Moving currents: on the Lie transport equation

Paolo Bonicatto

University of Warwick

Joint work with G. Del Nin and F. Rindler (Warwick)

www.ercsingularity.org

Ypatia 2022

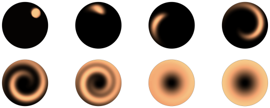
Rome, 8 June 2022

Introduction

Transport equations and **conservation laws** are ubiquitous in the mathematical description of physical phenomena, encompassing such diverse topics as fluid mechanics, electromagnetism, materials science and several others.

Introduction

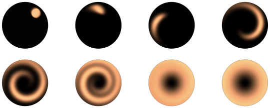
Transport equations and **conservation laws** are ubiquitous in the mathematical description of physical phenomena, encompassing such diverse topics as fluid mechanics, electromagnetism, materials science and several others.



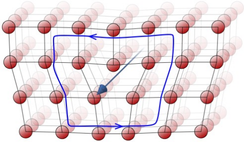
The evolution of an initial concentration subject to radial stirring and small diffusion (milk stirred in a coffee mug!). Picture by M. Coti Zelati et al.

Introduction

Transport equations and **conservation laws** are ubiquitous in the mathematical description of physical phenomena, encompassing such diverse topics as fluid mechanics, electromagnetism, materials science and several others.



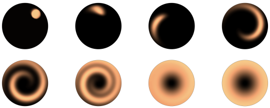
The evolution of an initial concentration subject to radial stirring and small diffusion (milk stirred in a coffee mug!). Picture by M. Coti Zelati et al.



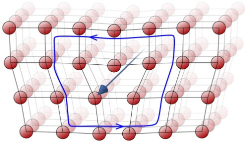
The movement of *dislocations* is at the basis of plastic deformation in crystalline materials. Picture by F. Rindler.

Introduction

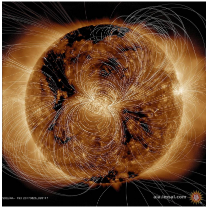
Transport equations and **conservation laws** are ubiquitous in the mathematical description of physical phenomena, encompassing such diverse topics as fluid mechanics, electromagnetism, materials science and several others.



The evolution of an initial concentration subject to radial stirring and small diffusion (milk stirred in a coffee mug!). Picture by M. Coti Zelati et al.



The movement of *dislocations* is at the basis of plastic deformation in crystalline materials. Picture by F. Rindler.



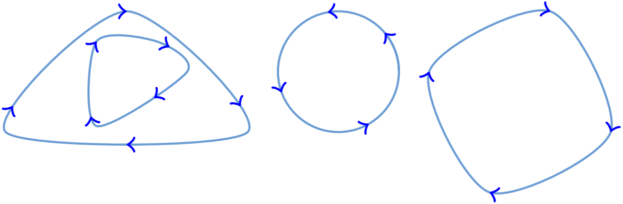
A computer-generated view of the Sun's magnetic field. Picture by the NASA's Solar Dynamics Observatory.

Moving loops: a dynamical theory for IR currents

Integer rectifiable (IR) currents are vector-valued measures T of the form

$$T := \vartheta\, v_1 \wedge \ldots \wedge v_k\, \mathscr{H}^k \llcorner E$$

where E is a k-countably rectifiable set, $v_1, \ldots, v_k$ are an orthonormal basis of the approximate tangent space of E and $\vartheta \in \mathrm{L}^1(E; \mathbb{Z}; \mathscr{H}^k)$.



General structure of boundaryless, IR 1-currents in $\mathbb{R}^d$.

Moving loops: a dynamical theory for IR currents

Let $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ be a given vector field. We want to consider the following **transport equation** on IR currents:

$$\begin{cases} \frac{d}{dt} T_t + \mathcal{L}_{\boldsymbol{b}} T_t = 0 & \text{in } (0,1) \times \mathbb{R}^d \\ T_0 = \overline{T} & \text{as currents in } \mathbb{R}^d \end{cases} \tag{PDE}$$

where $\overline{T}$ is a given initial IR current and

$$\mathcal{L}_{\boldsymbol{b}} T := -\boldsymbol{b} \wedge \partial T - \partial(\boldsymbol{b} \wedge T)$$

is the Lie derivative operator.

Remarks

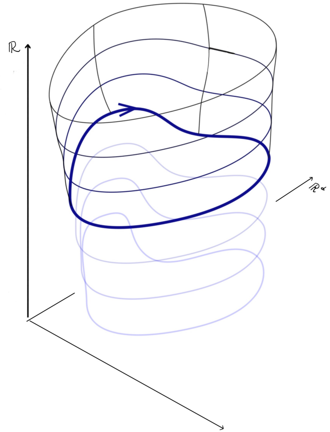
- If $\boldsymbol{b}$ is smooth $\implies \exists!$ of solution to (PDE). The solution is the transport of the initial current, i.e. $T_t = (\Phi_t)_* \overline{T}$, where Φ is the flow of $\boldsymbol{b}$.
- It seems rather difficult to develop a theory for **rough vector fields** (lack of regularity of the Regular Lagrangian Flow).

A levelset approach to the transport of currents

The levelset method is a useful approach to the description of evolving surfaces (e.g. mean curvature flow).

A levelset approach to the transport of currents

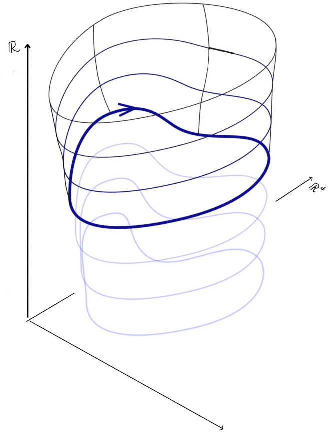
The levelset method is a useful approach to the description of evolving surfaces (e.g. mean curvature flow).



- Start from an **IR space-time** $(1+k)$**-current** S in $\mathbb{R} \times \mathbb{R}^d$;

A levelset approach to the transport of currents

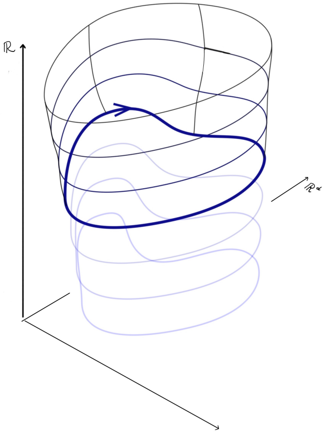
The levelset method is a useful approach to the description of evolving surfaces (e.g. mean curvature flow).



- Start from an **IR space-time** $(1+k)$-**current** S in $\mathbb{R} \times \mathbb{R}^d$;
- "slice" it with $\{t\} \times \mathbb{R}^d$, obtaining S_t;

A levelset approach to the transport of currents

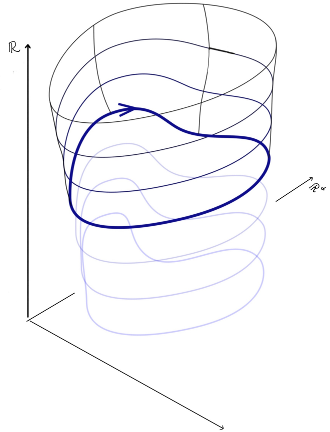
The levelset method is a useful approach to the description of evolving surfaces (e.g. mean curvature flow).



- Start from an **IR space-time** $(1+k)$-**current** S in $\mathbb{R} \times \mathbb{R}^d$;
- "slice" it with $\{t\} \times \mathbb{R}^d$, obtaining S_t;
- S_t can be identified with an IR k-current in $\mathbb{R}^d$;

A levelset approach to the transport of currents

The levelset method is a useful approach to the description of evolving surfaces (e.g. mean curvature flow).



- Start from an **IR space-time** $(1+k)$**-current** S in $\mathbb{R} \times \mathbb{R}^d$;
- "slice" it with $\{t\} \times \mathbb{R}^d$, obtaining S_t;
- S_t can be identified with an IR k-current in $\mathbb{R}^d$;
- taking advantage of the space-time structure, we can define a vector field $\boldsymbol{b}$ advecting S_t.

Negligible Criticality condition

In order to make this recipe work we need to impose (beside a mild regularity in time) a **Negligible Criticality** condition on the current S, which reads

$$\|S\| \llcorner \{\nabla^S \mathbf{t} = 0\} = 0, \tag{NC}$$

where $\mathbf{t} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the projection on the time variable.

Theorem (B.-Del Nin-Rindler '22)

Given an IR $(1+k)$-current S in $\mathbb{R} \times \mathbb{R}^d$, the slices $(S_t)_t$ solve the PDE

$$\frac{d}{dt} S_t + \mathcal{L}_{\boldsymbol{b}} S_t = 0$$

where $\boldsymbol{b}$ is a suitably defined vector field **if and only if** *S has (NC).*

Negligible Criticality condition

In order to make this recipe work we need to impose (beside a mild regularity in time) a **Negligible Criticality** condition on the current S, which reads

$$\|S\| \llcorner \{\nabla^S \mathbf{t} = 0\} = 0, \tag{NC}$$

where $\mathbf{t} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the projection on the time variable.

Theorem (B.-Del Nin-Rindler '22)

Given an IR $(1+k)$-current S in $\mathbb{R} \times \mathbb{R}^d$, the slices $(S_t)_t$ solve the PDE

$$\frac{d}{dt} S_t + \mathcal{L}_{\boldsymbol{b}} S_t = 0$$

where $\boldsymbol{b}$ is a suitably defined vector field **if and only if** *S has (NC).*

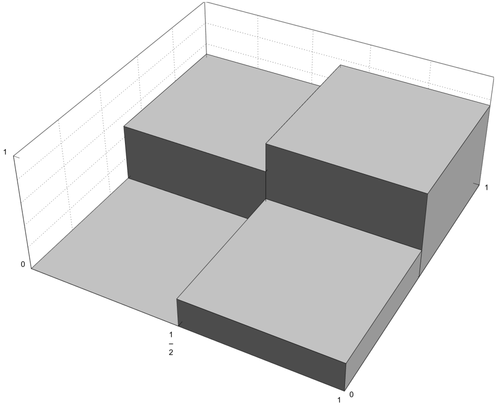
The condition (NC) is thus **necessary and sufficient** for the validity of the PDE within the space-time framework.

More on (NC)

⚠ Notice that (NC) is not always satisfied:

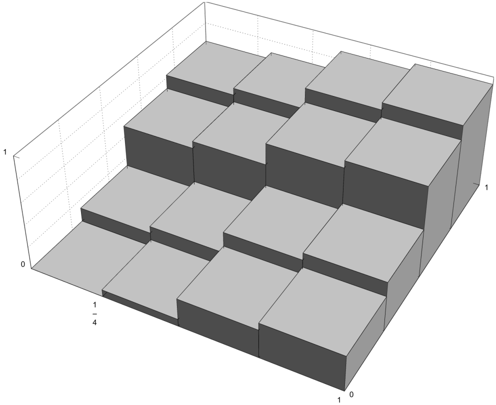
More on (NC)

Notice that (NC) is not always satisfied:



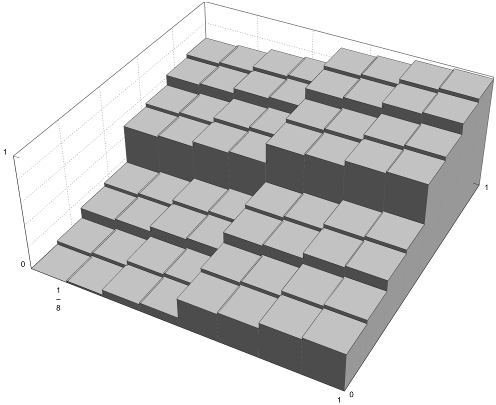
More on (NC)

⚠ Notice that (NC) is not always satisfied:



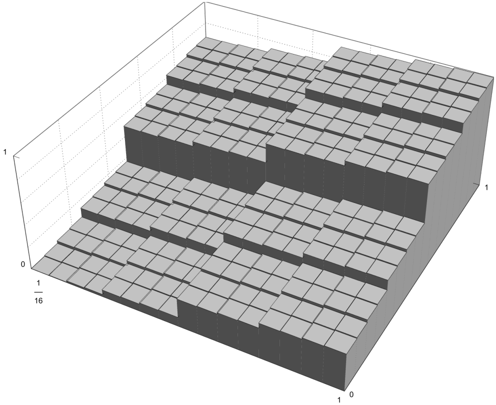
More on (NC)

⚠ Notice that (NC) is not always satisfied:



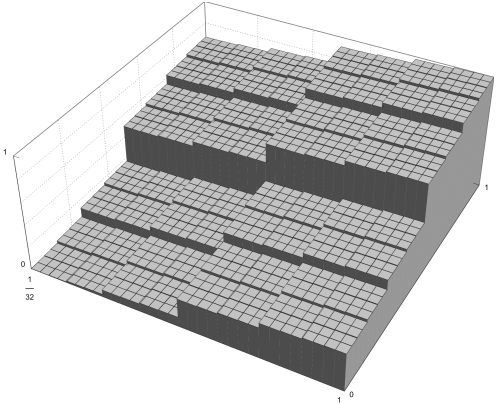
More on (NC)

⚠ Notice that (NC) is not always satisfied:



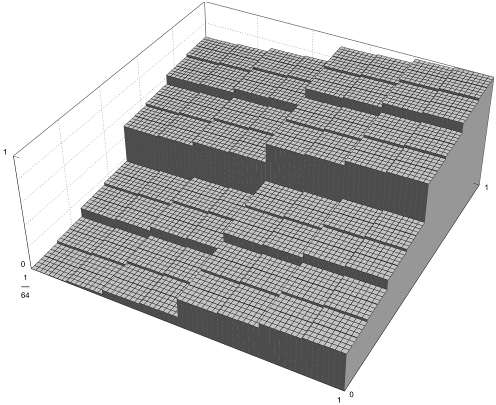
More on (NC)

⚠ Notice that (NC) is not always satisfied:



More on (NC)

⚠ Notice that (NC) is not always satisfied:



More on (NC)

⚠ Notice that (NC) is not always satisfied:

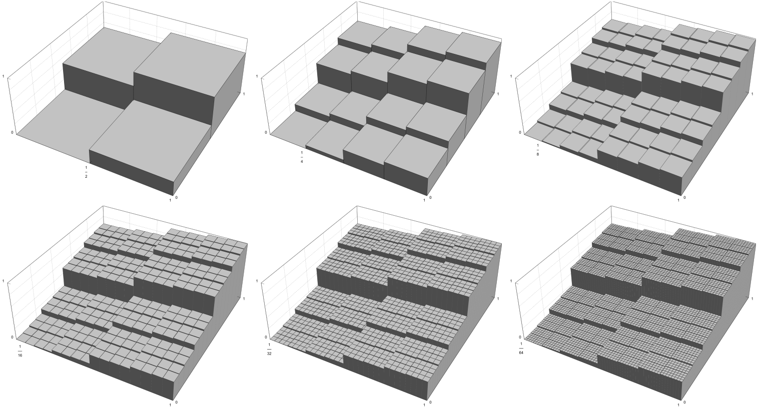


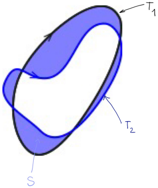
Figure: A space-time current without (NC).

Geometric Rademacher-type theorems

Combining the space-time analysis with appropriate “gluing” procedures, we obtain **Rademacher-type theorems for curves of currents**.

Geometric Rademacher-type theorems

Combining the space-time analysis with appropriate "gluing" procedures, we obtain **Rademacher-type theorems for curves of currents**.
Suppose we are given a Lipschitz map $t \mapsto T_t$, with $\partial T_t = 0$.

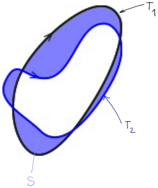


Lipschitz here is intended w.r.t. the homogeneous **flat distance**

$$d(T_1, T_2) := \inf\{\mathbf{M}(S) : \partial S = T_2 - T_1\}.$$

Geometric Rademacher-type theorems

Combining the space-time analysis with appropriate "gluing" procedures, we obtain **Rademacher-type theorems for curves of currents**.
Suppose we are given a Lipschitz map $t \mapsto T_t$, with $\partial T_t = 0$.



Lipschitz here is intended w.r.t. the homogeneous **flat distance**

$$d(T_1, T_2) := \inf\{\mathbf{M}(S) : \partial S = T_2 - T_1\}.$$

Theorem (B.-Del Nin-Rindler, '22)

Under suitable assumptions on the map $t \mapsto T_t$, we can find a **velocity field $\boldsymbol{b}$** *such that the T_t's solve* (PDE) *driven by $\boldsymbol{b}$.*

Such a field $\boldsymbol{b}$ can be thought of as a **geometric derivative of the Lipschitz map** $t \mapsto T_t$.

Thank you!