Extended Carroll symmetries and application on black holes

Loïc Marsot

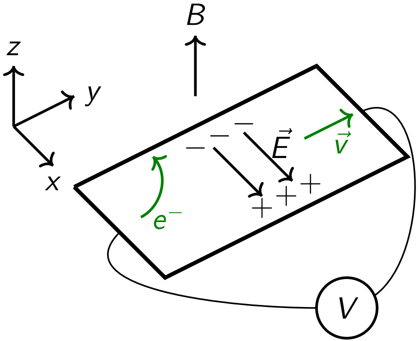
Aix-Marseille Université

Based on
L. M., J. Geom. Phys. **179** (2022), arXiv: 2110.08489,
L. M., PM. Zhang, P. Horvathy, arXiv: 2207.06302,
and w.i.p with PM. Zhang and P. Horvathy

The Hall Effect

Charges moving under a magnetic field B create an electric field.

An equilibrium is reached when the force of the electric field compensates that of the magnetic field.



Equilibrium condition:

$$v_y = \frac{E_x}{B_z}$$

Charge does not appear in the eom. but is needed.

The Carroll group

The Carroll group, “cousin” of the Galilei group, was originally introduced as a new contraction ($c \to 0$) of the Poincaré group [Lévy-Leblond, ’65].

Both the Galilei and Carroll groups act on spacetime:

- For the Galilei group,

$$\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} R\boldsymbol{x} + \boldsymbol{b}t + \boldsymbol{c} \\ t + e \end{pmatrix}$$

- For the Carroll group,

$$\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix} \mapsto \begin{pmatrix} R\boldsymbol{x} + \boldsymbol{c} \\ s - \langle \boldsymbol{b}, R\boldsymbol{x} \rangle + f \end{pmatrix},$$

  with s Carrollian “time”.

$\Rightarrow$ They differ by the way boosts act.

Contractions of the Poincaré algebra

Let us compare the Poincaré, Galilei, and Carroll algebras,

Poincaré	Galilei	Carroll
$[J_i, J_j] = \epsilon_{ijk} J_k,$	$[J_i, J_j] = \epsilon_{ijk} J_k,$	$[J_i, J_j] = \epsilon_{ijk} J_k,$
$[J_i, K_j] = \epsilon_{ijk} K_k,$	$[J_i, K_j] = \epsilon_{ijk} K_k,$	$[J_i, K_j] = \epsilon_{ijk} K_k,$
$[K_i, K_j] = -\epsilon_{ijk} J_k,$	$[K_i, K_j] = 0,$	$[K_i, K_j] = 0,$
$[J_i, P_j] = \epsilon_{ijk} P_k,$	$[J_i, P_j] = \epsilon_{ijk} P_k,$	$[J_i, P_j] = \epsilon_{ijk} P_k,$
$[K_i, P_j] = \delta_{ij} P_0,$	$[K_i, P_j] = 0,$	$[K_i, P_j] = \delta_{ij} P_0,$
$[J_i, P_0] = 0,$	$[J_i, P_0] = 0,$	$[J_i, P_0] = 0,$
$[K_i, P_0] = P_i,$	$[K_i, P_0] = P_i,$	$[K_i, P_0] = 0,$
$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$
$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$

Physical applications for the Carroll group

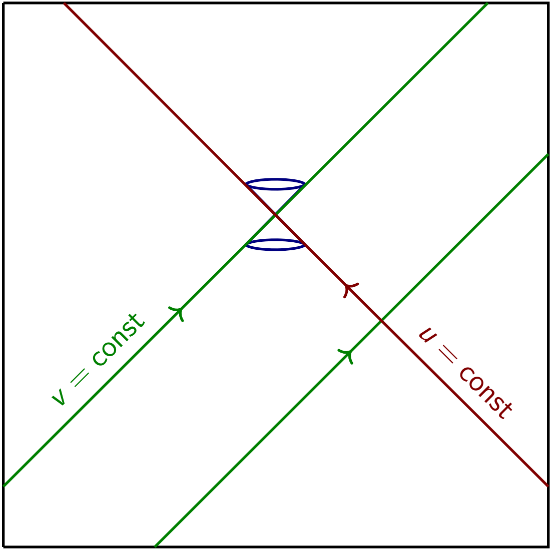
The Carroll group was originally dismissed as non physical, as people thought it would feature no motion.

Physical applications for the Carroll group were found much later, in General Relativity. See related topics,

- BMS symmetry
- Flat holography
- Black hole horizons
- Wave front of gravitational waves

Carroll physics take place on null hypersurfaces in General Relativity.

Naïve restriction to a null hypersurface



Equations of motion,

$$\dot{X}^\mu = P^\mu$$
$$\dot{P}^\mu = 0$$

Take $P = (0, 0, 1, 1)$, and use null coordinates (u, v),

$$u = \frac{t+z}{2}\,, \quad v = \frac{t-z}{2}\,.$$

The eom become,

$$\frac{dx^i}{du} = 0, \quad \frac{dv}{du} = 0\,.$$

Methods to obtain Carroll dynamics

There are three methods to obtain Carroll dynamics:

- Induce the motion of a bulk theory on a Carroll geometry ;
- Take the $c \to 0$ limit of a relativistic theory ;
- Work intrinsically with group theory.

Dynamical study from group theory

Studying the symmetry group of the eom allows us to:

- Classify elementary particles
- Obtain free equations of motion

Carroll elementary particles are classified by two numbers:

- mass m ;
- spin ℓ.

The eom can be obtained using, *e.g.*, the KKS (coadjoint orbits) method.

"Standard" Carrollian dynamics

- Free Carroll particles do not move [C. Duval, et. al., '14],

$$\frac{d\boldsymbol{x}}{ds} = 0 \qquad \& \qquad \frac{d\boldsymbol{p}}{ds} = 0$$

Note that we do not have $\frac{d\boldsymbol{x}}{ds} = \frac{\boldsymbol{p}}{m}$.

- However, Carrollian massless particles with charge do move in a background EM field [L. M., PM. Zhang, P. Horvathy., 'soon],

$$\frac{dx^i}{ds} = \epsilon^{ij}\frac{E^j}{B}$$

$\Rightarrow$ same equation as for Hall motion.

Carrollian motion can be non-trivial.

Carroll Hall motion

We show that Carroll motion is possible, however at the cost that massless particles need to be charged.

$\Rightarrow$ Could one obtain similar non trivial motion without electric charge?

Our clue: the discovery of anyons (see H. Bartolomei, et. al., Science **368** (2020)), featuring an infinite amount of spin statistics.

Effective dynamics can be very hard to infer from bulk dynamics. However, anyons in condensed matter can be described using group theory.

Are there more general Carroll particles and motion?
$\Rightarrow$ Yes, for planar ($2+1$-dimensional) systems.

The subtlety of central extensions on dynamical systems

Central extensions parametrize the obstruction to realize a representation of a group on *e.g.* phase space, and wavefunctions.

- Representations in Quantum Mechanics are often *projective*,

$$U_g U_{g'} = e^{i\xi(g,g')} U_{gg'}$$

- A Lagrangian may be invariant under a group, up to a total derivative,

$$L(g \cdot q, (g \dot{\cdot} q)) = L(q, \dot{q}) + \frac{d}{ds}\alpha(g; q(s))$$

The different classes of ξ and α are parametrized by some constant m.

The QM representation and the invariance of the Lagrangian become exact for the *central extension* $G \ltimes \mathbb{R}$ of G.

For $h \equiv (g, \theta) \in G \ltimes \mathbb{R}$, we have

$$(g, \theta) \cdot (g', \theta') = (gg', \theta + \theta' + \xi(g, g'))$$

Example: central extension of the Galilei group

Action on wavefunctions

$$U_{g'} U_g = e^{\frac{im}{\hbar}\left(\frac{1}{2}\|\boldsymbol{b}'\|^2 e + \boldsymbol{b}' \cdot R' \boldsymbol{c}\right)} U_{g'g}$$

Action on Lagrangian

$$L(g \cdot q, (g \dot{\cdot} q)) = L(q, \dot{q}) + m \frac{d}{ds} \left(\frac{1}{2} \|\boldsymbol{b}\|^2 t + \boldsymbol{b} \cdot R \boldsymbol{x} \right)$$

Depends on the parameter m: the mass

The representation and invariance become exact under the 11-dimensional Bargmann group, $\mathrm{Barg} = \mathrm{Gal} \ltimes \mathbb{R}$.

The extension parameter becomes a new moment, and a new Casimir invariant.

Carrollian central extensions

The amount of non-trivial central extensions depends on the dimension:

	Dimension of non-trivial central extensions	
spatial dimension	Galilei	Carroll
$d \geq 3$	1	0
$d = 2$	2	2

The Carroll group has a richer structure in spatial dimension 2.

Contractions of the Poincaré algebra, $2+1$ dimensions

The extended Galilei algebra can be found in [Lévy-Leblond, '72], and the extended Carroll algebra was computed in [de Azcarraga, et. al. '98; Ngendakumana, et. al. '14].

Poincaré	Extended Galilei	Extended Carroll
$[J_3, K_i] = \epsilon_{ij} K_j,$	$[J_3, K_i] = \epsilon_{ij} K_j,$	$[J_3, K_i] = \epsilon_{ij} K_j,$
$[K_i, K_j] = -\epsilon_{ij} J_3,$	$[K_i, K_j] = \epsilon_{ij} A_{exo},$	$[K_i, K_j] = \epsilon_{ij} A_{exo},$
$[J_3, P_i] = \epsilon_{ij} P_j,$	$[J_3, P_i] = \epsilon_{ij} P_j,$	$[J_3, P_i] = \epsilon_{ij} P_j,$
$[K_i, P_j] = \delta_{ij} P_0,$	$[K_i, P_j] = \delta_{ij} M,$	$[K_i, P_j] = \delta_{ij} P_0,$
$[J_3, P_0] = 0,$	$[J_3, P_0] = 0,$	$[J_3, P_0] = 0,$
$[K_i, P_0] = P_i,$	$[K_i, P_0] = P_i,$	$[K_i, P_0] = 0,$
$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$	$[P_i, P_j] = \epsilon_{ij} A_{mag},$
$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$

Planar Carroll's central extensions

Each central extension parameter becomes a new physical quantity to describe elementary particles.

Planar Carrollian particles are described by,

- mass m ;
- spin ℓ ;
- "exotic" charge κ_{exo}, $[\kappa_{exo}] = MT$;
- "magnetic" charge κ_{mag}, $[\kappa_{mag}] = MT^{-1}$.

κ_{exo}: "shadow of the spin", seems to be a general parameter for anyons.

κ_{mag}: akin to an intrinsic magnetic field.

The description of planar Carroll particles may require 2 new parameters.

Free, planar, massive, and spinless Carroll particles

The KKS method applied to particles with $m \neq 0$, $\ell = 0$, $\kappa_{exo} \neq 0$, $\kappa_{mag} \neq 0$ yield equations of motions that depend on an *effective mass*,

$$m^* = m\left(1 - \frac{\kappa_{mag}\kappa_{exo}}{m^2}\right)$$

$\boxed{m^* \neq 0}$

Free planar particles do not move:

$$\frac{d\boldsymbol{x}}{ds} = 0$$
$$\frac{d\boldsymbol{p}}{ds} = 0$$

$\boxed{m^* = 0}$

The equations degenerate:

$$\frac{dx^i}{ds} = -\frac{\kappa_{exo}}{m^2}\epsilon^{ij}\frac{dp_j}{ds}$$

$\Rightarrow$ Hamiltonian reduction, motion not completely defined

Planar Carroll particles in an EM field – anyons

A particle described by the Casimirs: $m \neq 0$, $\ell \neq 0$, $\kappa_{mag} \neq 0$, $\kappa_{exo} \neq 0$.

There is again an effective mass,

$$m^* := m\left(1 - (\kappa_{mag} + eB)\frac{\kappa_{exo}}{m^2}\right),$$

There equations of motion are, for $m^* \neq 0$,

$$\frac{m^*}{m}\frac{dx^i}{ds} = -\frac{\kappa_{exo}}{m^2}\epsilon^{ij}\left(eE_j + \mu\ell\nabla_j B\right),$$
$$\frac{m^*}{m}\frac{dp_i}{ds} = \left(eE_i + \mu\ell\nabla_i B\right).$$

Carrollian motion can be non-trivial.

Planar massless Carroll particles in an EM field

For a massless ($m = 0$), chargeless particle with anyonic spin and non vanishing magnetic moment $\mu \neq 0$,

$$\frac{dx^i}{ds} = \frac{\mu\ell}{\kappa_{mag}}\epsilon^{ij}\,\nabla_j B,$$

$$\kappa_{exo}\frac{dp_i}{ds} = 0.$$

Since $B^* := \kappa_{mag}$ is akin to a magnetic field, and $E^* = \nabla B$ to an effective electric field, we can write the eom as,

$$\frac{dx^i}{ds} = \mu\ell\,\epsilon^{ij}\,\frac{E^*_j}{B^*},$$

which is effectively Hall-like.

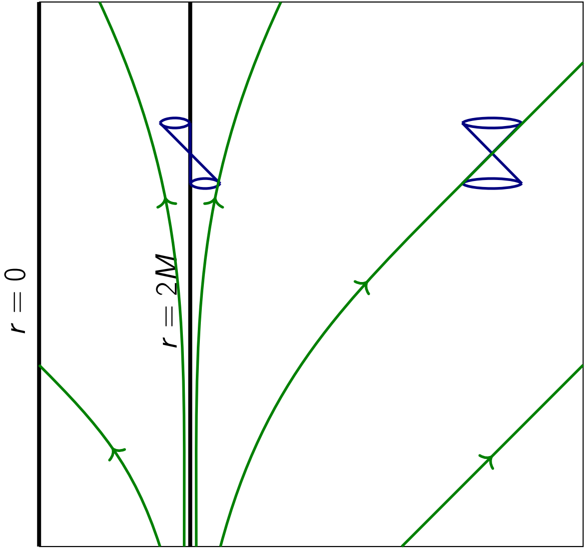
Mathematical motion?

- Mathematically, Carrollian dynamics allow for non trivial motion
  $\Rightarrow$ are these motions actually realized in Nature?

- To understand what could these motion look like, we will apply these equations to an hypothetical photon with non vanishing magnetic moment μ on the horizon of a black hole.

Trapping a photon on an horizon

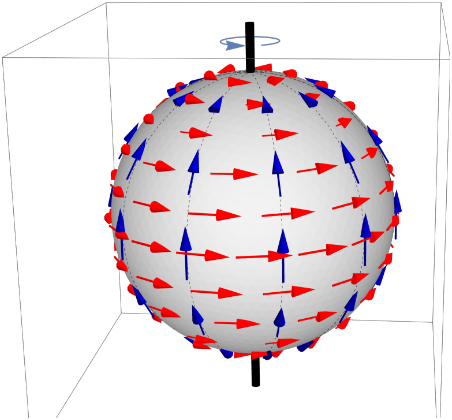
Eddington-Finkelstein diagram of a Schwarzschild black hole:



A photon emitted radially outward on the horizon stays in place.

The horizon being a Carroll hypersurface, this is a good example of the Carrollian "no motion" law

Velocity drift on the horizon of a Kerr-Newmann BH



Electromagnetic potential in a KN spacetime:

$$A = Qr\frac{a\sin^2\theta d\varphi - dt}{r^2 + a^2\cos^2\theta}.$$

The dr component of the magnetic field is induced on $r = \text{const}$ hypersurfaces:

$$B_r = \frac{2aQr\left(r^2 + a^2\right)\cos\theta}{\left(r^2 + a^2\cos^2\theta\right)^3}.$$

The eom $dx^i/ds = \frac{\mu\ell}{\kappa_{mag}}\epsilon^{ij}\nabla_j B$ are given by,

$$\dot{x}^\theta = 0\,, \quad \dot{x}^\varphi = 2aQr\frac{\mu\ell}{\kappa_{mag}}\frac{(r^2+a^2)(r^2-5a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^4}\sin\theta.$$

Symmetries

The dynamical symmetries of the above equations of motion are computed,

- Linear momentum

$$p_\varphi = \kappa_{mag}\theta$$

- Infinite dimensional symmetries: super translations $X = \mathcal{T}(\theta)\partial_s$,

$$\theta = \text{const}$$

$\Rightarrow$ The super translations bring no additional physical information.

Conclusion

- Carroll symmetry is an effective symmetry of null hypersurfaces in General Relativity
- In $2+1$ dimensions, one needs to consider the double central extension of the Carroll group to be the most general
- Two new Casimirs to describe elementary particles: κ_{mag} and κ_{exo}
- Carroll symmetry describes Hall motion for massless but charged particles, and also Hall-like motion for massless particles without charge but with anyonic spin.

Backup slides

Link between Galilei and Carroll groups

The Carroll group is a subgroup of the Bargmann group, which is the central extension of the Galilei group:

$$
\underset{\text{Bargmann}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & e \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}} \longrightarrow \underset{\text{Galilei}}{\begin{pmatrix} R & \boldsymbol{b} & \boldsymbol{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}}
$$

$$
\uparrow
$$

$$
\underset{\text{Carroll}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & 0 \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}} \cong \underset{\text{Carroll}}{\begin{pmatrix} R & 0 & \boldsymbol{c} \\ -\overline{\boldsymbol{b}}R & 1 & f \\ 0 & 0 & 1 \end{pmatrix}}
$$

Explicit example: the horizon of Kerr-Newman black holes

The Kerr-Newman metric,

$$g = -\frac{\Delta}{\Sigma}\left(-dv + \frac{\Sigma}{\Delta}dr + a\sin^2\vartheta d\varphi\right)^2 + \frac{\sin^2\vartheta}{\Sigma}\left(adv - (r^2+a^2)d\varphi\right)^2 + \Sigma d\vartheta^2 + \frac{\Sigma}{\Delta}dr^2$$

The induced metric on this horizon, at $\Delta = 0$ with $r = \text{const}$, is,

$$\widetilde{g} = \frac{\sin^2\theta}{\Sigma}\left(a\, dv - (r^2+a^2)d\varphi\right)^2 + \Sigma d\theta^2$$

The vector field ξ such that $g(\xi) = 0$ and $L_\xi g = 0$ is,

$$\xi = \partial_v + \frac{a}{r^2+a^2}\partial_\varphi$$

Change of coordinates $(\theta, \varphi, t) \mapsto (\theta, \widetilde{\varphi} = \varphi - \frac{a}{r^2+a^2}s, s = v)$,

$$\widetilde{g} = \frac{(r^2+a^2)\sin^2\theta}{\Sigma}d\widetilde{\varphi}^2 + \Sigma d\theta^2 \quad \& \quad \xi = \partial_s$$