Conformal bridge transformation

Mikhail S. Plyushchay

Universidad de Santiago de Chile

Exotic symmetries and spacetimes

• *Conference in honor of Peter Horvathy* •

on the occasion of his retirement

Institut Denis Poisson, Tours

November 16, 2022

Peter A. Horvathy, MP, • "*Non-relativistic anyons, exotic Galilean symmetry and noncommutative plane*," JHEP 06 (2002) 033; • "*Anyon wave equations and the noncommutative plane*," PLB 595 (2004) 547; • "*Nonrelativistic anyons in external electromagnetic field,*" NPB 714 (2005) 269; Peter A. Horvathy, MP, M. Valenzuela, • "*Bosonized supersymmetry of anyons and supersymmetric exotic particle on the non-commutative plane,*" NPB 768 (2007) 247; • "*Bosonized supersymmetry from the Majorana-Dirac-Staunton theory, and massive higher-spin fields,*" PRD 77 (2008) 025017; P. D. Alvarez, J. L. Cortes, Peter A. Horvathy, MP, • "*Super-extended noncommutative Landau problem and conformal symmetry*," JHEP 03 (2009) 034; Peter A. Horvathy, MP, M. Valenzuela, • "*Bosons, fermions and anyons in the plane, and supersymmetry*," Ann. Phys. 325 (2010) 1931; • "*Supersymmetry of the Planar Dirac - Deser-Jackiw-Templeton System, and of Its Non-Relativistic Limit*," J.Math.Phys. 51 (2010) 092108; • "*Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons*," PRD 81 (2010) 127701

Luis Inzunza, MP (U. de Santiago de Chile), Andreas Wipf (Jena U.)

PRD **101** (2020) 105019, JHEP **04** (2020) 028,

PRD **103** (2021) 106004;
J.Phys.Conf.Ser. **2038** (2021) 012014;
JHEP **05** (2021) 165, **01** (2022) 179, **08** (2022) 228

• Peculiar properties of various classical and quantum systems can be related to/derived from those of a free particle

• Free particle $\rightarrow$ Darboux transformations $\rightarrow$ reflectionless quantum systems:

• Employing Darboux covariance of Lax pair representation of the KdV equation

•$\Longrightarrow$ Potentials of reflectionless quantum systems can be promoted to multi-soliton solutions of the classical KdV equation

(• Inverse scattering theory: KdV evolution = isospectral deformation of the corresponding multi-soliton Schrödinger potential)

• By periodization of reflectionless systems, finite-gap quantum systems can be obtained
Potentials = solutions of stationary equations of the KdV hierarchy

• Darboux covariance of Lax representation $\bullet\!\!\Longrightarrow$ cnoidal type solutions of the KdV equation

• Darboux transformations applied to finite-gap systems $\bullet\!\!\Longrightarrow$ solutions to the KdV and mKdV equations which represent soliton defects propagating in the periodic finite-gap background

Another exactly solvable system:

quantum harmonic oscillator

??? Free particle $\longrightarrow$ QHO ???

Darboux transformation cannot make this job

Conformal bridge transformation (CBT)

$1d$ Free particle: $so(2,1) \cong sl(2,\mathbb{R})$ conformal symmetry:

$$\hat{H}_0 = -\tfrac{1}{2}\tfrac{d^2}{dx^2}\,, \qquad \hat{K} = \tfrac{1}{2}x^2\,, \qquad \hat{D} = -\tfrac{i}{2}\left(x\tfrac{d}{dx} + \tfrac{1}{2}\right)$$

$$[\hat{H}_0, \hat{D}] = -i\hat{H}_0\,, \qquad [\hat{H}_0, \hat{K}] = -2i\hat{D}\,, \qquad [\hat{K}, \hat{D}] = i\hat{K}$$

$$\hat{J}_0 = \frac{1}{2}(\hat{H}_0 + \hat{K}) = \hat{H}_{osc}\,, \quad \hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2 = -\frac{1}{2}(\hat{H}_0 - \hat{K} \pm i2\hat{D})$$

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm\,, \qquad [\hat{J}_-, \hat{J}_+] = 2\hat{J}_0$$

$\hat{H}_0$ (like $\hat{D}$ and $\hat{K}$) is a non-compact $sl(2,\mathbb{R})$ generator $\Rightarrow$ continuous spectrum

Define a *non-unitary* operator

$$\hat{\mathfrak{S}} = e^{-\hat{K}} e^{i \ln 2 \cdot \hat{D}} e^{\hat{H}_0} = \exp(\hat{J}_1 - \hat{J}_0) \cdot \exp(i \ln 2 \cdot \hat{J}_2) \cdot \exp(\hat{J}_0 + \hat{J}_1)$$

It is *nonlocal* (compare with Darboux transformations generators)

Notice that

$$\hat{\mathfrak{S}} = \exp\left(\frac{\pi}{4}(\hat{H}_0 - \hat{K}_0)\right)$$

= the evolution operator of the *inverted harmonic oscillator* system taken for complex time value $t = i\pi/4$.

It generates a canonical transformation identified as the fourth order root of the space reflection operator $\mathcal{P}$,

$$\hat{\mathfrak{S}} : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (\hat{a}^+, -i\hat{a}^-, -i\hat{p}, x)$$
$$\hat{\mathfrak{S}}^2 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (-i\hat{p}, -ix, -\hat{a}^-, \hat{a}^+)$$
$$\hat{\mathfrak{S}}^4 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (-x, -\hat{p}, -\hat{a}^+, -\hat{a}^-)$$

$\Rightarrow \quad \hat{\mathfrak{S}}^8 = 1$.

It produces the non-unitary automorphism of $\mathfrak{sl}(2, \mathbb{R})$:

$$\hat{\mathfrak{S}} \hat{J}_0 \hat{\mathfrak{S}}^{-1} = i\hat{J}_2 \,, \qquad \hat{\mathfrak{S}} \hat{J}_1 \hat{\mathfrak{S}}^{-1} = -\hat{J}_1 \,, \qquad \hat{\mathfrak{S}} \hat{J}_2 \hat{\mathfrak{S}}^{-1} = -i\hat{J}_0$$

The action of the $\hat{\mathfrak{S}}^2$ on $\mathfrak{sl}(2, \mathbb{R})$ generators is like a rotation by π about J_1: $\quad \hat{\mathfrak{S}}^2 : (\hat{J}_0, \hat{J}_1, \hat{J}_2) \to (-\hat{J}_0, \hat{J}_1, -\hat{J}_2)$

• $\hat{\mathfrak{S}}$ is $\mathcal{PT}$-invariant operator: $[\mathcal{PT}, \hat{\mathfrak{S}}] = 0$

• Since $\hat{\mathfrak{S}}(2i\hat{D})\hat{\mathfrak{S}}^{-1} = 2\hat{J}_0 = \hat{H}_{\text{osc}}$, it changes the form of dynamics in the sense of Dirac

• Classical analog of $\hat{\mathfrak{S}}$ generates complex canonical transformation in the phase space: $\widetilde{x} = a^+ \in \mathbb{C}$, $\widetilde{p} = -ia^- \in \mathbb{C}$

• Under complex conjugation, $\overline{\tilde{x}} = a^- = i\tilde{p}$, $\Rightarrow$ at the quantum level we pass over from the coordinate representation to representation in which $\hat{\tilde{x}} = \hat{a}^+$ acts as the operator of multiplication by $z \in \mathbb{C}$, $\hat{a}^+\psi(z) = z\psi(z)$, while $i\hat{\tilde{p}} = \hat{a}^-$ acts as $\hat{a}^-\psi(z) = \frac{d}{dz}\psi(z)$

Replacing

$$(\psi_1, \psi_2) = \int_{-\infty}^{+\infty} \overline{\psi_1(x)} \psi_2(x) dx$$

by

$$(\psi_1, \psi_2) = \frac{1}{\pi} \int_{\mathbb{R}^2} \overline{\psi_1(z)} \psi_2(z) e^{-\bar{z}z} d^2 z, \quad d^2 z = d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)\,,$$

$\Rightarrow$ we arrive at the Fock-Bargmann representation where $\hat{a}^+ = z$, $\hat{a}^- = \frac{d}{dz}$, $(\hat{a}^+)^\dagger = \hat{a}^-$ in correspondence with the classical relation $\overline{\tilde{x}} = a^- = i\tilde{p}$

In this representation

$$2i\widehat{\widetilde{D}} = \hat{H}_{\text{osc}} = \left(z\frac{d}{dz} + \frac{1}{2}\right), \qquad \widehat{\widetilde{H}}_0 = -\frac{1}{2}\frac{d^2}{dz^2}\,, \qquad \widehat{\widetilde{K}} = \frac{1}{2}z^2$$

• ⇒ Transformed operators in the Fock-Bargmann representation can be obtained from the corresponding initial generators of conformal symmetry of the quantum free particle by a formal change of x to z.

• The change of the scalar product transmutes the non-unitary similarity transformation into the *unitary transformation from the coordinate to the holomorphic representation* for the Heisenberg algebra in correspondence with the Neumann-Stone theorem

The conformal bridge generated by $\hat{\mathfrak{S}}$ transforms

- Jordan states : x^n,

$(\hat{H}_0)^{k+1}x^{2k} = (\hat{H}_0)^{k+1}x^{2k+1} = 0$, $k = 0, 1, \ldots$, of $\hat{H}_0$ for $\underline{E = 0}$,

($x^0 = 1$ and x^1 are its physical and nonphysical eigenstates), which simultaneously are the formal eigenstates of $2i\hat{D}$ of eigenvalues $(n + 1/2)$, into eigenstates of $\hat{H}_{\text{osc}}$ of energies

$E_n = n + 1/2$: $\quad \hat{\mathfrak{S}}(x^n) \propto \psi_n(x) = C_n e^{-x^2/2} H_n(x)$

The CBT allows to define the pseudo-Hermitian inner product

$$(\phi_{n_1}, \phi_{n_2}) := \langle \phi_{n_1} | \hat{\Theta} | \phi_{n_2} \rangle = \langle \psi_{n_1} | \psi_{n_2} \rangle = \delta_{n_1 n_2} \,, \quad \hat{\Theta} = \hat{\mathfrak{S}}^2 \,,$$

which reduces to a usual scalar product for orthonormalized eigenfunctions of the harmonic oscillator ($\phi_n = C_n x^n$)

$\Longrightarrow$ The CBT is the *Dyson map* that relates the non-Hermitian Hamiltonian $2i\hat{D}$ with the harmonic oscillator Hamiltonian

Under CBT,

- Plane waves eigenstates $e^{ikx} \longrightarrow$ *coherent states* of QHO

- Gaussian wave packets of the quantum free particle $\longrightarrow$ the *squeezed states* of the QHO

- Generalizations:

- $1d$ conformal mechanics ($V(x) = g/x^2$) $\longrightarrow$ de Alfaro, Fubini, Furlan model ($V(x) = g/x^2 + \frac{1}{2}x^2$)

- $2d$ free particle $\longrightarrow$ Landau problem (based on conformal invariance of the angular momentum): $\hat{\mathfrak{S}}(2i\hat{D} - \hat{M})\hat{\mathfrak{S}}^{-1} = \hat{H}_L$

- $3d$ conformal mechanics in a monopole background $\longrightarrow$ de Alfaro, Fubini, Furlan model in $3d$ monopole background

- free particle $\longrightarrow$ QHO in the cosmic string background; $\Rightarrow$ vortices in a rotating cosmic string background

- $2d$ free particle $\longrightarrow$ exotic rotationally invariant QHO

- Some supersymmetric systems with linear and nonlinear SUSY

- Vortex dynamics and non-commutative QM (in progress)

Good health to you, Peter!,

And new creative achievements!