbf{x}_{d} - \nabla_{\mathbf{x}} m_{k}(\mathbf{x})) \\ &+ (\mathbf{x}_{d} - \nabla_{\mathbf{x}} m_{k}(\mathbf{x}))^{T} \partial_{\mathbf{x}\mathbf{y}}^{2} R(\mathbf{x}, \mathbf{x}) (\mathbf{x}_{d} - \nabla_{\mathbf{x}} m_{k}(\mathbf{x})). \end{split}$$

# Consistency with the macro data

$$\chi_k(\mathbf{x}) = \frac{1 + \operatorname{sgn}(g_{\mathbf{x}})}{2}$$

• Parameters of  $g_{\mathbf{x}}$ :  $m_k(\mathbf{x})$ , and R (more precisely,  $\check{\Sigma} = \partial_{\mathbf{x}\mathbf{y}}R(\mathbf{x}, \mathbf{x})$ )

Volume fraction

$$\alpha_k(\mathbf{x}) = \mathbb{E}\left[\chi_k(\mathbf{x})\right]$$
$$m_k(\mathbf{x}) = \sqrt{2} \operatorname{erf}^{-1}(2\alpha_k(\mathbf{x}) - 1).$$

### Gradient of the volume fraction

$$\nabla_{\mathbf{x}}\alpha_k(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbb{E}\left[\chi_k(\mathbf{x})\right] = \mathbb{E}\left[\nabla_{\mathbf{x}}\chi_k(\mathbf{x})\right]$$

• Wrong for the Hessian!

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### Interfacial area density

$$\begin{aligned} \mathcal{A}_{l} &= \mathbb{E}\left[ \|\nabla \chi_{k}(\mathbf{x})\| \right] \\ &= \frac{\exp(-m_{k}(\mathbf{x})^{2}/2)}{((2\pi)^{d+1} \det \check{\Sigma})^{1/2}} \int_{\mathbf{x}_{d} \in \mathbb{R}^{d}} \|\mathbf{x}_{d}\| e^{-\frac{(\mathbf{x}_{d} - \nabla m_{k}(\mathbf{x}))^{T} \check{\Sigma}^{-1}(\mathbf{x}_{d} - \nabla m_{k}(\mathbf{x}))}{2}} d\mathbf{x}_{d}. \end{aligned}$$



• Does not provide any global interpretation of  $\check{\Sigma}$ , except when it is isotropic

### Efficient interfacial area density

- $\check{\Sigma}$  is symmetric and determines *d* orthonormal eigendirections  $\check{\mathbf{e}}_i$ .
- Eigenvalues can be recovered with the half efficient interfacial area density.

$$\mathbb{E}\left[\max(\mathbf{0}, \nabla \chi_k \cdot \check{\mathbf{e}}_i)\right]$$



## Consistency with the macro data

- We have been concerned only on interfacial area density
- We want to investigate the inclusion size, with  $\nabla_{\mathbf{x}} \alpha_k = 0$ .
- One dimensional simulations of a Gaussian process. Square exponential autocorrelation. Numerical evaluation of the length of an inclusion



## Consistency with the macro data



• Looks like a  $\gamma$ -law, but is not a  $\gamma$ -law.

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# Baer-and-Nunziato models obtained and their properties

$$\begin{aligned} \partial_t \left( \alpha_k \hat{\mathbf{U}}_k \right) &+ \operatorname{div}_{\mathbf{x}} \left( \alpha_k \hat{\mathbf{F}}_k (\hat{\mathbf{U}}_k) \right) \\ &= \frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1} \operatorname{det} \check{\boldsymbol{\Sigma}})^{1/2}} \int_{\mathbf{x}_d \in \mathbb{R}^d} e^{-\frac{(\mathbf{x}_d - \nabla m_k(\mathbf{x}))^T \check{\boldsymbol{\Sigma}}^{-1}(\mathbf{x}_d - \nabla m_k(\mathbf{x}))}{2}} \hat{\mathbf{F}}_{\bar{k}k}^{lag} \left( \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d. \end{aligned}$$

- Address particular cases
  - Relative state of the two phases
  - $\implies$  Simplify  $\hat{\mathbf{F}}_{\bar{k}k}^{lag}$ 
    - Topology of the flow: relative value of  $\check{\Sigma}$  and  $\nabla_x m$ .
  - $\implies$  Linearize the exponential.

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det\check{\Sigma})^{1/2}}\int_{\mathbf{x}_d\in\mathbb{R}^d} \mathrm{e}^{-\frac{(\mathbf{x}_d-\nabla m_k(\mathbf{x}))^T\check{\Sigma}^{-1}(\mathbf{x}_d-\nabla m_k(\mathbf{x}))}{2}}\hat{\mathbf{F}}_{\bar{k}k}^{lag}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

### Local contact

If for a given  $\boldsymbol{x}$ 

• 
$$\mathbf{u}_k(\mathbf{x}) = \mathbf{u}_{\bar{k}}(\mathbf{x}) = \mathbf{u}_0$$

• 
$$P_k(\mathbf{x}) = P_{\bar{k}}(\mathbf{x}) = P_0$$

then  $P_I = P_0$  and  $\mathbf{u}_I = \mathbf{u}_0$  whatever the topology.  $\lambda$  and  $\mu$  are undetermined (as they are not active).

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det\check{\Sigma})^{1/2}}\int_{\mathbf{x}_d\in\mathbb{R}^d} e^{-\frac{(\mathbf{x}_d-\nabla m_k(\mathbf{x}))^T\check{\Sigma}^{-1}(\mathbf{x}_d-\nabla m_k(\mathbf{x}))}{2}}\hat{\mathbf{F}}_{\bar{k}k}^{lag}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

### One dimensional case

In one dimension, setting

$$w := \operatorname{erf}\left(\frac{|\partial_x m_k(x)|}{\lambda_1\sqrt{2}}\right),$$

the system is closed by the following interfacial quantities

$$\begin{cases} \mathbf{u}_{l} = \operatorname{sgn}(\partial_{x}\alpha_{k}(x)) \left( \frac{1+w}{2} u_{\tilde{k}k}^{\star}(\partial_{x}\alpha_{k}(x)) + \frac{w-1}{2} u_{\tilde{k}k}^{\star}(-\partial_{x}\alpha_{k}(x)) \right) \\ P_{l} = \operatorname{sgn}(\partial_{x}\alpha_{k}(x)) \left( \frac{1+w}{2} P_{\tilde{k}k}^{\star}(\partial_{x}\alpha_{k}(x)) + \frac{w-1}{2} P_{\tilde{k}k}^{\star}(-\partial_{x}\alpha_{k}(x)) \right) \\ (P\mathbf{u})_{l} = \operatorname{sgn}(\partial_{x}\alpha_{k}(x)) \left( \frac{1+w}{2} P_{\tilde{k}k}^{\star}(\partial_{x}\alpha_{k}(x)) u_{\tilde{k}k}^{\star}(\partial_{x}\alpha_{k}(x)) + \frac{w-1}{2} P_{\tilde{k}k}^{\star}(-\partial_{x}\alpha_{k}(x)) + \frac{w-1}{2} P_{\tilde{k}k}^{\star}(-\partial_{x}\alpha_{k}(x)) \right) \end{cases}$$

and includes the following nonlinear relaxation term

...

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$$w := \operatorname{erf}\left(\frac{|\partial_x m_k(x)|}{\lambda_1\sqrt{2}}\right),$$

the system is closed by the following interfacial quantities

and includes the following nonlinear relaxation term

$$\lambda_{1} \mathrm{e}^{-\frac{\partial_{x}m_{k}(x)^{2}}{2\lambda_{1}^{2}}} \frac{\mathrm{e}^{-\frac{m_{k}(x)^{2}}{2}}}{2\pi} \begin{pmatrix} u_{kk}^{\star}(-\partial_{x}\alpha(x)) - u_{kk}^{\star}(\partial_{x}\alpha(x)) \\ 0 \\ P_{kk}^{\star}(\partial_{x}\alpha(x)) - P_{kk}^{\star}(-\partial_{x}\alpha(x)) \\ P_{kk}^{\star}(\partial_{x}\alpha(x)) - P_{kk}^{\star}(-\partial_{x}\alpha(x)) u_{kk}^{\star}(-\partial_{x}\alpha(x)) \end{pmatrix}.$$

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det\check{\Sigma})^{1/2}}\int_{\mathbf{x}_d\in\mathbb{R}^d} e^{-\frac{(\mathbf{x}_d-\nabla m_k(\mathbf{x}))^T\check{\Sigma}^{-1}(\mathbf{x}_d-\nabla m_k(\mathbf{x}))}{2}}\hat{\mathbf{F}}_{\bar{k}k}^{lag}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

### Long memory case

When  $\partial_{xy}^2 R(\mathbf{x}, \mathbf{x}) \to 0$ , the system obtained is the Baer-and-Nunziato model with  $\lambda = \mu = 0$ , and the following interfacial velocity and pressure

$$\mathbf{u}_{I} = u^{\star} \left( \frac{\nabla \alpha(\mathbf{x})}{\|\nabla \alpha(\mathbf{x})\|} \right) \frac{\nabla \alpha(\mathbf{x})}{\|\nabla \alpha(\mathbf{x})\|} \quad \text{and} \quad P_{I} = P^{\star} \left( \frac{\nabla \alpha(\mathbf{x})}{\|\nabla \alpha(\mathbf{x})\|} \right).$$

- First proposed in
  - E. Franquet and V. Perrier, Runge-Kutta discontinuous Galerkin method for the approximation of Baer and Nunziato type multiphase models, JCP, 2012.
- Matches with interfacial flows

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det\check{\Sigma})^{1/2}}\int_{\mathbf{x}_d\in\mathbb{R}^d} e^{-\frac{(\mathbf{x}_d-\nabla m_k(\mathbf{x}))^T\check{\Sigma}^{-1}(\mathbf{x}_d-\nabla m_k(\mathbf{x}))}{2}}\hat{\mathbf{F}}_{\bar{k}k}^{lag}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

## Short memory case

### Short memory case

$$\check{\Sigma} = Q^T \Lambda^2 Q$$

If we suppose that  $\Lambda^{-1} 
abla m_k(\mathbf{x}) \ll 1$ , the interfacial terms are

$$\begin{cases} \mathbf{u}_{I} = \frac{\check{\boldsymbol{\Sigma}}^{-1}}{\left|\mathbb{V}_{d}\right| \left|\det\check{\boldsymbol{\Sigma}}\right|^{1/2}} \int_{\mathbf{x}_{d} \in Q^{T} \wedge \left(\mathbb{S}^{d-1}\right)} u_{\check{k}k}^{\star} \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \|\mathbf{x}_{d}\| \mathbf{x}_{d} d\mathbf{x}_{d} \\ \mathbf{P}_{I} = \frac{1}{\left|\mathbb{V}_{d}\right| \left|\det\check{\boldsymbol{\Sigma}}\right|^{1/2}} \left(\int_{\mathbf{x}_{d} \in Q^{T} \wedge \left(\mathbb{S}^{d-1}\right)} P_{\check{k}k}^{\star} \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \mathbf{x}_{d} \mathbf{x}_{d}^{T} d\mathbf{x}_{d}\right) \check{\boldsymbol{\Sigma}}^{-1} \\ (P\mathbf{u})_{I} = \frac{\check{\boldsymbol{\Sigma}}^{-1}}{\left|\mathbb{V}_{d}\right| \left|\det\check{\boldsymbol{\Sigma}}\right|^{1/2}} \int_{\mathbf{x}_{d} \in Q^{T} \wedge \left(\mathbb{S}^{d-1}\right)} (Pu)_{\check{k}k}^{\star} \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \|\mathbf{x}_{d}\| \mathbf{x}_{d} d\mathbf{x}_{d}, \end{cases} \end{cases}$$

and the nonlinear relaxation terms are

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...

If we suppose that  $\Lambda^{-1} \nabla m_k(\mathbf{x}) \ll 1$ , the interfacial terms are

#### and the nonlinear relaxation terms are

$$\begin{aligned} \mathcal{R}_{k}^{(\alpha)} &= \frac{\exp(-m_{k}(\mathbf{x})^{2}/2)}{2\pi \left\|\mathbb{V}_{d-1}\right\| \left|\det\check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_{d} \in Q^{T} \wedge (\mathbb{S}^{d-1})} (-u_{kk}^{\star}) \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \|\mathbf{x}_{d}\| \, \mathrm{d}\mathbf{x}_{d} \\ \mathcal{R}_{k}^{(\rho\mathbf{u})} &= \frac{\exp(-m_{k}(\mathbf{x})^{2}/2)}{2\pi \left\|\mathbb{V}_{d-1}\right\| \left|\det\check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_{d} \in Q^{T} \wedge (\mathbb{S}^{d-1})} P_{kk}^{\star} \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \mathbf{x}_{d} \mathrm{d}\mathbf{x}_{d} \\ \mathcal{R}_{k}^{(\rho E)} &= \frac{\exp(-m_{k}(\mathbf{x})^{2}/2)}{2\pi \left\|\mathbb{V}_{d-1}\right\| \left|\det\check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_{d} \in Q^{T} \wedge (\mathbb{S}^{d-1})} (Pu)_{kk}^{\star} \left(\frac{\mathbf{x}_{d}}{\|\mathbf{x}_{d}\|}\right) \|\mathbf{x}_{d}\| \, \mathrm{d}\mathbf{x}_{d}. \end{aligned}$$

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$$\check{\Sigma} = Q^T \Lambda^2 Q$$

. . .

If we suppose that  $\Lambda^{-1} \nabla m_k(\mathbf{x}) \ll 1$ , the interfacial terms are

and the nonlinear relaxation terms are

- Matches with flows with very small inclusions
  - $P_k$  and  $P_{\bar{k}}$  are supposed to be close
  - $\mathbf{u}_k$  and  $\mathbf{u}_{\bar{k}}$  are supposed to be close

### Acoustic approximation

Once the waves curves are linearized, the solution of the Riemann problem is

$$\begin{pmatrix}
 u_{\bar{k}k}^{\star}(\mathbf{n}_{\bar{k}k}) = \frac{Z_{k}\mathbf{u}_{k} + Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \cdot \mathbf{n}_{\bar{k}k} + \frac{P_{\bar{k}} - P_{k}}{Z_{k} + Z_{\bar{k}}} \\
 P_{\bar{k}k}^{\star}(\mathbf{n}_{\bar{k}k}) = \frac{Z_{\bar{k}}P_{k} + Z_{k}P_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} + \frac{Z_{k}Z_{\bar{k}}(\mathbf{u}_{\bar{k}} - \mathbf{u}_{k})}{Z_{k} + Z_{\bar{k}}} \cdot \mathbf{n}_{\bar{k}k}.$$

where  $Z_k = \rho_k c_k$  is the acoustic impedance.

### Short memory, linearized case

$$\mathbf{u}_{I}^{L} = \frac{Z_{k}\mathbf{u}_{k} + Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \qquad P_{I}^{L} = \frac{Z_{\bar{k}}P_{k} + Z_{k}P_{\bar{k}}}{Z_{k} + Z_{\bar{k}}},$$
$$\mathscr{L}(\Lambda) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \|\Lambda \mathbf{x}_{d}\| \, \mathrm{d}\mathbf{x}_{d} \quad \tilde{\Lambda} = Q^{T}\Lambda^{2} \left(\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \frac{\mathbf{y}_{d}\mathbf{y}_{d}^{T}}{\|\Lambda \mathbf{y}_{d}\|} \, \mathrm{d}\mathbf{y}_{d}\right) Q$$

. . .

. . .

Then the interfacial terms are

and the relaxation terms are

### Short memory, linearized case

$$\mathbf{u}_{I}^{L} = \frac{Z_{k}\mathbf{u}_{k} + Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \qquad P_{I}^{L} = \frac{Z_{\bar{k}}P_{k} + Z_{k}P_{\bar{k}}}{Z_{k} + Z_{\bar{k}}},$$
$$\mathscr{L}(\Lambda) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \|\Lambda \mathbf{x}_{d}\| \, \mathrm{d}\mathbf{x}_{d} \quad \tilde{\Lambda} = Q^{T}\Lambda^{2} \left(\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \frac{\mathbf{y}_{d}\mathbf{y}_{d}^{T}}{\|\Lambda \mathbf{y}_{d}\|} \mathrm{d}\mathbf{y}_{d}\right) Q$$

Then the interfacial terms are

$$\begin{aligned} \mathbf{P}_{I} &= P_{I}^{L} \mathbf{I}_{d} \\ \mathbf{u}_{I} &= \mathbf{u}_{I}^{L} \\ (P\mathbf{u})_{I} &= \frac{(Z_{\bar{k}}P_{k} + Z_{k}P_{\bar{k}})(Z_{k}\mathbf{u}_{k} + Z_{\bar{k}}\mathbf{u}_{\bar{k}}) + Z_{k}Z_{\bar{k}}(P_{\bar{k}} - P_{k})(\mathbf{u}_{\bar{k}} - \mathbf{u}_{k})}{(Z_{k} + Z_{\bar{k}})^{2}}, \end{aligned}$$

and the relaxation terms are

...

### Short memory, linearized case

$$\mathbf{u}_{I}^{L} = \frac{Z_{k}\mathbf{u}_{k} + Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \qquad P_{I}^{L} = \frac{Z_{\bar{k}}P_{k} + Z_{k}P_{\bar{k}}}{Z_{k} + Z_{\bar{k}}},$$
$$\mathscr{L}(\Lambda) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \|\Lambda\mathbf{x}_{d}\| \, \mathrm{d}\mathbf{x}_{d} \quad \tilde{\Lambda} = Q^{T}\Lambda^{2} \left(\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_{d} \in \mathbb{S}^{d-1}} \frac{\mathbf{y}_{d}\mathbf{y}_{d}^{T}}{\|\Lambda\mathbf{y}_{d}\|} \, \mathrm{d}\mathbf{y}_{d}\right) Q$$

. . .

Then the interfacial terms are

and the relaxation terms are

$$\begin{pmatrix}
\mathcal{R}_{k}^{(\alpha)} = \frac{\exp(-m_{k}(\mathbf{x})^{2}/2) \left| \mathbb{S}^{d-1} \right| \mathscr{L}(\Lambda)}{2\pi \left| \mathbb{V}_{d-1} \right|} \frac{P_{k} - P_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \\
\mathcal{R}_{k}^{(\rho \mathbf{u})} = \frac{\exp(-m_{k}(\mathbf{x})^{2}/2) \left| \mathbb{S}^{d-1} \right|}{2\pi \left| \mathbb{V}_{d-1} \right|} \frac{Z_{k} Z_{\bar{k}}}{Z_{k} + Z_{\bar{k}}} \tilde{\Lambda} \left( \mathbf{u}_{\bar{k}} - \mathbf{u}_{k} \right) \\
\mathcal{R}_{k}^{(\rho E)} = \mathbf{u}_{I}^{L} \cdot \mathcal{R}_{k}^{(\rho \mathbf{u})} - P_{I}^{L} \mathcal{R}_{k}^{(\alpha)}.
\end{cases}$$

### Short memory, linearized, isotropic case

Same interfacial terms as the non isotropic case;

$$\check{\Sigma} = \nu I_d$$

The initial model is exactly found with

$$\begin{cases} \mu = \frac{\exp(-m_k(\mathbf{x})^2/2) \left| \mathbb{S}^{d-1} \right| \nu}{2\pi \left| \mathbb{V}_{d-1} \right|} \frac{1}{Z_k + Z_{\bar{k}}} \\ \lambda = \frac{\exp(-m_k(\mathbf{x})^2/2) \left| \mathbb{S}^{d-1} \right| \nu}{2\pi \left| \mathbb{V}_{d-1} \right|} \frac{Z_k Z_{\bar{k}}}{Z_k + Z_{\bar{k}}}. \end{cases}$$

### Hyperbolicity

Except when resonant, the system is hyperbolic.

### Linear degeneracy of $\mathbf{u}_I$

- In general, the filed is genuinely nonlinear
- Maybe the relaxation (Pu)<sub>I</sub> ≠ P<sub>I</sub>u<sub>I</sub> can be exploited for ensuring linear degeneracy of the field.

- Entropy dissipation
  - Because of velocity and pressure disequilibrium, entropy conservation is not expected.
  - What is expected is phasic entropy dissipation.
    - The initial averaged approximated model is globally entropy dissipative
  - Results

Models	Hyperbolic terms	Relaxation terms
Long memory model	✓	✓
One dimensional model	×	✓
Short memory model	×	✓
Short memory, linearized model	×	✓

- However
  - One dimensional model is globally entropy dissipative
  - For short memory models, relaxation terms are zeroth order, whereas hyperbolic terms are first order.

## Possible extensions

## Two-phase flows with capillarity



• For a known topology: the Riemann problem should be solved, but with a tension at the interface depending on the curvature.

$$P_k^{\star} - P_{\bar{k}}^{\star} = \sigma \kappa$$

• The non averaged system is

$$\partial_t \left( \chi_k \hat{\mathbf{U}}_k \right) + \operatorname{div}_{\mathbf{x}} \left( \chi_k \hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) \right) = \hat{\mathbf{F}}_{\bar{k}k}^{lag} \left( \mathbf{n}(f), \kappa(f) \right) \| \nabla_{\mathbf{x}} f(\mathbf{x}) \|$$

• Raises other derivatives of *R* (that should be physically interpreted).

# Multiphase flows

- How to extend to more than two phases?
- Three phase flow are intrisically more complicated
  - Do all the phases have a similar role?
- Two-phase stochastic model relies on a process with value in  $\mathbb{S}^0$



• Diffusive terms are physically strongly different

- Regularizing effect: continuity of temperature and velocity, and of diffusive fluxes.
- Infinite velocity of propagation
- Our aim is slightly different from classical homogenization of heat equation
  - Classical (stochastic) homogenization of Laplace equation: find the equation for an averaged temperature.
  - Baer-and-Nunziato approach: find an equation for each of temperature of the phases.
- Didier Bresch and Matthieu Hillairet, A compressible multifluid system with new physical relaxation terms, Ann. Sci. ENS, 2019.
  - Single velocity
  - Pressure relaxation induced by viscosity change between the "phases".

## Numerical schemes

- Difficulties:
  - Define jump relations for Riemann-solver based finite volume methods
    - If the nonconservative terms match with genuinely nonlinear fields, this is not enough for ensuring convergence.
  - Positivity of the volume fraction [Relaxation terms].
  - Positivity of the temperature [Harten phase-entropy inequalities].
- Rémi Abgrall and Richard Saurel, Discrete equations for physical and numerical compressible multiphase mixtures, JCP 2003.
- Erwin Franquet, Vincent Perrier, Runge-Kutta discontinuous Galerkin method for the approximation of Baer and Nunziato type multiphase models, JCP 2012.
  - Clear stochastic modeling.
  - Only interface flows (R = 0).
  - If R ≠ 0: not done yet; but not hard. The major problem is the evolution of the second order derivatives of R.
  - If  $\partial_{xy}R \to \infty$ : consistency with Kapila's model?
- High order schemes
  - Avoid a splitting between hyperbolic and relaxation terms

# Conclusion

- Systematic way of deriving compressible multiphase models
  - "Exact" model derivation
  - Repartition of the phases based on an explicit stochastic model
- Models obtained
  - Closures depend on the topology of the flow
  - Closures are only asymptotic formulations of the exact model. Closures have a range of validity.
  - A lot of prospects: capillarity, diffusive flows, multiphase models.
  - Mathematical parameters are not straightforward to interpret physically.
    - Geometry of level sets of Gaussian process....
  - Evolution of R (or  $\partial_{xy}R$ ) is still an open question.
- Published in
  - V. Perrier and E. Gutiérrez, Derivation and Closure of Baer and Nunziato Type Multiphase Models by Averaging a Simple Stochastic Model, SIAM MMS, 2021.
- Prospects for numerical approximation without asymptotic expansion
  - Ensure positivity of volume fraction, temperature.
  - "Avoid" the problem of nonconservative terms.