Quantitative De Giorgi method for kinetic Fokker-Planck equation

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Summary

- Study of quantitative regularity properties (Hölder continuity, Harnack and weak Harnack) of the kinetic Fokker-Planck equation
- $f = f(t, x, v)$ solution of

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$$

- Rough coefficients (merely measurable) such that

$$\begin{cases} A = A(t, x, v) \text{ satisfies } 0 < \lambda \leq A \leq \Lambda \\ B = B(t, x, v) \text{ satisfies } |B| \leq \Lambda \\ S = S(t, x, v) \text{ is in } L^\infty \end{cases}$$

- Focus on the De Giorgi method
- Non divergence form still open question

Historical overview: Hölder continuity results

Equations with **bounded merely measurable** coefficients:

1. Elliptic equation: $-\nabla \cdot A(x)\nabla u = 0$

   - [1957, De Giorgi] 19th Hilbert problem: **Analiticity** of

     Local minimizers of $\mathcal{E}(w) = \int_\Omega F(\nabla w)\mathrm{d}x$

     Euler-Lagrange equation $-\nabla \cdot DF(\nabla w) = 0$

     Derive EL wrt ∂_{x_i} $-\nabla \cdot D^2F(\nabla w)\nabla(\partial_{x_i} w) = 0$

     Hölder Continuity $-\nabla \cdot A(x)\nabla u = 0$

2. Parabolic equation: $\partial_t u - \nabla_x \cdot A(t,x)\nabla_x u = 0$

   - [1958, Nash]: Independently, elliptic and parabolic equations
   - [1960, Moser]: New approach, Harnack inequality

Motivations in kinetic theory

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

- Long range interactions means grazing collision dominates and leads to singular Boltzmann collision operator
- Coulomb interactions ill-defined for the Boltzmann collision operator but Landau 1936 derived

$$Q(f, f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \mathbf{P}\Big(f(v_*)\nabla_v f(v) - f(v)\nabla_v f(v_*) \Big) |v - v_*|^{-1} \, \mathrm{d}v_* \right)$$

where $\mathbf{P}$ orthogonal projection on $(v - v_*)^\perp$

- Rewrites as a nonlinear local drift-diffusion operator

$$Q(f, f) = \nabla_v \cdot \left(A[f]\nabla_v f + B[f] f \right)$$

$$\begin{cases} A[f](v) = a \displaystyle\int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{-1} f(t, x, v - w) \, \mathrm{d}w \\ B[f](v) = b \displaystyle\int_{\mathbb{R}^3} |w|^{-3} \, w \, f(t, x, v - w) \, \mathrm{d}w \end{cases}$$

Regularity for kinetic Fokker-Planck solutions

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + S, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$$

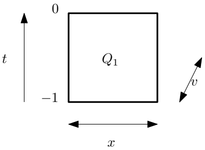
Theorem (Interior Hölder regularity)

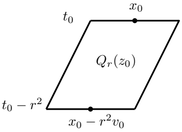
$f : Q_2 \to \mathbb{R}$ *solution* $\Rightarrow \begin{cases} f \in C^\alpha(Q_1) \\ \|f\|_{C^\alpha(Q_1)} \lesssim_{d,\lambda,\Lambda} \|f\|_{L^2(Q_2)} \end{cases}$

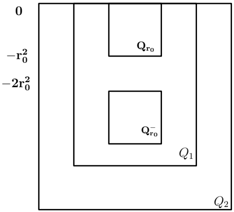
Theorem (Harnack inequalities)

There exists $r_0 \in (0,1)$, $f \geq 0$

- $f : Q_2 \to \mathbb{R}$ *super-solution* $\Rightarrow \left(\int_{Q^-_{r_0}} f^q \right)^{1/q} \lesssim \inf_{Q_{r_0}} f$ *(Weak Harnack)*
- $f : Q_2 \to \mathbb{R}$ *solution* $\Rightarrow \sup_{Q^-_{r_0}} f \lesssim \inf_{Q_{r_0}} f$







Main result and litterature about two methods

Quantitative Krüzkov-Moser method:

- Pascucci-Polidoro (2004): Boundedness of solution
- Wang-Zhang (2009): Hölder continuity (solution point of view) without B and S
- G. - Imbert (2021): New weak Harnack inequality and quantitative Hölder and Harnack inequality (sub/super-sol point of view)

Question: Quantitative De Giorgi method?

Elliptic	Yes DG
Parabolic	Yes G. 2019
Kinetic Fokker-Planck	No GIMV 2017

→ No because non-quantitative step: Intermediate value lemma

Main new result:

Quantitative kinetic FP intermediate value lemma

Comparison of methods

De Giorgi method (eq or DG classes)

- 1st Lm: $L^2 - L^\infty$ estimate
- 2nd Lm: Intermediate value lemma
  $$|f \le k||f \ge l| \le C|k < f < l|^\alpha$$

applied to linear transformations

Moser-Krüzkhov method(eq)

- Estimation $L^2 - L^\infty$
- Poincaré inequality

applied to log of solution

Measure to pointwise estimate

"$f \le 1$ with enough mass below $0 \Rightarrow f \le 1 - \mu$ in small cylinder"

scaling arguments

Hölder continuity (sol)

Scaling and covering argument

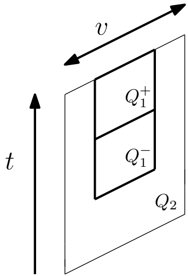
Weak Harnack (super-sol)

$L^2 - L^\infty$ estimate

Harnack (sol)

Parabolic IVL and counterexamples

Parabolic case: $\partial_t f \leq \nabla_v \cdot (A \nabla_v f)$

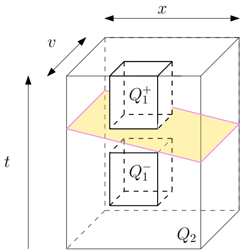


IVL (G. 2020): For $a < b$

$$|f \leq a, Q_1^-||f \geq b, Q_1^+| \leq C|a < f < b, Q_2|^{1/6}$$

$$f(t,v) = \begin{cases} b & \text{for } t < -1 \\ a & \text{for } t \geq -1 \end{cases}$$

Kinetic FP case: $\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot (A \nabla_v f)$



For $c > 2$,

$$f(t,x,v) = \begin{cases} b & \text{for } x + ct < -2 \\ a & \text{for } x + ct \geq -2 \end{cases}$$

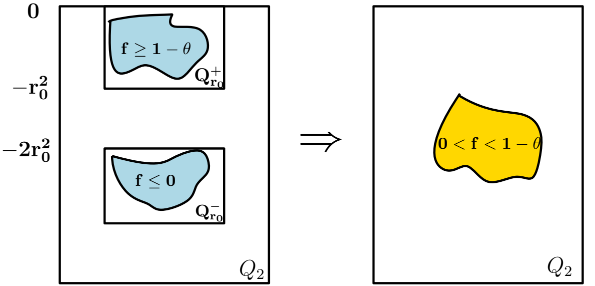
Kinetic Fokker-Planck intermediate value lemma

Second Lemma of De Giorgi

Lemma (Intermediate value lemma (G. Mouhot 2021))

$\forall \delta_1, \delta_2 \in (0,1)$, $\exists r_0, \theta, \nu \in (0,1)$ *where* $\nu \gtrsim (\delta_1\delta_2)^{5d+8}$ *st for all* $f : Q_2 \to \mathbb{R}$ *subsol st* $f \leq 1$ *in* Q_1 ,

$$\begin{cases} |\{f \leq 0\} \cap Q^-_{r_0}| & \geq & \delta_1 |Q^-_{r_0}| \\ |\{f \geq 1-\theta\} \cap Q^+_{r_0}| & \geq & \delta_2 |Q^+_{r_0}| \end{cases} \Rightarrow |\{0 < f < 1-\theta\} \cap Q_2| \geq \nu |Q_2|$$



Trajectory method for the IVL

Main step Poincaré inequality: $f : Q_5 \to \mathbb{R}$ non-negative sub-solution

$$\int_{Q_1^+} \left(f - \int_{Q_1^-} f \right)_+ \lesssim \frac{1}{\varepsilon^\alpha} \int_{Q_5} |\nabla_v f| \; (+\varepsilon^\beta \|f\|_{L^2})$$

It's enough to estimate the difference $f(t,x,v) - f(s,y,w)$ for that we introduce $V = \frac{x-y}{t-s}$ to construct a "good" trajectory.

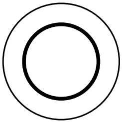
(t,x,v) — (t,x,V) — (s,y,V) — (s,y,w)

$$\underbrace{f(t,x,v) - f(t,x,V)}_{\lesssim \int |\nabla_v f|} + \underbrace{f(t,x,V) - f(s,y,V)}_{\lesssim \int \partial_t f + V \cdot \nabla_x f \lesssim \int \nabla_v \cdot (A \nabla_v f)} + \underbrace{f(s,y,V) - f(s,y,w)}_{\lesssim \int |\nabla_v f|}$$

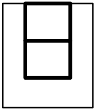
Intuition: Hörmander commutators with ∇_v et $\partial_t + v \cdot \nabla_x$ span all the directions

Advantage of the method: adaptation to other equations

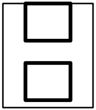
Elliptique



Parabolique



Cinétique



Thanks for your attention!