Eisenhart-Duval lift and plane gravitational waves

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Outline

1. Circularly polarized periodic gravitational wave
2. Bargmann framework and Pais-Uhlenbeck oscillator
3. Conserved charges of the Pais-Uhlenbeck oscillator
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Plane gravitational waves

In Brinkmann coordinates $\{X^i, U, V\}$, a pp-wave spacetime is

$$ds^2 = \delta_{ij} dX^i dX^j + 2dUdV - 2\Phi(U, X^i) dU^2, \quad i = 1, 2 \tag{1}$$

with a covariantly constant null Killing vector $\xi = \partial_V$.

- Models for gravitational radiation with the speed of light.
- Symmetry algebra of non-conformally flat pp-waves can be at most 7 dimensional[1].
- Isometries/conformal symmetries $\implies$ conserved quantities.

[1] H. Stephani et al., CUP, 2003

Plane gravitational waves

A particular class of pp-waves (1) is plane gravitational waves

$$ds^2 = d\boldsymbol{X}^2 + 2dUdV + K_{ij}(U)X^iX^jdU^2, \qquad (2)$$

where 2×2 matrix $K_{ij}(U)$ determines the profile.

- If $\mathrm{Tr(K)} = 0$, then (2) is an exact plane gravitational wave.
- Plane gravitational waves (2) are endowed with a 5-parameter isometry group[2] identified as the $2+1$ dimensional Carroll group with broken rotations[3].

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[2] H. Bondi, F.A.E. Pirani and I. Robinson, 1958

[3] J-M. Souriau, 1973
C. Duval, G. W. Gibbons, P. Horvathy and PM. Zhang, 2014

Associated Killing vector is (pulled back from BJR coordinates)

$$\begin{aligned} Y_C &= h\frac{\partial}{\partial V} + c^i \left(L_{ji}\frac{\partial}{\partial X^j} - L'_{ji}X^j\frac{\partial}{\partial V} \right) \\ &+ b_i \left(L_{jk}S^{ki}\frac{\partial}{\partial X^j} - (L_{jk}S^{ki})'X^j\frac{\partial}{\partial V} \right) \\ &\equiv hY^v + c^iY^c_i + b^iY^b_i, \end{aligned} \tag{3}$$

where $L(U)$ is a 2×2 matrix which satisfies a Sturm -Liouville (S-L) equation,

$$L''_{ij} = K_{ik}L_{kj}, \quad L^TL' = (L^T)'L, \tag{4}$$

with $\{\,\}' = \frac{d}{dU}\{\,\}$. The 2×2 matrix

$$S^{ij}(U) = \int^U a^{ij}(\tilde{U})d\tilde{U} \tag{5}$$

is the *Souriau matrix* with $a^{ij} = (L^TL)^{-1}_{ij}$.

Motion of a test particle in (2) is found with geodesic Lagrangian

$$L_B = \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = \frac{1}{2} \dot{X}^i \dot{X}^i + \dot{U}\dot{V} + \frac{1}{2} K_{ij} X^i X^j \dot{U}^2, \qquad (6)$$

where $\dot{X}^\mu = \frac{dX^\mu}{d\lambda}$ and λ is an affine parameter.

Equations of motion for X^i and U are decoupled from that of V

$$\ddot{X}^i = K_{ij} X^j \dot{U}^2, \qquad (7)$$

$$\ddot{U} = 0. \qquad (8)$$

So that we can choose U as affine parameter and express (7) as

$$(X^i)'' = K_{ij}(U) X^j, \qquad (9)$$

where $\{\}' = \frac{d}{dU}\{\}$.

- (9) describes coupled and "time" dependent oscillators ($V-$independent)
- It is another S-L equation.

Conserved quantities

Related Hamiltonian is

$$H_B = \frac{1}{2}g^{\mu\nu}P_\mu P_\nu = \frac{P_iP_i}{2} + P_UP_V - \frac{1}{2}K_{ij}(U)X^iX^jP_V^2, \tag{10}$$

where canonical momenta P_μ are

$$P_i = \dot{X}^i, \quad P_U = \dot{V} + K_{ij}X^iX^j\dot{U}, \quad P_V = \dot{U}. \tag{11}$$

Carroll symmetry (3) yields 5 conserved charges along the geodesic motion,

$$\begin{aligned}\mathcal{Q}_C &= hP_V + c^i\left(P_jL_{ji} - P_VX^jL'_{ji}\right) + b_i\left(P_jL_{jk}S^{ki} - P_VX^j(L_{jk}S^{ki})'\right) \\ &\equiv hQ^v + c^i\mathcal{Q}^c_i + b^i\mathcal{Q}^b_i,\end{aligned} \tag{12}$$

with

$$\{\mathcal{Q}_C, H_B\} = 0. \tag{13}$$

Non-vanishing Poisson bracket are

$$\{\mathcal{Q}^c_i, \mathcal{Q}^b_j\} = \delta_{ij}\mathcal{Q}^v = \delta_{ij}P_V. \tag{14}$$

Circularly polarized periodic GW

Circularly polarized periodic gravitational wave (CPP GW):

$$
\begin{aligned}
ds^2 &= d\boldsymbol{X}^2 + 2dUdV + K_{ij}(U)X^iX^jdU^2 \\
K_{ij}(U)X^iX^j &= \frac{A_0}{2}\Big(\cos(\omega U)\big((X^1)^2-(X^2)^2\big) + 2\sin(\omega U)X^1X^2\Big)dU^2,
\end{aligned}
$$

with $A_0 > 0$ amplitude and ω frequency.

- It is a vacuum solution i.e., $R_{\mu\nu} = 0$.
- It is maximally symmetric i.e., it has a $7-$d conformal algebra.
- In addition to Carroll (3), it has a screw isometry

$$Y_s = \partial_U + \frac{\omega}{2}\epsilon_{ij}X^i\partial_j. \tag{16}$$

- Together with the homothetic vector field

$$Y_h = 2V\partial_V + X^i\partial_i, \quad \mathcal{L}_{Y_h}g_{\mu\nu} = 2g_{\mu\nu}, \tag{17}$$

  it enjoys a 7-parameter conformal algebra

$$[Y_i^c, Y_j^b] = -\delta_{ij}Y^v, \quad [Y_h, Y_C] = -Y_C - hY^v. \tag{18}$$

Rotating coordinates

To solve geodesic equations (9), we switch to rotating coordinates $Y^i = (R^{-1})_{ij} X^j$

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \begin{pmatrix} \cos(\omega U/2) & \sin(\omega U/2) \\ -\sin(\omega U/2) & \cos(\omega U/2) \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \tag{19}$$

Then (15) becomes

$$ds^2 = d\mathbf{Y}^2 + 2dUdV + \omega\epsilon_{ij} Y^i dY^j dU + \left(\Omega_+^2 (Y^1)^2 + \Omega_-^2 (Y^2)^2\right) dU^2, \tag{20}$$

with

$$\Omega_\pm^2 = \frac{\omega^2}{4} \pm \frac{A_0}{2}. \tag{21}$$

Now, Carroll symmetries (3) become

$$\begin{aligned} Y_C &= h\frac{\partial}{\partial V} + c^i \left(L_{ji} R^{jk} \frac{\partial}{\partial Y^k} - L'_{ji} R^{jk} Y^k \frac{\partial}{\partial V} \right) \\ &+ b_i \left(L_{jk} S^{ki} R^{jm} \frac{\partial}{\partial Y^m} - (L_{jk} S^{ki})' R^{jm} Y^m \frac{\partial}{\partial V} \right), \end{aligned} \tag{22}$$

Beginning with an affinely parametrized Lagrangian $L_{4d}^{rot} = \frac{1}{2} g_{\mu\nu} \dot{Y}^{\mu} \dot{Y}^{\nu}$, we obtain its Hamiltonian as

$$H_{4d}^{rot} = \frac{(P_i + \frac{\omega}{2}\epsilon_{ij} Y^j P_V)^2}{2} + P_U P_V - \frac{1}{2}\big(\Omega_+^2 (Y^1)^2 + \Omega_-^2 (Y^2)^2\big) P_V^2, \tag{23}$$

with canonical momenta P_μ

$$\begin{aligned} P_i &= \dot{Y}^i - \frac{\omega}{2}\epsilon^{ij} Y^j \dot{U}, \quad P_V = \dot{U} \\ P_U &= \dot{V} + \frac{\omega}{2}\epsilon^{ij} Y^i \dot{Y}^j + \big(\Omega_+^2 (Y^1)^2 + \Omega_-^2 (Y^2)^2\big) \dot{U} \end{aligned} \tag{24}$$

P_V is a constant. It is one of the conserved Carroll charges $\mathcal{Q}_C$

$$\dot{\mathcal{Q}}_C = \{\mathcal{Q}_C, H_{4d}^{rot}\} = 0, \tag{25}$$

obtained from Y_C via $\partial_\mu \to P_\mu$.
Y_s becomes

$$Y_s = \partial_U, \tag{26}$$

implying the U-independence of (20). Thus, P_U is also conserved.

Non-relativistic dynamics

Considering null geodesics, we project to the underlying NR space

$$H_{4d}^{rot} = \frac{g_{rot}^{\mu\nu} P_\mu P_\nu}{2} \equiv 0, \tag{27}$$

and obtain a lower dimensional and NR Hamiltonian

$$H_{NR}^{rot} = \frac{(P_i + \frac{M\omega}{2}\epsilon_{ij} Y^j)^2}{2M} - \frac{M}{2}\left(\Omega_+^2 (Y^1)^2 + \Omega_-^2 (Y^2)^2\right) = -P_U. \tag{28}$$

U becomes Newtonian time and $P_V = M$ becomes the NR mass.

In our specific problem Hamilton equations are found to be

$$(Y^i)' = \frac{P_i + \frac{M\omega}{2}\epsilon_{ij}Y^j}{M}, \tag{29a}$$

$$(P_i)' = \frac{\omega}{2}\epsilon_{ij}(P_j + \frac{M\omega}{2}\epsilon_{jk}Y^k) + M(\Omega_+^2 Y^1\delta_{i,1} + \Omega_-^2 Y^2\delta_{i,2}). \tag{29b}$$

- From (29a) we obtain two coupled, second order Euler - Lagrange equations

$$(Y^1)'' - \omega(Y^2)' - \Omega_+^2 Y^1 = 0 \tag{30a}$$

$$(Y^2)'' + \omega(Y^1)' - \Omega_-^2 Y^2 = 0, \tag{30b}$$

  with *time-*U *independent* coefficients.
- The dynamics generated by H_{NR}^{rot} (28) is a combination of oscillators plus a Coriolis force.
- H_{NR}^{rot} is conserved as implied by the screw symmetry (26).

Motivation

- A 4-dimensional pp-wave metric (1) can be viewed as the Bargmann space[4] for a NR system in 2 dimensions.
- Conversely, the NR motion can be Eisenhart-Duval lifted to (1) as null geodesics.
- Symmetries of Schroedinger-Newton equation[5], NR spinors coupled to Chern-Simons gauge fields[6], and cosmology[7]
- Periodic waves $\leftrightarrow$ Ion traps[8].
- Underlying NR system for $4-$d CPP GW?
- Role of Caroll symmetry?

[4] L. P. Eisenhart 1928,
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, 1985
[5] C. Duval and S. Lazzarini, 2015
[6] C. Duval, P. Horvathy and L. Palla, 1995
[7] M. Cariglia, A. Galajinsky, G. W. Gibbons and P. A. Horvathy, 2018
[8] PM Zhang et. al. 2018

Before solving them, let us make an important observation.

Equations (30) can be cast into the form of a *4th order differential equation* for either of the components :

$$(Y^i)'''' + (\Omega_+^2 + \Omega_-^2)(Y^i)'' + \Omega_+^2 \Omega_-^2 Y^i = 0, \qquad (31)$$

which is the 1 dimensional Pais-Uhlenbeck oscillator[9].

Thus we conclude that, in rotating coordinates, *the underlying NR system of a CPP GW is actually* 1*- dimensional PU oscillator (31)*.

Lets analyze this result by relating H_{NR}^{rot} to the PU Hamiltonian.

[9] A. Pais and G. E. Uhlenbeck, 1950

Chiral decomposition

Equations (30) can be solved by *chiral decomposition*. The related symplectic structure is

$$\begin{aligned} \sigma &= d\Pi_1 \wedge dY^1 + d\Pi_2 \wedge dY^2 + \omega M dY^1 \wedge dY^2, && (32) \\ H_{NR}^{rot} &= \frac{\mathbf{\Pi}^2}{2M} - \frac{1}{2} M(\Omega_+^2 (Y^1)^2 + \Omega_-^2 (Y^2)^2), && (33) \end{aligned}$$

where the $\Pi_i = P_i + \frac{M\omega}{2}\epsilon_{ij} Y^j$ are the kinematical momenta. Chiral decomposition is based on finding "good coordinates". One set of such coordinates is $(X_+^{1,2},\ X_-^{1,2})$ such that

$$\Pi_1 = \frac{M\omega}{2}\left(X_+^2 + \frac{4\Omega_-^2}{\omega^2} X_-^2\right), \quad \Pi_2 = -\frac{M\omega}{2}\left(\frac{4\Omega_+^2}{\omega^2} X_+^1 + X_-^1\right), \tag{34a}$$

$$Y^1 = X_+^1 + X_-^1, \quad Y^2 = X_+^2 + X_-^2. \tag{34b}$$

Then both the symplectic form σ (32) and the Hamiltonian H_{NR}^{rot} (33) are decomposed into separate $\pm$ sectors,

$$\sigma = -\frac{M(\Omega_+^2 - \Omega_-^2)}{\sqrt{2(\Omega_+^2 + \Omega_-^2)}}(dX_+^1 \wedge dX_+^2 - dX_-^1 \wedge dX_-^2), \tag{35}$$

$$\begin{aligned} H_{NR}^{rot} &= \frac{M(\Omega_+^2 - \Omega_-^2)}{2}\left(\frac{\Omega_+^2}{\Omega_+^2 + \Omega_-^2}(X_+^1)^2 + \frac{1}{2}(X_+^2)^2\right) \\ &- \frac{M(\Omega_+^2 - \Omega_-^2)}{2}\left(\frac{1}{2}(X_-^1)^2 - \frac{\Omega_-^2}{\Omega_+^2 + \Omega_-^2}(X_-^1)^2\right). \end{aligned} \tag{36}$$

We can easily find the equations of motion as

$$(X_\pm^{1,2})'' + \Omega_\pm^2 X_\pm^{1,2} = 0, \tag{37}$$

which are *simple harmonic motions with* $\Omega_\pm$.

The solutions for $Y^{1,2}$ are found as,

$$Y^1(U) = A_1 \cos(\Omega_+ U) + B_1 \sin(\Omega_+ U) + C_1 \cos(\Omega_- U) + D_1 \sin(\Omega_- U), \tag{38a}$$

$$Y^2(U) = -\Omega_+ \left(\frac{2}{\Omega_+^2 + \Omega_-^2}\right)^{\frac{1}{2}} \big(A_1 \sin(\Omega_+ U) - B_1 \cos(\Omega_+ U)\big) - \frac{1}{\Omega_-}\left(\frac{\Omega_+^2 + \Omega_-^2}{2}\right)^{\frac{1}{2}} \big(C_1 \sin(\Omega_- U) - D_1 \cos(\Omega_- U)\big), \tag{38b}$$

- A_1, B_1, C_1, D_1 are constants to be determined.
- $Y^{1,2}$ (38) also solve the 4th order PU equation (31).
- The geodesics of the original Brinkmann metric (15) can be found via (19).
- We emphasize that σ (35) and H_{NR}^{rot} (36) never vanish since $\Omega_+ > \Omega_-$.

Pais-Uhlenbeck oscillator

The following redefinition

$$x^1 = \sqrt{\frac{M(\Omega_+^2 - \Omega_-^2)}{\Omega_+^2 + \Omega_-^2}} X_+^1, \quad p_1 = \sqrt{\frac{1}{2} M(\Omega_+^2 - \Omega_-^2)} X_+^2, \tag{39a}$$

$$x^2 = \sqrt{\frac{M(\Omega_+^2 - \Omega_-^2)}{\Omega_+^2 + \Omega_-^2}} X_-^2, \quad p_2 = \sqrt{\frac{1}{2} M(\Omega_+^2 - \Omega_-^2)} X_-^1, \tag{39b}$$

allows us to pass to *canonical coordinates* $\{x^i, p_i\}$ such that

$$H_{NR}^{rot} \equiv H_{o\pm} = \frac{p_1^2 + \Omega_+^2 (x^1)^2}{2} - \frac{p_2^2 + \Omega_-^2 (x^2)^2}{2}, \quad \sigma = dp_i \wedge dx^i. \tag{40}$$

$H_{o\pm}$ consists of two uncoupled harmonic oscillators but *with a relative minus sign* between them. Hamilton equations of motion:

$$(x^1)' = p_1, \quad (p_1)' = -\Omega_+^2 x^1, \qquad (x^2)' = -p_2, \quad (p_2)' = \Omega_-^2 x^2. \tag{41}$$

A final canonical transformation[10]

$$x^1 = \frac{p_x + \Omega_+^2 v}{\Omega_+ \sqrt{(\Omega_+^2 - \Omega_-^2)}}, \quad p_1 = \frac{\Omega_+(p_v + \Omega_-^2 x)}{\sqrt{(\Omega_+^2 - \Omega_-^2)}}, \tag{42a}$$

$$x^2 = \frac{p_v + \Omega_+^2 x}{\sqrt{(\Omega_+^2 - \Omega_-^2)}}, \quad p_2 = \frac{p_x + \Omega_-^2 v}{\sqrt{(\Omega_+^2 - \Omega_-^2)}}, \tag{42b}$$

yields the Ostrogradsky Hamiltonian for the PU oscillator,

$$H_{o\pm} = H_{PU} = p_x v + \frac{p_v^2}{2} + \frac{(\Omega_+^2 + \Omega_-^2)v^2}{2} - \frac{\Omega_+^2 \Omega_-^2 x^2}{2}. \tag{43}$$

- Therefore, the underlying NR Hamiltonian (28) can also be mapped to the PU Hamiltonian (43).
- Put differently: if we begin with PU Hamiltonian H_{PU} (43) and use the inverse of (42), (39) and (34), we end up with H_{NR}^{rot} (28). Unrotating with inverse of (19) and ED lifting it, we obtain (15).
- Thus, *the PU oscillator is the NR system of the CPP GW*.

[10] A. Smilga, 2017

We note the associated Lagrangian

$$L_{PU} = \frac{1}{2}\Big(x''^2 - (\Omega_+^2 + \Omega_-^2)x'^2 + \Omega_+^2\Omega_-^2 x^2\Big), \tag{44}$$

and the equation of motion

$$\Big(\frac{d^4}{du^4} + (\Omega_+^2 + \Omega_-^2)\frac{d^2}{du^2} + \Omega_+^2\Omega_-^2\Big)x = 0, \tag{45}$$

cf. (31) with $u = U$ and $\Omega_+ > \Omega_-$. The PU oscillator phase space is 4 dimensional with coordinates $\{x, p_x, v, p_v\}$ where $v = x'$. The definition of the canonical momenta ,

$$p_v = \frac{\partial L_{PU}}{\partial x''}, \quad p_x = \frac{\partial L_{PU}}{\partial x'} - \frac{d}{dt}\frac{\partial L_{PU}}{\partial x''} = \frac{\partial L_{PU}}{\partial x'} - {p_v}', \tag{46}$$

yields the Ostrogradsky Hamiltonian (43). The Hamilton equations of motion are

$$x' \equiv \frac{\partial H_{PU}}{\partial p_x} = v, \quad v' \equiv \frac{\partial H_{PU}}{\partial p_v} = p_v, \tag{47a}$$

$$p_x{}' \equiv -\frac{\partial H_{PU}}{\partial x} = \Omega_+^2\Omega_-^2 x, \quad p_v{}' \equiv -\frac{\partial H_{PU}}{\partial v} = -p_x - (\Omega_+^2 + \Omega_-^2)v. \tag{47b}$$

Conserved charges for PU

- CPP GW (15) has a 7-dimensional conformal algebra.
- The isometries are generated by the 5 dimensional Carroll algebra (3).
- We also have the screw isometry (16) and a homothetic vector field (17).
- All these yield conserved quantities for a null geodesic in the CPP GW background.
- We will derive conserved charges of the PU oscillator from those of a spinless test particle in a CPP GW background.
- Note that these charges will have an explicit time dependence,

$$\frac{d\mathcal{Q}}{dU} = \frac{\partial \mathcal{Q}}{\partial U} + \{\mathcal{Q}, H_{NR}\} = 0. \tag{48}$$

Screw charge

- In Brinkmann coordinates (15), the screw charge is

$$\mathcal{Q}_s = P_U + \frac{\omega}{2}\epsilon_{ij}X^i P_j\,, \tag{49}$$

  where P_U and P_i are the canonical momenta.

- In rotating coordinates, it becomes just P_U.
- When projected, it corresponds to (minus) the *total conserved energy* of the NR system H_{NR}^{rot}.
- After chiral decomposition it becomes $\mathcal{Q}_s = -H_{o\pm}$.
- Finally, performing a second transformation (42), it yields the conserved energy of the PU oscillator up to a minus sign,

$$\mathcal{Q}_s = -H_{PU}. \tag{50}$$

Homothetic charge

- The homothetic vector field (17) does not change in rotating coordinates.
- When projected, it becomes

$$\mathcal{Q}_h = Y^i P_i + 2MV, \quad V(U) = V_0 - \frac{1}{M}\int^U L_{NR}\, d\tilde{U}, \quad (51)$$

with L_{NR} being the associated NR Lagrangian

$$L_{NR}^{rot} = \frac{1}{2}M(\boldsymbol{Y}')^2 - \frac{M\omega}{2}\epsilon_{ij}(Y^i)'Y^j + \frac{M}{2}\big(\Omega_+^2(Y^1)^2 + \Omega_-^2(Y^2)^2\big) \quad (52)$$

and V_0 is a constant.

- For various physical systems it is related to *virial theorem* [11].

[11] P.M. Zhang, M. Cariglia, M. E. and P. Horvathy, 2020

In PU-phase space $\{x, v, p_x, p_v\}$ it consists of two parts which are separately conserved

$$\mathcal{Q}_h = \mathcal{Q}_{h1} + \mathcal{Q}_{h2} \tag{53}$$

While $\mathcal{Q}_{h2} = 0$ is trivial, the virial charge $\mathcal{Q}_{h1}$ (54)

$$\mathcal{Q}_{h1} = p_x x + p_v v - 2 \int d\tilde{u}\ L_{PU}, \tag{54}$$

originates from the scaling symmetry of the PU action (44)

$$x \to \lambda x, \quad L_{PU}\ dt \to \lambda^2 L_{PU}\ dt, \tag{55}$$

where λ is a constant scale factor.

Carroll

In rotating coordinates (20), the S-L problem (4) is modified as

$$O''_{ij} = \omega\epsilon_{ik}O'_{kj} + \Omega_{ik}O_{kj}, \tag{56}$$

where

$$\Omega = \begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix}. \tag{57}$$

Above 2×2 $O(U)$ matrix is defined via the rotation in (19)

$$O_{ij} = (R^{-1})_{ik}L_{kj}. \tag{58}$$

On the other hand, the subsidiary condition $L^TL' = (L^T)'L$ turns out to be

$$\big(O^TO' - (O^T)'O\big)_{ij} = \omega O^T_{ik}\epsilon_{km}O_{mj}, \tag{59}$$

which, in explicit form, becomes

$$O_{11}O'_{12} - O_{12}O'_{11} + O_{21}O'_{22} - O_{22}O'_{21} = \omega \det O. \tag{60}$$

Using the O-matrix (58), the projected Carroll charges in rotating coordinates become

$$\begin{aligned}\mathcal{Q}_C &= hM + c^i\left(P_kO_{ki} - MY^k(O'_{ki} - \frac{\omega}{2}\epsilon_{kj}O_{ji})\right)\\ &+ b_i\left(P_jO_{jk}S^{ki} - MY^j(O'_{jk}S^{ki} - \frac{\omega}{2}\epsilon_{jm}O_{mk}S^{ki} + O^{-1}_{ij})\right)\\ &\equiv h\mathcal{Q}_V + c^i\mathcal{Q}^c_i + b^i\mathcal{Q}^b_i. \qquad (61)\end{aligned}$$

We note that (56) has the same structure with (30). Thus, we may derive O_{ij} by using the solutions (38) with

$$O = \begin{pmatrix} Y^1 & \tilde{Y}^1 \\ Y^2 & \tilde{Y}^2 \end{pmatrix} \qquad (62)$$

where $\tilde{Y}^i$ are of the same form as Y^i but with different coefficients A_2, B_2, C_2, D_2.

When projected, P_V becomes the NR mass, i.e., $\mathcal{Q}_V = M$, a constant in PU phase space.

In PU coordinates $\{x, v, p_x, p_v\}$, $\mathcal{Q}_i^c$ and $\mathcal{Q}_i^b$ become

$$\begin{aligned}
\mathcal{Q}_i^{c,b} &= \Omega_+\Big(\big(2\Omega_+^3+\sqrt{2(\Omega_+^2+\Omega_-^2)}\Omega_-^2\big)A_{1i}+\big(2\Omega_-^2+\sqrt{2(\Omega_+^2+\Omega_-^2)}\Omega_+\big)A'_{2i}\Big)x \\
&- \frac{1}{\Omega_+}\Big(\big(2\Omega_-^2+\sqrt{2(\Omega_+^2+\Omega_-^2)}\Omega_+\big)A_{2i}-\big(2\Omega_++\sqrt{2(\Omega_+^2+\Omega_-^2)}\big)A'_{1i}\Big)p_x \\
&- \Big(\big(2\Omega_-^2\Omega_++\sqrt{2(\Omega_+^2+\Omega_-^2)}\Omega_-^2\big)A_{2i}-\big(2\Omega_-^2+\sqrt{2(\Omega_+^2+\Omega_-^2)}\Omega_+\big)A'_{1i}\Big)v \\
&+ \Big(\big(2\Omega_++\sqrt{2(\Omega_+^2+\Omega_-^2)}\big)(\Omega_+A_{1i}+A'_{2i})\Big)p_v, \qquad (63)
\end{aligned}$$

where

$$A_{ji} = O_{ji}, \qquad A'_{ji} = O'_{ji} \qquad \text{for} \quad \mathcal{Q}_i^c, \qquad (64)$$

$$A_{ji} = O_{jk}S^{ki}, \qquad A'_{ji} = O'_{jk}S^{ki} + O_{ij}^{-1} \qquad \text{for} \quad \mathcal{Q}_i^b. \qquad (65)$$

- (63) are conserved via the Hamilton equations (47) when (56) and (59) are satisfied.
- This generic form is complicated if the O-matrix (62) is directly substituted.
- We simplify it by using the initial condition

$$(X^i)'(0) = 0, \tag{66}$$

of the original 4 dimensional problem (30).

- This initial condition implies

$$O(0) = 1_{2\times 2}, \quad O'(0) = \frac{\omega}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{67}$$

and simplifies the original complicated O-matrix

$$O = \begin{pmatrix} \cos(\Omega_- u) & \frac{\omega}{2\Omega_+}\sin(\Omega_+ u) \\ -\frac{\omega}{2\Omega_-}\sin(\Omega_- u) & \cos(\Omega_+ u) \end{pmatrix}, \quad \omega = \sqrt{2(\Omega_+^2 + \Omega_-^2)}. \tag{68}$$

Projected Carroll charges become

$$\mathcal{Q}_1^c = -\sqrt{\frac{M}{2}}\left[\cos(\Omega_- u)\left(\Omega_+^2 x + p_v\right) + \sin(\Omega_- u)\left(\frac{p_x}{\Omega_-} + \Omega_- v\right)\right], \tag{69a}$$

$$\mathcal{Q}_2^c = -\sqrt{\frac{M}{2}}\left[\cos(\Omega_+ u)\left(\frac{p_x}{\Omega_+} + \Omega_+ v\right) - \sin(\Omega_+ u)\left(\Omega_-^2 x + p_v\right)\right], \tag{69b}$$

$$\mathcal{Q}_1^b = -\frac{\sqrt{2M}}{(\Omega_+^2 - \Omega_-^2)}\left[\cos(\Omega_- u)\left(\Omega_-^2 v + p_x\right) - \Omega_- \sin(\Omega_- u)\left(p_v + \Omega_+^2 x\right)\right] \tag{69c}$$

$$\mathcal{Q}_2^b = -\frac{\sqrt{2M}}{(\Omega_+^2 - \Omega_-^2)}\left[\Omega_+ \cos(\Omega_+ u)\left(\Omega_-^2 x + p_v\right) + \sin(\Omega_+ u)\left(\Omega_+^2 v + p_x\right)\right] \tag{69d}$$

We emphasize that $(\mathcal{Q}_i^c, \mathcal{Q}_i^b)$ are conserved in PU phase space (47), i.e.,

$$\frac{d\mathcal{Q}}{du} = \frac{\partial \mathcal{Q}}{\partial u} + \{\mathcal{Q}, H_{PU}\} = 0. \tag{70}$$

In PU phase space, the relation between $\mathcal{Q}_i^c$ and $\mathcal{Q}_i^b$ becomes similar to EM duality (rotation). For instance

$$\cos(\Omega_- u) \to -\frac{2\Omega_-}{(\Omega_+^2 - \Omega_+^2)} \sin(\Omega_- u), \;\; \sin(\Omega_- u) \to \frac{2\Omega_-}{(\Omega_+^2 - \Omega_+^2)} \cos(\Omega_- u) \tag{71}$$

cf. (12).

Non-vanishing Poisson brackets of M, $\mathcal{Q}_i^c$ and $\mathcal{Q}_i^b$ (69)

$$\{\mathcal{Q}_i^c, \mathcal{Q}_j^b\} = M\delta_{ij}. \tag{72}$$

Lastly, Poisson brackets of the Carroll charges with the screw charge $\mathcal{Q}_s$ are found to be

$$\{\mathcal{Q}_{1,2}^c\,, H_{PU}\} = \pm\frac{(\Omega_+^2 - \Omega_-^2)}{2}\mathcal{Q}_{1,2}^b\,, \quad \{\mathcal{Q}_{1,2}^b\,, H_{PU}\} = \mp\frac{2\Omega_\mp^2}{(\Omega_+^2 - \Omega_-^2)}\mathcal{Q}_{1,2}^c\,.$$

Elements of Newton-Hooke group "downstairs".

Discussion

- Relation between two distant systems:
  PU oscillator is the underlying non-relativistic system of 4-dimensional CPP GW.

- Conversely, when the PU oscillator is ED lifted, it becomes a CPP GW endowed with a Carroll symmetry.

- Conserved quantities:
  We derive conserved charges for the PU oscillator for generic frequencies from the symmetries of the gravitational wave.

- Comparison with literature:
  In line with Andrzejewski[12] *for generic frequencies* $\Omega_+ > \Omega_-$.
  Adopting $\Omega_+ = 3\Omega_-$, $\mathcal{Q}_i^{c,b}$ *yields translation, boost and two accelerations*[13], *elements of N-H group.*

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[12] K. Andrzejewski, 2014
[13] Andrzejewski et al., 2014

- There are some similar systems: Penning trap, Lukash plane wave, em. vortex etc. In which manner, the systems can be described as a higher order theory? CPP may not be the only example.
- Perhaps, correspondence between CPP GW and PU oscillator can be built entirely "upstairs" to link it to the work[14].
- Coordinate transformations from B to BJR provides a link between different NR systems.
- Any other physical systems ? :
  Electromagnetic vortex and CPP GW via double copy[15]*, see* [16]
- Most general version of Eisenhart-Duval lift and its relation to double copy etc.

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[14] A. Galajinsky and I. Masterov, 2016

[15] A. Ilderton 2018

[16] K. Andrzejewski and S. Prencel, 2018