Kerr-Schild Ansatz for Black Holes with Matter Source

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Plan of the talk

1. Kerr-Schild (KS) Ansatz : Definitions, Motivations and Examples.
2. Role of symmetries in the KS derivation of the Kerr metric (vacuum) [E. Ayón-Beato, M. Hassaine and D. Higuita-Borja, Phys. Rev. D **94**, no.6, 064073 (2016).]
3. KS construction of static (regular) black holes with scalar field source [E. Babichev, C. Charmousis, A. Cisterna and M. Hassaine, JCAP **06**, 049 (2020).]
4. KS construction of spinning black holes with source (Kerr-Sen solution), work in progress [E. Ayón-Beato, M. Hassaine and D. Higuita-Borja.]

A little historical overview

• 1963 [Roy P. Kerr, Phys. Rev. Lett. 11, 237 (1963)] : stationary axisymmetric solution of the Einstein's vacuum equations with no charge (Petrov classification of spacetimes by studying the algebraic character of the Weyl tensor).

• 1965 [E. T. Newman et. *al*, J. Math. Phys. 6, 918 (1965)] : the charged extension of the Kerr's solution.

• 1965 [R. P. Kerr, A. Schild] Both metric Kerr-(Newman) can be put in the same form, the so-called Kerr-Schild form with a flat base metric (seed metric).

• 1924 [Eddington, A., Nature 113, 192 (1924)]. Schwarzschild metric can be written in the "Kerr-Schild form"

$$ds^2 = (-dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2) - (2m/r)(dt - dr)^2$$

Kerr-Schild Ansatz : Definitions, Motivations and Examples

• Motivations

$\longrightarrow$ highly nonlinear character of the Einstein's equations

$\longrightarrow$ To look for metrics describing "exact perturbations" of flat spacetime propagating along a null direction (the calculations are considerably simplified in this case)

• (Generalized) Kerr-Schild Ansatz

$\longrightarrow$ A (generalized) Kerr Schild perturbation has the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} - 2H(x)\, l_\mu l_\nu,$$

where $g^{(0)}_{\mu\nu}$ is the seed metric, and l is a null vector $g^{\mu\nu} l_\mu l_\nu = {g^{(0)}}^{\mu\nu} l_\mu l_\nu = 0$. Originally $g^{(0)}_{\mu\nu}$ was a flat spacetime, and its generalization for a non flat seed metric was first given in [B. C. Xanthopoulos, Journal of Mathematical Physics 19, 1607 (1978);].

Kerr-Schild Ansatz : Definitions, Motivations and Examples

$\longrightarrow$ Any "perturbation" of this form $g^{(0)}_{\mu\nu} - 2H(x)\, l_\mu l_\nu$ will only appear quadratically in the Einstein equations, and only linearly if the null vector is also geodesic, i. e.

$$(\nabla_\nu l_\mu) l^\nu = (\nabla^{(0)}_\nu l_\mu) l^\nu = 0.$$

$\longrightarrow$ For a null and geodesic vector field, the Ricci tensor has the following form

$$R^\mu_\nu = R^{(0)\mu}{}_\nu + 2h^\mu_\sigma R^{(0)\sigma}{}_\nu - \nabla^{(0)}_\nu \nabla^{(0)}_\sigma h^{\sigma\mu} - \nabla^{(0)\mu} \nabla^{(0)}_\sigma h^\sigma_\mu + \Box^{(0)} h^\mu_\nu$$

with $h^{\mu\nu} = H l^\mu l^\nu$.

$\longrightarrow$ Einstein's equations reduce to a linear system without any approximation.

Kerr-Schild Ansatz : Definitions, Motivations and Examples

- Examples of Kerr-Schild metrics

1. The pp wave

$$ds^2 = -F(u,\vec{x})du^2 - 2dudv + d\vec{x}^2 = g^{\text{flat}}_{\mu\nu} - F l_\mu l_\nu$$

where $l^\mu\partial_\mu = \partial_v$ is a covariantly constant null vector.

2. The AdS wave

$$\begin{aligned} ds^2 &= \frac{l^2}{y^2}\left[-F(u,y,\vec{x})du^2 - 2dudv + dy^2 + d\vec{x}^2\right] \\ &= g^{\text{AdS}}_{\mu\nu} - \frac{y^2 F}{l^2} l_\mu l_\nu \end{aligned}$$

where $l^\mu\partial_\mu = \partial_v$ is a null Killing field.

3. In vacuum, most of the metrics describing black holes are of the Kerr-Schild form (Schwarzschild, Kerr-(Newman), Kerr-(A)dS in arbitrary D).

4. Five-dimensional black ring solution is not of the Kerr-Schild form. [R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002)]

Main ingredients for the Kerr-Schild procedure for black holes

1. The **seed metric** corresponds to the massless spacetime, and for spinning solutions, the angular momenta must be codified into it. For example, in the derivation of the Kerr spacetime, a judicious choice for the seed metric is provided by flat spacetime written in ellipsoidal coordinates

$$ds_0^2 = -dt^2 + (r^2 + a^2\cos^2(\theta))\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2(\theta)d\varphi^2$$

2. The appropriate null and geodesic vector field. Physically, geodesic null vector field can be interpreted as the tangent vector of optical rays.
3. Sachs [Sachs, R.K., Proc. R. Soc. London, Ser. A, 270, 103,Äì126, (1962).] $\longrightarrow$ optical parameters : the divergence, the twist and the shear of the congruence and classify congruences by the vanishing of one or more of these parameters.

Main ingredients for the Kerr-Schild procedure for black holes

1. For examples, in flat space, divergence-free congruences always exist while general vacuum spacetimes do not allow shear-free congruences.
2. For example, the Kerr's theorem gives (in an implicit form) the general form of a geodesic and shear-free null congruence of the flat metric $ds_0^2 = -2dudv + d\xi d\bar{\xi}$, i.e.

$$l = dv$$

or

$$l = du + Y\bar{Y}dv + \bar{Y}d\xi + Yd\bar{\xi},$$

where $Y = Y(u, v, \xi, \bar{\xi})$ is defined implicitly by $F(Y, \bar{\xi}Y + u, vY + \xi) = 0$, and where F is an arbitrary analytic function. But only a specific election of F will yield the Kerr's solution.

Kerr-Schild ansatz in higher dimension

$\longrightarrow$ In higher dimension D, there is the possibility of $[\frac{D-1}{2}]$ independent angular momenta.

$\longrightarrow$ The higher-dimensional extension of the Kerr solution, namely the Myers-Perry solution [R. C. Myers and M. J. Perry, Annals Phys. 172, 304 (1986)] was achieved thanks to the Kerr-Schild representation.

$\longrightarrow$ The higher-dimensional Kerr-(A)dS metric was only recently discovered by means of a Kerr-Schild ansatz [G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, Phys. Rev. Lett. **93** (2004), 171102]. *"We have been led to these metrics by putting the previously-known $D = 4$ and $D = 5$ Kerr-de Sitter metrics into Kerr-Schild form, and making natural generalizations to higher dimensions. We have explicitly checked that they obey the Einstein equations for all physically interesting cases $D \leq 11$".*

Role of symmetries in the KS derivation of the Kerr metric [E. Ayón-Beato, M. Hassaine and D. Higuita-Borja, Phys. Rev. D **94**, no.6, 064073 (2016).]

New derivation of the Kerr's stationary axisymmetric (SA) metric

$\longrightarrow$ Well-educated guess for the analytic function F would yield the Kerr's metric.

$\longrightarrow$ Other option : assume beforehand the relevant symmetries that the final state of the gravitational collapse is supposed to enjoy (SA)$\longrightarrow$ (SA) version of the Kerr's theorem.

- First step : Flat seed metric where the (SA) is manifest (cylindrical)

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2,$$

where $k = \partial_t$ and $m = \partial_\phi$ are the two Killing fields, and $l = l_\mu dx^\mu$ with $l_\mu = l_\mu(\rho, z)$.

• Second step : The most general (SA) shear free and geodesic null congruence

$$l = dt + \frac{r\rho}{r^2+a^2}d\rho - \frac{a\rho^2}{r^2+a^2}d\phi + \frac{z}{r}dz$$

where r is the affine parameter

$$\frac{\rho^2}{r^2+a^2} + \frac{z^2}{r^2} = 1, \quad \text{ellipsoids of revolution}$$

and where a is an integration constant (angular momentum).

• Third step : Coordinates adapted to these ellipsoids, $\rho = \sqrt{r^2+a^2}\sin(\theta)$ and $z = r\cos(\theta)$

$$ds_0^2 = -dt^2 + (r^2 + a^2\cos^2(\theta))\left(\frac{dr^2}{r^2+a^2} + d\theta^2\right) + (r^2+a^2)\sin^2(\theta)d\phi^2$$

and

$$l = dt + \frac{r^2 + a^2\cos^2(\theta)}{r^2+a^2}dr - a\sin^2(\theta)d\phi$$

- Fourth step : The general (SA) KS ansatz is naturally

$$ds^2 = ds_0^2 + 2H(r,\theta)\left(dt + \frac{r^2 + a^2\cos^2(\theta)}{r^2+a^2}dr - a\sin^2(\theta)d\phi\right)^2$$

where the wave profile has to be determined.

- Fifth step : Circularity theorem :

Any (SA) with commuting Killing vectors k and m satisfies

$$d*(k\wedge m\wedge dk) = 2*(k\wedge m\wedge R(k))\,,$$
$$d*(k\wedge m\wedge dm) = 2*(k\wedge m\wedge R(m))\,.$$

For vacuum solutions, the lhs vanish
$\Longrightarrow \partial_\theta\Big[\left(r^2+a^2\cos^2(\theta)\right)H(r,\theta)\Big]=0 \Longrightarrow H(r,\theta)=\frac{rM(r)}{r^2+a^2\cos^2(\theta)}\Longrightarrow$
Boyer-Lindquist coordinates $\exists$ And $G_r^r = 0 \propto M'(r) = 0$

Extension for the Kerr-Newman solution

1. The Froebenius identities are now given by

$$-\frac{1}{4}d*(k\wedge m\wedge dk) = *F(k,m)E_k + F(k,m)B_k,$$
$$-\frac{1}{4}d*(k\wedge m\wedge dm) = *F(k,m)E_m + F(k,m)B_m,$$

2. For a (SA) electromagnetic field, $\mathcal{L}_k F = 0 = \mathcal{L}_m F$, it follows from the Maxwell equations that $F(k,m) = 0 = *F(k,m) \Longrightarrow$ the same wave profile $H(r,\theta) = \frac{rM(r)}{r^2+a^2\cos^2(\theta)}$.
3. The novelty is to fix an ansatz for the Maxwell field. A (SA) ansatz $A = \hat{S}(r,\theta)l$ is not circular by construction unless $\hat{S} = \frac{rQ}{r^2+a^2\cos^2(\theta)}$ and $E_r^r = 0 \propto M'(r) = \frac{Q^2}{2r^2}$

Kerr-Schild Ansatz in presence of matter source

• The Kerr-Schild procedure was originally considered for vacuum solutions or in presence of a Maxwell gauge field (only in $D = 4$), and later in the presence of a cosmological constant in higher D [G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, Phys. Rev. Lett. **93**, 171102 (2004)]. How to implement the Kerr-Schild procedure in presence of matter?

• The difficulty with the presence of source is to find an appropriate ansatz for the extra dynamical fields to be fully compatible with the equations of motion.

• The most appealing example where the Kerr-Schild procedure works is the Kerr-Newman solution in $D = 4$

$$ds^2 = ds_0^2 + \frac{2r}{\Sigma(r,\theta)}\left(\mathcal{M} - \frac{\mathcal{Q}^2}{2r}\right) l \otimes l, \quad A = \frac{r\mathcal{Q}}{\Sigma(r,\theta)} l$$

• BUT the higher-dimensional Kerr-Newman solution is not yet known ($A \propto l$ is incompatible with the EOM).

Kerr-Schild Ansatz in presence of matter source

[E. Babichev, C. Charmousis, A. Cisterna and M. Hassaine, JCAP **06** (2020), 049]

• Investigate the feasibility of the Kerr-Schild procedure in the case of Scalar Tensor Theories (STT), i. e. $\mathcal{L} = \mathcal{L}(g, \phi)$ in order to construct black holes with a scalar field source.

1. From a simple seed configuration (g_0, ϕ_0), generate a nontrivial BH configuration (g, ϕ) by means of the Kerr-Schild procedure.
2. Construct BHs with interesting features (e. g. regular BHs).
3. The main idea will be to fix the desired properties of the solution and by "engineering inverse" to determine the STT susceptible to sustain this solution.

Presentation of the Kerr-Schild procedure

1. Consider a STT $\mathcal{L}(g, \partial g, \partial^2 g, \partial\phi, \partial^2\phi)$ invariant under the shift symmetry of the scalar field $\phi \to \phi$+cst,

$$\begin{aligned}\mathcal{L} = \quad & K + G\,R + A_1\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right)\\ & + A_3\Box\phi\,\phi^\mu\phi_{\mu\nu}\phi^\nu + A_4\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho + A_5\left(\phi^\mu\phi_{\mu\nu}\phi^\nu\right)^2\end{aligned}$$

   where the coupling functions K, G and A_i are arbitrary functions of $X = \phi_\mu\phi^\mu$, and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$.

2. Implement the KS Ansatz in the static case with a seed configuration $(g_0, \phi^{(0)})$

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2 d\Sigma_2^2,\ \phi^{(0)}(t, r) := qt + \psi^{(0)}(r)$$

3. KS transformation for the metric

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \mu a(r)\, l_\mu l_\nu, \quad l = dt - \frac{dr}{\sqrt{f_0(r)\, h_0(r)}}$$

Presentation of the Kerr-Schild procedure

• Working hypothesis : We will demand that the kinetic term of the scalar field remains unchanged (but not necessarily constant) under the Kerr-Schild transformation of the metric

$$X^{(0)} := g^{(0)\mu\nu}\,\partial_\mu\phi^{(0)}\,\partial_\nu\phi^{(0)} = X := g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi.$$

• This condition has to be put in parallel with the Kerr-Newman case ; the seed configuration is given by a flat metric $g^{(0)}$ with a zero gauge field $A_\mu^{(0)} = 0$ while the gauge field after the KS transformation $A_\mu \propto l_\mu \Longrightarrow$ the norm (scalar quantity) of the gauge field remains the same under a KS transformation, i. e.

$$|A^{(0)}| := g^{(0)\mu\nu} A_\mu^{(0)} A_\nu^{(0)} = 0 = |A| := g^{\mu\nu} A_\mu A_\nu.$$

• This observation will also be important in the case of the Kerr-Sen solution.

Presentation of the Kerr-Schild procedure

• We will say that the static Kerr-Schild transformation $g = g^{(0)} + \mu a(r) l \otimes l$ leaving invariant the kinetic term $X = X^{(0)}$ is a symmetry if the variation of the action

$$S(g,\phi) - S(g^{(0)},\phi^{(0)}) = \int dr\, \mathcal{E}\left(r, a(r), a'(r), X(r), X'(r)\right) + b.t.,$$

vanishes for arbitrary μ (up to a boundary term) for a specific election of the mass term $a(r) = a(r, X^{(0)}(r))$ such that $\mathcal{E} = 0$.

• For the STT defined previously

$$a(r) = \frac{1}{r} e^{\int \frac{3(A_3 X + 4G_X - 2A_1)}{8(A_1 X - G)}\, dX}.$$

• Standard mass fall off $a \sim \frac{1}{r}$ for $X =$cst or for $X \neq$cst with a STT satisfying the constraint $A_3 X + 4G_X - 2A_1 = 0$.

Construction of regular black hole solutions by means of the Kerr-Schild procedure

- Construction of asymptotically flat regular with a flat seed metric $h_0 = f_0 = 1$

1. The final metric obtained through the Kerr-Schild transf. reads

$$h(r) = f(r) = 1 - \frac{\mu}{r} e^{\int \frac{3(A_3 X + 4G_X - 2A_1)}{8(A_1 X - G)} dX}.$$

2. Make the following simple hypothesis,

$$\frac{3(A_3 X + 4G_X - 2A_1)}{8(A_1 X - G)} = \frac{1}{X} \Longrightarrow h(r) = f(r) = 1 - \frac{X(r)}{r}.$$

3. Fix appropriately $X(r)$ for the metric to be : asy. flat, outer event horizon and satisfy the Sakharov criterion

$$f(r) \underset{r\sim 0}{\sim} 1 - f_0 r^p, \qquad p \geq 2,$$

4. By inverse engineering $\longrightarrow$ specify the corresponding STT theory, i. e. K, G, A_1 and A_3 (as functions of X only).

Brief presentation of the Kerr-Sen solution

• The Kerr-Sen BH is an exact solution of the $4D$ low-energy heterotic string theory,

$$\begin{aligned} L &= \sqrt{-g}\Big[R - \frac{1}{2}(\partial\phi)^2 - e^{-\phi}F^2 - \frac{1}{12}e^{-2\phi}H^2\Big] \\ &= \sqrt{-g}\Big[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 - e^{-\phi}F^2\Big] + \frac{\chi}{2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}\,, \end{aligned}$$

where ϕ is the dilaton scalar field, χ is the axion pseudoscalar field dual to the $3-$form antisymmetric tensor $H = -e^{2\phi}\star d\chi$.

• The Kerr-Sen black hole solution can be expressed in the Boyer-Lindquist coordinates

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\varphi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[adt - (r^2 + 2br + a^2)d\varphi\right]^2$$

$$A = \frac{\sqrt{2mb}\,r}{\Sigma}\left(dt - a\sin^2\theta\, d\varphi\right), \quad e^{\phi} = \left(\frac{r^2 + a^2\cos^2\theta}{\Sigma}\right), \quad \chi = \frac{2ba\cos\theta}{r^2 + a^2\cos^2\theta}$$

where $\Delta = r^2 - 2mr + 2br + a^2$ and $\Sigma = r^2 + 2br + a^2\cos^2\theta$.

Brief presentation of the Kerr-Sen solution

1. For $b = 0$, the solution reduces to Kerr but there is no limit to Kerr-Newman since $A \propto b$
2. The Kerr-Sen metric is of type I in the Petrov classification (algebraically general)
3. As shown below, the Kerr-Sen metric possesses a shear-free null geodesic congruence but it is not in contradiction with the Goldberg-Sachs theorem since it is not a vacuum metric.
4. **Goldberg-Sachs Theorem** [Goldberg, J.N. and Sachs, R.K., Acta Phys. Pol., 22, 13‚Äì23, (1962).] : *A vacuum spacetime metric is algebraically special if and only if it possesses a shear-free null geodesic congruence.*

KS construction of the Kerr-Sen solution, [Work in progress with E. Ayón-Beato and D. Higuita]

- What are the difficulties to implement a KS construction

1. We have to specify the seed configuration ; not only the seed metric but also the form of the seed dilaton $\phi^{(0)}$, axion $\chi^{(0)}$ and the Maxwell potential $A^{(0)}$.
2. A priori, there is no guarantee of the existence of a shear-free null geodesic congruence.
3. What is the interpretation of the constant b. Should it be part of the seed configuration or not. If $b = 0$ from the seed configuration $\longrightarrow \phi^{(0)} = \chi^{(0)} = A^{(0)} = 0$ (useless and not in the spirit of the KS construction). Hence $b \neq 0$ and only require that $A^{(0)} = 0$ as in the 4D Kerr-Newman case and for the Maxwell-Chern-Simons in $D = 5$.

KS construction of the Kerr-Sen solution

- What are our working hypothesis

1. $K = \partial_t$ and $m = \partial_\varphi$ are two Killing fields $\longrightarrow$ same symmetry for the source, i. e. $\mathcal{L}_K(\text{fields}) = 0 = \mathcal{L}_m(\text{fields})$ where $(\text{fields}) = \{\phi, \chi, F\}$. The seed configuration $\{\phi^{(0)}, \chi^{(0)}\}$ and the Einstein EOM

$$R^{(0)}_{\mu\nu} = \frac{1}{2}\left[\partial_\mu\phi^{(0)}\partial_\nu\phi^{(0)} + e^{2\phi^{(0)}}\partial_\mu\chi^{(0)}\partial_\nu\chi^{(0)}\right] \neq 0 \Longrightarrow$$

2. The seed metric is not flat and not a vacuum metric but its Ricci tensor satisfies $R^{(0)}(K,K) = R^{(0)}(K,m) = R^{(0)}(m,m) = 0$.

KS construction of the Kerr-Sen solution

- Fixing the seed metric :

1. Carter's class of metric : metrics and electromagnetic fields in which Schrodinger's (wave's) equation is separable and other nice properties (existence of a Killing tensor) depending on $\Delta_r(r), \Delta_\theta(\theta), C_r(r), C_\theta(\theta), Z_r(r)$ and $Z_\theta(\theta)$,

$$ds^2_{(0)} = P_{r,\theta}\left[\frac{dr^2}{\Delta_r} + \frac{\sin^2\theta d\theta^2}{\Delta_\theta}\right] + \frac{1}{P_{r,\theta}}\left[\Delta_\theta\left(C_r dt - Z_r d\varphi\right)^2 - \Delta_r\left(C_\theta dt - Z_\theta d\varphi\right)^2\right],$$

   where $P_{r,\theta} = (C_\theta Z_r - C_r Z_\theta)$

2. Imposing the conditions $R^{(0)}(K,K) = R^{(0)}(K,m) = R^{(0)}(m,m) = 0 \Longrightarrow$ 2−parametric (a,b) seed metric with $b \geq 0$

$$ds^2_{(0)} = -dt^2 + \left(r^2 + a^2\cos^2\theta + 2br\right)\left[\frac{dr^2}{r^2 + a^2 + 2br} + d\theta^2\right] + \left(r^2 + a^2 + 2br\right)\sin^2\theta d\varphi^2$$

KS construction of the Kerr-Sen solution

- Fixing the seed configuration :

1. With the 2−parametric seed metric, resolve the reduced field equations ($A_\mu^{(0)}=0$) to find $\phi^{(0)}$ and $\chi^{(0)}$.
2. Complexification of the field equations with the axi-dilaton $\tau^{(0)}$

$$\tau^{(0)}=\chi^{(0)}+ie^{-\phi^{(0)}}, \qquad z=r+i\mu, \qquad \mu=a\cos\theta.$$

3. The scalar quantities $\phi^{(0)},\chi^{(0)}$ and $A_\mu^{(0)}A^{(0)\mu}$ are Kerr-Schild invariant, i. e.

$$\phi=\phi^{(0)}, \quad \chi=\chi^{(0)}, \quad g^{\mu\nu}A_\mu A_\nu=g^{(0)\mu\nu}A_\mu^{(0)}A_\nu^{(0)}=0.$$

KS construction of the Kerr-Sen solution

- Fixing the congruence : Kerr-Sen metric is of type I

Looking for a congruence with the same symmetry than the metric

$$l = dt + l_r(r,\theta)dr + l_\theta(r,\theta) + l_\varphi(r,\theta)$$

imposing the null, geodesic and shear-free conditions $\Longrightarrow$ unique congruence given by

$$l = dt + \frac{(r^2 + a^2 + 2br)}{r^2 + a^2\cos^2\theta + 2br}dr - a\sin^2\theta d\varphi$$

KS construction of the Kerr-Sen solution

We now have all the ingredients : seed metric $g^{(0)}$, the seed configuration and the appropriate congruence l

1. The final configuration is given by

$$ds^2 = ds^2_{(0)} + \frac{2rM}{r^2 + a^2\cos^2\theta + 2br} l \otimes l,$$

$$\phi = \phi^{(0)}, \quad \chi = \chi^{(0)}, \quad A = \frac{\sqrt{m\,br}}{2(r^2 + a^2\cos^2\theta + 2br)} l$$

2. The parameter b which was seen to be positive can now be interpreted as the electromagnetic charge $b \propto Q^2$ while the parameter a (as in the Kerr case) is identified with the angular momentum.