Quasi-periodic Traveling Waves on an Infinitely Deep Perfect Fluid Under Gravity

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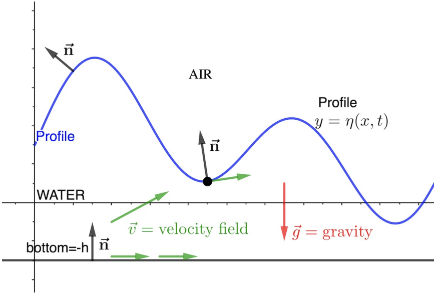
Goal: to understand fluid motion, such as the Water Waves on the surface of the ocean

- many questions and possible applications :
  - the equation governing the fluid motions are **quasilinear PDEs**;
  - to study the evolution for **long times** is challenging;
  - to study instabilities and breakdown phenomena is very hard;
- several possible scenarios:
  - Rotational vs irrotational fluids;
  - gravity and/or surface tension;
  - different types of fluid domains.

Aim of the talk: to present a result about existence and stability of quasi-periodic motions on the surface of a fluid which is

- incompressible, irrotational, perfect,
- two dimensional ($1d$ interface),
- infinitely deep,
- subject to the gravity force,
- under spatial periodic boundary conditions.



The model

We study the motion of an incompressible and irrotational perfect fluid, in the time dependent domain

$$\Omega_t := \{(x, y) \in \mathbb{T} \times \mathbb{R} ; \ -h < y < \eta(t, x)\},$$

where:

- $\eta : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is a smooth enough function (shape of the water);
- $x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, (periodic setting);
- h depth of the fluid. We assume $h = \infty$,

under the action of the gravity. We assume:

$$\text{velocity field : } v(t, x, y) = \nabla_{x,y}\Phi(t, x, y), \quad \Phi = \text{velocity potential}$$

(the flow is irrotational).

The free surface Bernoulli equations

Bernoulli equation

$$\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + g\eta = 0 \quad \text{at} \quad y = \eta(t,x)$$

$$\Delta \Phi = 0 \quad \text{on} \ \ \Omega_t, \quad \Phi = \psi \ \text{ on } \ \{y = \eta(t,x)\}, \ \rightsquigarrow \ \textit{incompressibility}$$

Boundary conditions

$$\partial_y \Phi \to 0\,, \quad \text{as} \quad y = -\infty$$

$$\partial_t \eta = \partial_y \Phi - \eta_x(\partial_x \Phi) \quad \text{at} \quad y = \eta(t,x)$$

Rmk:

- g: gravity. We set $g = 1$.
- $v = \nabla \Phi$ velocity potential $\rightsquigarrow$ $\mathrm{curl} v = 0$ (the fluid is irrotational)
- The boundary conditions imply that the fluid particles do not cross the bottom/surface.

Zakharov/Craig-Sulem formulation

The unknowns of the problem are the free surface $y = \eta(t,x)$ and the velocity potential $\Phi(t,x,y)$.
Zakharov/Craig-Sulem $\rightsquigarrow$ use variables (η, ψ) where $\psi(t,x) = \Phi(t,x,\eta(t,x))$.

Vector field

$$\begin{cases} \partial_t \eta = G(\eta)\psi\,, \\ \partial_t \psi = -g\eta - \frac{1}{2}\psi_x^2 + \frac{1}{2}\frac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2}\,, \end{cases}$$

where $G(\eta)\psi$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := \sqrt{1+\eta_x^2}(\partial_n \Phi)|_{y=\eta(t,x)} = (\partial_y \Phi - \eta_x \partial_x \Phi)(t,x,\eta(t,x))$$

and ∂_n is the outward unit normal at the free interface $y = \eta(t,x)$.
From (η, ψ) one recovers $\Phi(t,x,y)$ by solving the elliptic problem

$$\begin{cases} \Delta\Phi = 0\,, & -\infty < y < \eta(t,x)\,, \\ \Phi(t,x,\eta(t,x)) = \psi(t,x) & \\ (\partial_y \Phi) \to 0\,, & as\ y \to -\infty\,, \end{cases}$$

Hamiltonian structure of the water waves

The WW can be written as

$$(\eta_t, \psi_t) = X_H(\eta, \psi) = (\nabla_\psi H, -\nabla_\eta H)$$

with

$$H(\eta, \psi) = \frac{1}{2} \int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx + \frac{g}{2} \int_{\mathbb{T}} \eta^2\, dx$$

Remarks:

- Poisson bracket: $\{F, H\} = \int_{\mathbb{T}} \big(\nabla_\eta H \nabla_\psi F - \nabla_\psi H \nabla_\eta F\big) dx$
- Constants of motion:
  - the "mass" $\mathtt{L} := \int_{\mathbb{T}} \eta(x) dx$,
  - the "momentum" $\mathtt{M} := \int_{\mathbb{T}} \eta_x(x)\psi(x) dx$
- invariant subspace $\int_{\mathbb{T}} \eta\, dx = \int_{\mathbb{T}} \psi\, dx = 0$. (recall $h = -\infty$!!)
- other symmetries:
  - Reversibility: let $S : (\eta, \psi)(x) \to (\eta, -\psi)(-x)$, $\Rightarrow X_H \circ S = -S \circ X_H$
  - parity-preserving: preserves the subspace of even functions of x;

Linearized problem at the flat interface

We look for small amplitude solutions! The linearized equations at $(\eta, \psi) = 0$ are

$$\begin{cases} \partial_t \eta = G(0)\psi \\ \partial_t \psi = -\eta \,, \end{cases} \qquad G(0) = |D| = |-\mathrm{i}\partial_x| \tag{1}$$

Complex coordinates: eq. (1) can be written as

$$u = \frac{1}{\sqrt{2}}\big(|D|^{-\frac14}\eta + \mathrm{i}|D|^{\frac14}\psi\big) \qquad \rightsquigarrow \qquad \partial_t u = -\mathrm{i}|D|^{\frac12} u \,.$$

Linear solutions

$$u(t,x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} u_j(0) e^{-\mathrm{i}\sqrt{|j|}t + \mathrm{i}jx} \,, \qquad u_j(0) := \frac{1}{\sqrt{2}}\big(|j|^{-\frac14}\eta_j(0) + \mathrm{i}|j|^{\frac14}\psi_j(0)\big)$$

Dispersion law: $j \mapsto \sqrt{|j|}, \quad j \in \mathbb{Z}\setminus\{0\}$.

Linearized problem at the flat interface: remarks

- Completely resonant: if $S \subset \mathbb{Z} \setminus \{0\}$ is a finite set of modes then the linear solutions

$$\sum_{j \in S} c_j \, e^{\mathrm{i}(jx - \sqrt{|j|}t)}$$

are periodic or quasi-periodic (depending on $\sqrt{|j_i|}$, $j_i \in S$). Nevertheless we notice that solutions initially Fourier supported on the infinite-dimensional invariant subspace $\{e^{\mathrm{i}n^2 x} \, : \, n \in \mathbb{Z}\}$ are periodic in time. We say that the WW are completely resonant

- Weak dispersion: $j \mapsto \sqrt{|j|}$ is sub-linear. To develop KAM theory is harder...

- NO external parameters to impose non-resonances among linear frequencies of oscillation $\sqrt{|j_i|}$, $j_i \in S$.
  - ex.: the dispersion law for the gravity and capillary water waves is

$$j \mapsto \sqrt{\kappa |j|^3 + g|j|}$$

where κ is the surface tension parameter.

Traveling quasi-periodic solutions

Quasi-periodic solutions

We say that a function

$$(\eta(t,x),\psi(t,x)) : \mathbb{R}\times\mathbb{T}\to\mathbb{R}^2$$

is a quasi-periodic solution of the WW equations with irrational frequency vector $\omega\in\mathbb{R}^{\nu}$ (i.e. $\omega\cdot\ell\neq 0$ for all $\ell\in\mathbb{Z}^{\nu}\setminus\{0\}$) if there is a smooth embedding

$$\mathbb{T}^{\nu}\to H^1_0(\mathbb{T};\mathbb{R})\times H^1_0(\mathbb{T};\mathbb{R}) \qquad \varphi \quad \mapsto U(\varphi,x):=(\tilde\eta(\varphi,x),\tilde\psi(\varphi,x))$$

such that $(\eta(t,x),\psi(t,x))=U(\omega t,x)$ solves the WW.

Traveling quasi-periodic

A quasi-periodic solution is traveling with velocity vector $\mathrm{v}\in\mathbb{Z}^{\nu}$ if there is a function $\widetilde{U}:\mathbb{T}^{\nu}\to\mathbb{R}^2$ such that

$$(\eta(t,x),\psi(t,x))=U(\omega t,x)=\widetilde{U}(\omega t-\mathrm{v}x)\,.$$

Remarks

QP sol's $\leftrightsquigarrow$ tori embedded in the phase space invariant for the flow of H.

We look for $\varphi \to U(\varphi; x)$ with the following properties :

1. $U(\omega t, x)$ is a QP sol. for the WW, i.e. if $\Psi^t_\omega(\varphi_0) := \varphi_0 + \omega t$, $\Phi^t_H \sim$ flow of H,

$$U \circ \Psi^t_\omega = \Phi^t_H \circ U \quad \rightsquigarrow \quad \omega \cdot \partial_\varphi U - X_H(U) = 0\,. \tag{2}$$

2. $U(\omega t, x)$ is traveling $\rightsquigarrow$ additional constraint on $\varphi \to U(\varphi, x)$: the embedding is invariant for the flow of $\mathtt{M} = \int \eta_x \psi$, i.e.

$$U \circ \Psi^s_{-\mathtt{v}} = \Phi^s_{\mathtt{M}} \circ U \quad \rightsquigarrow \quad \mathtt{v} \cdot \partial_\varphi U + X_{\mathtt{M}}(U) = 0\,, \quad X_{\mathtt{M}}(U) = \partial_x U \tag{3}$$

- It is possible to solve at the same time (2), (3) since H, $\mathtt{M}$ poisson commutes.
- Notice that (3) $\Leftrightarrow U(\varphi, x) = \widetilde{U}(\varphi - \mathtt{v}x)$.

Main result

We shall construct solutions localized in Fourier space at ν distinct *tangential sites*:

$$S := \{\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu\} \subset \mathbb{Z}\,, \quad \textit{velocity vector}: \mathtt{v} := (\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu) \in \mathbb{Z}^\nu$$

$$\textit{frequency vector}: \overline{\omega} := \left(\sqrt{|\bar{\jmath}_1|}, \ldots, \sqrt{|\bar{\jmath}_\nu|}\right) \in \mathbb{R}^\nu$$

We construct small amplitude, q-p traveling waves solutions "close" to the linear ones, namely they will be of the form

$$\begin{aligned} \eta(t,x) &= \sum_{j\in S} \sqrt{2\zeta_j}|j|^{\frac{1}{4}} \cos\left(\omega_j t - jx\right) + o(\sqrt{|\zeta|})\,, \\ \psi(t,x) &= -\sum_{j\in S} \sqrt{2\zeta_j}|j|^{-\frac{1}{4}} \sin\left(\omega_j t - jx\right) + o(\sqrt{|\zeta|})\,, \end{aligned} \tag{4}$$

where $\zeta = (\zeta_j)_{j\in S}$ with $\zeta_j > 0$, $\omega = \bar{\omega} + O(|\zeta|)$ and $o(\sqrt{|\zeta|})$ in the H^s-topology.
$(\sqrt{\zeta_j})_{j\in S} \sim$ amplitudes of the approximate solution from which we have the bifurcation.

Quasi-periodic traveling gravity waves. (F- Giuliani, (Memoires of AMS, to appear))

For a generic choice of $\mathtt{v} \in \mathbb{Z}^{\nu}$ there exist a Cantor-like set $\mathfrak{A}$ of small amplitudes $\zeta \in \mathbb{R}^{\nu}_{+}$ with density 1 at $\zeta = 0$ such that the following holds.
For any $\zeta \in \mathfrak{A}$ the WW equation possesses a small amplitude, linearly stable, quasi-periodic solution

$$(\eta, \psi)(t, x; \zeta) = U(\omega t, x; \zeta) \in H^{s}(\mathbb{T}^{\nu+1}, \mathbb{R}^{2}), \quad s \gg 1,$$

which is a traveling wave with velocity vector $\mathtt{v}$ and diophantine frequency vector $\omega := \omega(\zeta) \in \mathbb{R}^{\nu}$.

- generic choice= the tangential sites $\bar{\jmath}_i$ are chosen in such a way that the vector $\mathtt{v} := (\bar{\jmath}_1, \ldots, \bar{\jmath}_{\nu})$ is not a zero of a certain non-trivial polynomial $\mathbb{C}^{\nu} \to \mathbb{C}$.
- the frequency vector $\omega(\zeta) = \bar{\omega} + O(|\zeta|)$ is a perturbation of the linear freq.. The corrections $O(|\zeta|)$ comes from the nonlinearity.
- the generic choice of $\mathtt{v}$ guarantees some non-degeneracy conditions. Then one proves $\zeta \to \omega(\zeta)$ is a diffeomorphism.

Quasi-periodic traveling gravity waves. (F- Giuliani, (Memoires of AMS, to appear))

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Non-resonance conditions

The parameters ζ will be selected from a complicated Cantor-like set in order to impose Melnikov conditions as

$$|\omega \cdot l| \gtrsim \frac{\gamma}{|l|^\tau}, \quad \forall l \in \mathbb{Z}^\nu, \qquad (\textit{diophantine condition})$$

$$|\omega \cdot l + \sqrt{|j|} - \sqrt{|k|}\,| \gtrsim \frac{\gamma}{|l|^\tau}, \quad \forall l \in \mathbb{Z}^\nu, \ \ j, k \in S^c, \qquad (2^{nd} \ \textit{Melnikov conditions})$$

Some literature: periodic sol.

- Traveling waves: stationary w.r.t. a moving reference frame.
    - Stokes (Trans. Cambridge Phil, 1847) approximate flows...
    - Levi-Civita (Math. Ann. 1925), Struik (Math. Ann. 1926) $\rightsquigarrow$ $2d$-traveling gravity WW
    - Craig-Nicholls (SIAM J. Math. Anal. 2000) $\rightsquigarrow$ $2/3d$-gravity-capillary WW
    - Ioss-Plotnikov (Mem. Amer. Math. Soc. 2009)-(Arch. Rat. Mech. Anal. 2011) $\rightsquigarrow$ $3d$-gravity WW $\leftrightarrow$ small divisors
    - Lokharu-Seth-Wahlén (Arch. Rat. Mech. Anal. 2020) $\rightsquigarrow$ $3d$-gravity-capillary+vorticity
    - many other $\rightsquigarrow$ approach with variational methods.... (see Buffoni, Grooves, Sun, Wahlén...)
- Standing waves: NOT stationary w.r.t. a moving reference frame, but even in x
    - Plotnikov-Toland (Arch. Rat. Mech. Anal. 2001) $\rightsquigarrow$ $2d$-gravity, finite depth
    - Iooss-Plotnikov-Toland (Arch. Ration. Mech. Anal. 2005) $\rightsquigarrow$ $2d$-gravity, ∞ depth $\leftrightarrow$ completely resonant
    - Iooss-Plotnikov (J. math. fluid mech. 2005) $\rightsquigarrow$ $3d$-gravity, ∞ depth $\leftrightarrow$ completely resonant
    - Alazard-Baldi (Arch. Ration. Mech. Anal. 2015) $\rightsquigarrow$ $2d$-gravity-capillary, ∞ depth $\leftrightarrow$ almost all κ

Some literature: quasi-periodic sol.

- Standing waves:
  - Berti-Montalto (Memoires of AMS, 2020) $\rightsquigarrow$ $2d$-gravity-capillary, ∞ depth $\leftrightarrow$ almost all κ
  - Baldi-Berti-Haus-Montalto (Invent. math. 2018) $\rightsquigarrow$ $2d$-gravity, finite depth $\leftrightarrow$ for most values of the depth h
- Traveling waves:
  - Berti-Franzoi-Maspero (Arch. Rat. Mech. Anal. 2021) $\rightsquigarrow$ $2d$-gravity-capillary, with constant vorticity $\leftrightarrow$ for most values of κ
- KAM for other fluid models:
  - Baldi-Montalto qp sol. for incompressible Euler flows in 3d (Advances in Math. 2021) $\rightsquigarrow$ in presence of an quasi-periodic external forcing
  - F-Giuliani-Procesi (Comm. Math. Phys 2020) (+series of papers) $\rightsquigarrow$ Hamiltonian perturbation of Degapseris-Procesi equation $\leftrightarrow$ weakly dispersive, completely resonant

Some comments

1. we do not use "physical" parameters (such as κ, g, h...)
    - to tune the linear frequencies we use the "intitial data" or the amplitude of a suitable approximate sol. as internal parameters
2. to deal with item (1) we use *Birkhoff normal form* techniques:
    - not new in KAM theory for completely resonant PDEs $\rightsquigarrow$ KdV, NLS eq's...(semilinear) (see for instance Kuksin-Pöschel '96)
    - more difficulties in the context of quasi-linear PDEs: weak version of BNF (see Baldi-Berti-Montalto A.I.H.P. '16)
    - new approach for gravity WW which has non-trivial 4-wave resonant interactions
3. Overall strategy: BNF + Nash-Moser scheme + Reducibility
    - quite well-established in KAM lit. $\rightsquigarrow$ highly non-trivial for WW eq's
    - Dyachenko-Zakharov / Craig-Worfolk conjecture on approx. integrability at order 4 (Phys. Let. '94, Phys, D '95) $\rightsquigarrow$ rigorous proof Berti, F., Pusateri (CPAM '22).
    - integrability do not implies non-degeneracy!
    - technical difficulties in reducibility (quasi-linear eq's, see BM '20, BBHM '18) plus bad diophantine estimates: $\omega(\zeta) = \bar{\omega} + O(\varepsilon^2)$ when $|\zeta| \sim \varepsilon^2$. Then

$$\omega(\zeta) = \bar{\omega} + O(\varepsilon^2) \quad \Rightarrow \quad |\omega \cdot l| \geq \gamma |\ell|^{-\tau}, \;\; \gamma \sim \varepsilon^2 \sim |\zeta| \sim \textit{amplitude}$$

Thank you for the attention!